TORI IN CERTAIN ASPHERICAL FOUR–MANIFOLDS

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Abstract. The homology classes represented by embedded or mapped tori in the product of two surfaces are completely characterized.

If $X^4$ is a 4-manifold, then any element $\xi \in H_2(X^4)$ can be represented by a map from a closed, oriented surface, and also by an embedded, closed, oriented surface (perhaps of bigger genus). If $X^4$ is simply connected then $H_2(X^4) \approx \pi_2(X^4)$ by the Hurewicz Theorem, so that every class $\xi$ is represented by a map $S^2 \to X^4$. It has been an intensively studied fundamental problem to determine which classes are represented by smooth or topologically embedded 2-spheres, and, more generally, to determine the minimum genus of an embedded surface representing $\xi$. In general there are striking differences between the smooth and topological categories. See, for example, [2] and [3] for some of the latest results in this area.

In this note we begin to address similar questions in the aspherical case, in which there are no interesting 2-spheres at all, apart from “local knot theory.”

We determine which 2-dimensional homology classes can be represented by embedded tori in $X^4 = F^2 \times G^2$, where $F^2$ and $G^2$ are closed orientable surfaces of genus greater than 1. It turns out that there is no difference between the topological and differentiable categories in this case.

Theorem 1. Let $F^2$ and $G^2$ be closed orientable surfaces of genus greater than 1. An element $\xi \in H_2(F^2 \times G^2)$ is represented by a topological or differentiable embedded torus $T^2 \subset F^2 \times G^2$ if and only if $\xi = \alpha \times \beta$ for some $\alpha \in H_1(F^2)$ and $\beta \in H_1(G^2)$.

This follows immediately from the following two more precise propositions.

Proposition 1. Any map $f : T^2 \to F^2 \times G^2$ represents a homology class of the form $\alpha \times \beta$ for some $\alpha \in H_1(F^2)$ and $\beta \in H_1(G^2)$.

Proposition 2. If $\alpha \in H_1(F^2)$ and $\beta \in H_1(G^2)$ are given, then the homology class $\alpha \times \beta$ is represented by a smooth embedding $f : T^2 \to F^2 \times G^2$.

Lemma 1. If $F^2$ is a surface of negative euler characteristic, then any two commuting elements in $\pi_1(F^2)$ generate a cyclic subgroup of $\pi_1(F^2)$.

Proof. Let $\alpha, \beta \in \pi_1(F^2)$ be commuting elements and let $S \to F^2$ be the covering map, where $\pi_1(S)$ corresponds to the subgroup of $\pi_1(F^2)$ generated by $\alpha$ and $\beta$. If it is a finite-sheeted covering, then $S \cong T^2$ by the classification of surfaces, and

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Corollary 1. If $F^2$ is a closed orientable surface of negative euler characteristic, and $f : T^2 \rightarrow F^2$ is a map, then there is a primitive homology class $\alpha \in H_1(F^2)$ such that the image of $f_\ast : H_1(T^2) \rightarrow H_1(F^2)$ is contained in the cyclic subgroup generated by $\alpha$.

Proof. Since $\pi_1(T^2) \approx H_1(T^2)$, the image of $H_1(T^2)$ in $H_1(F^2)$ is the image under the Hurewicz homorphism of the image of $\pi_1(T^2)$ in $\pi_1(F^2)$, which is cyclic by Lemma 1, and hence contained in a cyclic subgroup generated by a primitive element. □

We will need the following well-known result. For a proof, see [4].

Lemma 2. If $\alpha \in H_1(F^2)$ is a nonzero homology class, then $\alpha$ is represented by a simple closed curve embedded in $F^2$ if and only if $\alpha$ is primitive. □

Proof of Proposition 1. Let $f : T^2 \rightarrow F^2 \times G^2$ represent a nonzero homology class and let $p_F : F^2 \times G^2 \rightarrow F^2$ and $p_G : F^2 \times G^2 \rightarrow G^2$ denote the projections onto the two factors. By Corollary 1 the image of $(p_F \circ f)_\ast : H_1(T^2) \rightarrow H_1(F^2)$ is contained in a cyclic subgroup $\langle \alpha \rangle$ generated by a primitive element $\alpha \in H_1(F^2)$. Similarly, the image of $(p_G \circ f)_\ast : H_1(T^2) \rightarrow H_1(G^2)$ is contained in a cyclic subgroup $\langle \beta \rangle$ generated by a primitive element $\beta \in H_1(G^2)$. Now by Lemma 2 both $\alpha$ and $\beta$ are represented by simple closed curves $A$ and $B$ in $F^2$ and $G^2$, respectively. It follows that $f_\ast[T^2]$ is a multiple of $[A \times B]$ in $H_2(F^2 \times G^2)$. But $n(\alpha \times \beta) = (mn) \alpha \times (mn) \beta$, which is what is needed. □

Proof of Proposition 2. If we write $\alpha = m\alpha^* = n\beta^*$, where $\alpha^*$ and $\beta^*$ are primitive homology classes, then $\alpha \times \beta = mn \alpha^* \times \beta^* = (mn \alpha^*) \times \beta^*$. Now, by Lemma 2, $\alpha^*$ and $\beta^*$ can be represented by simple closed curves $A^*$ and $B^*$, respectively. Since $A^*$ and $B^*$ have trivial normal bundles, so does the torus $A^* \times B^* \subset F^2 \times G^2$. The proof then is completed by Lemma 3 below. □

Lemma 3. The map $T^2 \rightarrow T^2 \times D^2$ given by $(z, w) \mapsto (z^k, w, 0)$ is homotopic to an embedding.

Proof. Just define an explicit embedding by $(z, w) \mapsto (z^k, w, z)$. □

Remark 1. Lemma 3 generalizes to state that if $F^2$ is a closed oriented surface of genus $g$ and $n$ is a positive integer, then the homology class in $H_2(F^2 \times D^2)$ corresponding to $n[F^2 \times \{0\}]$ is represented by an embedding of the surface of genus $ng - (n - 1)$. This is the least possible genus that can even represent $n[F^2 \times \{0\}]$ by a map, by Kneser’s inequality. For a modern exposition of Kneser’s theorem, see [5].

Remark 2. Any map of nonzero degree $T^2 \rightarrow T^2$ is homotopic to a covering, and any covering $T^2 \rightarrow T^2 \times \{0\} \subset T^2 \times D^2$ is homotopic to an embedding. The proof is somewhat more difficult than that of Lemma 3 and depends on lifting a pair of commuting permutations $\sigma, \tau \in S_n$, that generate a transitive subgroup of $S_n$, to a pair of commuting braids $b_\sigma$ and $b_\tau$ over them in the braid group $B_n$.
Remark 3. No nontrivial connected covering $F^2$ of the 0-section of a nontrivial oriented 2-disk bundle $E$ over a closed orientable surface $G^2$ is homotopic to an embedding of $F^2$ in $E$. See [1], for example.

It follows from the results of this note that a map $T^2 \to F^2 \times G^2$ is homologous to an embedding $T^2 \to F^2 \times G^2$. Is it possible that a map $T^2 \to F^2 \times G^2$ is actually homotopic to an embedding? It seems unlikely.

References


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