DERIVATIONS IMPLEMENTED BY LOCAL MULTIPLIERS

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Abstract. A condition on a derivation of an arbitrary $C^*$-algebra is presented entailing that it is implemented as an inner derivation by a local multiplier.

It is an outstanding open question whether every derivation of a $C^*$-algebra $A$ can be implemented as an inner derivation by a local multiplier, that is, an element in the direct limit of the multiplier algebras of the closed essential ideals of $A$. An affirmative answer was given by Elliott [4] for AF-algebras, and by Pedersen [11] for general separable $C^*$-algebras. In fact, it suffices to assume that every closed essential ideal of $A$ is $\sigma$-unital; hence Pedersen's result entails Sakai's theorem that every derivation of a simple unital $C^*$-algebra is inner. But only an affirmative answer in the non-separable case would cover, extend and unify the results that every derivation of a simple $C^*$-algebra is inner in the multiplier algebra [13] and that all derivations of von Neumann algebras [6], [12] and $AW^*$-algebras [10] are inner. This quest becomes even more attractive by the recent results in [9] and [14] implying that, if a derivation $\delta$ on $A$ is inner in the multiplier algebra, then there is a local multiplier $a$ of $A$ implementing $\delta$ such that $\|\delta\| = 2\|a\|$.

No progress on the above question seems to have been made since it was raised in [11] (see also [4]). The purpose of this note is to present a criterion on a given derivation $\delta$ of a (possibly non-separable) $C^*$-algebra $A$ implying that $\delta$ is inner in the local multiplier algebra $M_{loc}(A)$. Though this criterion, inspired by Herstein’s work [5], is rather algebraic in nature, it is hoped that some approximate version may eventually yield a positive solution of the general problem.

1. Notation and preliminaries

Throughout this paper, $M(A)$ will denote the multiplier algebra of the $C^*$-algebra $A$. A left ideal $L$ of $A$ is said to be essential if its left annihilator $L^\perp = \{a \in A \mid aL = 0\}$ is zero. For a (closed) two-sided ideal $I$, the left annihilator coincides with the right annihilator, and $I + I^\perp$ is a (closed) essential ideal. Given two closed essential ideals $I$, $J$ in $A$ such that $J \subseteq I$, $J$ is an essential ideal in $M(I)$ and hence $M(I)$ embeds isometrically into $M(J)$. Forming the $C^*$-direct limit of the directed family of multiplier algebras so obtained yields the local multiplier algebra of $A$. 

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denoted by $M_{\text{loc}}(A)$. If we merely take the algebraic direct limit, we obtain a dense *-subalgebra of $M_{\text{loc}}(A)$ which is called the bounded symmetric algebra of quotients, $Q_b(A)$ of $A$. The reason for this terminology is that $Q_b(A)$ is the bounded part of the purely algebraic version, the symmetric ring of quotients $Q_s(A)$ of $A$ in the sense of Kharchenko, where $A$ is considered as a semiprime ring only. For more details on $Q_s(A)$ we refer to [7]. Another important interrelation between $Q_s(A)$ and $Q_b(A)$ is noted in [1]: every element $q \in Q_s(A)$ can be written as $e^{-1}q_0$, where $q_0 \in Q_b(A)$, $e$ belongs to $C_b$, the center of $Q_b(A)$, and is not a divisor of zero. The commutative *-algebra $C_b$ is dense in the center of $M_{\text{loc}}(A)$ [2] and is the bounded part of the center $C$ of $Q_s(A)$; thus it is called the bounded extended centroid of $A$. Whenever $J$ is an ideal of $A$, there is a unique projection $c(J)$ in $C_b$ such that the annihilator of $JC$ in $AC$ is $(1 - c(J))AC$; we call $c(J)$ the central support of $J$. If $x \in A$ then $c_x := c(AxA)$ is the central support of $x$ (which is in fact the central support projection of $x$ within the AW*-algebra $Z(M_{\text{loc}}(A))$). Whenever $a, b \in M(A)$, we shall denote by $M_{a,b}$ the two-sided multiplication $x \mapsto axb$ on $A$, and by $\delta_a$ the inner derivation $x \mapsto xa - ax$.

It has emerged that, in working with local multipliers, it is often rather expedient and sometimes inevitable to also appeal to the surrounding algebraic framework, that is, to work within $Q_s(A)$ instead of $Q_b(A)$ only. The reason is the following. There is no way of making a non-invertible element of a $C^*$-algebra $A$ invertible by enlarging $A$ to a bigger $C^*$-algebra, but in $Q_s(A)$ such an element may become invertible, and hence many more equations can be solved within the non-$C^*$-algebra $Q_s(A)$. At the end, an additional argument is then needed to finally find the solution (to the original problem) within the $C^*$-algebraic frame, that is, $M_{\text{loc}}(A)$. Thus, working with local multipliers typically divides into two steps, a first purely algebraic one and a second, entirely independent analytic argument. This route is very well illustrated in [3], and we shall follow it subsequently again.

2. The results

The analytic step in our arguments is provided by the following observation.

Lemma. Let $L$ be an essential left ideal in a $C^*$-algebra $A$. Let $f : J \to A$ be a linear mapping defined on a subspace $J$ of $A$. If, for some derivation $\delta : A \to A$, the identity

$$f(x)u = -x\delta(u) \quad (x \in J, \ u \in L)$$

holds, then $f$ is bounded with norm at most $\|\delta\|$.

Proof. Let $\pi$ be an irreducible representation of $A$. By hypothesis,

$$\pi(f(x)y)\pi(z)\pi(u) = -\pi(x)\delta_\pi(\pi(yzu))$$

for all $x \in J$, $y, z \in A$ and $u \in L$, where $\delta_\pi$ denotes the induced derivation on $\pi(A)$.

Hence,

$$||M_{\pi(f(x)y),\pi(u)}\pi(z)|| \leq ||\pi(x)|| ||\delta_\pi|| ||\pi(y)|| ||\pi(z)|| ||\pi(u)||$$

$$\leq ||x|| ||\delta|| ||y|| ||\pi(z)|| ||\pi(u)||,$$

wherefore

$$||M_{\pi(f(x)y),\pi(u)}|| \leq ||x|| ||\delta|| ||y|| ||\pi(u)||.$$
for all \( x \in J, \ y \in A \) and \( u \in L \). Let \( I \) be the closed ideal \( \overline{IL} \). If \( \ker \pi \) does not contain \( I \), there is \( u \in L \) such that \( \pi(u) \neq 0 \). By [8, Proposition 2.3],
\[
\|M_{\pi(f(x)y), \pi(u)}\| = \|\pi(f(x)y)\|\|\pi(u)\|,\n\]
whence the above inequality entails that
\[
\|\pi(f(x)y)\| \leq \|x\| \|\delta\| \|y\|.
\]
Since each irreducible representation of \( I \) extends to an irreducible representation of \( A \) not vanishing on \( I \), it follows that \( \|f(x)y\| \leq \|x\| \|\delta\| \|y\| \) for all \( x \in J \) and \( y \in I \). Since \( I \) is essential (as \( L \) is essential), we conclude that
\[
\|f(x)\| = \sup \{\|f(x)y\| \mid y \in I, \ \|y\| \leq 1\} \leq \|\delta\| \|x\|
\]
for all \( x \in J \), as required. 

We shall apply this lemma below to show that a certain derivation that is inner when extended to \( Q_s(A) \), is in fact inner in \( M_{loc}(A) \). The most general result on innerness of derivations in the local multiplier algebra so far has been Pedersen’s result [11, Proposition 2]. (We use this occasion to note that one of the assertions in [11, Lemma 1], viz. the absolute summability of \( (y_n)_{n \in \mathbb{N}} \), is not proved and in fact cannot be proven, as simple counterexamples show. Fortunately, this does not interfere with the subsequent applications of [11, Lemma 1].) Pedersen’s condition is on the algebra \( A \) has to be separable), whereas our condition is on the derivation itself. Possibly a synthesis of weakened versions of both may result in the solution of the general question.

**Theorem A.** Let \( \delta \) be a derivation of a \( C^* \)-algebra \( A \). Suppose there exist an essential left ideal \( L \) of \( A \) and an element \( a \in A \) satisfying \( a\delta L = 0 \) and \( (1-e_a)\delta L = 0 \). Then there is \( h \in Q_b(A) \) such that
\[
\delta = \delta_h, \ ah = 0, \ Lh = 0, \ \text{and} \ \|h\| \leq \|\delta\|.
\]

**Proof.** For all \( u \in L \) and \( y \in A \) we have
\[
a y \delta u + a(\delta y) u = a \delta(y u) = 0
\]
by assumption, whence
\[
(1) \quad M_{a, \delta u} + M_{a, u} \circ \delta = 0 \quad (u \in L).
\]

On the ideal \( J = AaA \) we define \( f: J \to A \) by \( \sum_i x_i a y_i \mapsto \sum_i x_i a \delta y_i \) whenever \( x_i, y_i \) are finitely many elements in \( A \). Note that, by (1),
\[
\sum_i x_i a (\delta y_i) u = - \sum_i x_i a y_i \delta u,
\]
whence
\[
(2) \quad f(x) u = -x \delta u \quad (x \in J, \ u \in L)
\]
and
\[
(3) \quad f(x) y u = -x \delta(y u) \quad (x \in J, \ y \in A, \ u \in L).
\]

By (2), \( (f(x_1 + \lambda x_2) - f(x_1) - \lambda f(x_2)) u = 0 \) for all \( x_1, x_2 \in J, \ \lambda \in \mathbb{C} \) and \( u \in L \), whereas \( x = 0 \) implies that \( f(x) u = 0 \) for all \( u \in L \). Since \( L \) is essential, it follows that \( f \) is a well-defined linear mapping on \( J \).

Applying the Lemma to (2), we conclude that \( f \) is bounded with norm at most \( \|\delta\| \). Hence, replacing \( J \) by its closure, we may assume that \( J \) is closed.
Let \( J^\perp \) denote the annihilator of \( J \) in \( A \). If \( x_1 \in J, x_2 \in J^\perp \), we put \( \tilde{f}(x_1 + x_2) = f(x_1) \). Then, as \( (1 - e_a)\delta L = 0 \),
\[
\tilde{f}(x_1 + x_2)u = f(x_1)u = -(x_1 + x_2)e_a\delta u = -(x_1 + x_2)\delta u \quad (u \in L).
\]
Hence, replacing \( J \) by \( J + J^\perp \) and \( f \) by \( \tilde{f} \), we may assume that \( J \) is an essential closed ideal in \( A \).

By (2),
\[
(f(yx) - yf(x))u = -(yx - yx)\delta u = 0 \quad (x \in J, y \in A, u \in L),
\]
whence \( f \) is a left \( A \)-module map. Put \( g = f - \delta \). Then,
\[
g(xy)u = f(xy)u - \delta(xy)u = -xy\delta u - \delta(xy)u \\
= -\delta(xyu) \\
= -(\delta x)yu - x\delta(yu) = f(xy) - (\delta x)yu \\
= g(xy)u
\]
for all \( x \in J, y \in A \) and \( u \in L \) so that \( g \) is a right \( A \)-module map from \( J \) into \( A \). Moreover, if \( x, y \in J \), then, by (3),
\[
f(xy)u = -(\delta y)u = -xy\delta u - x(\delta y)u = x(f(y) - \delta y)u = xg(y)u \quad (u \in L),
\]
and thus \( f(xy) = xg(y) \). As a result, \( (f, g) \) is a double centralizer of \( J \) represented by an element \( h \in M(J) \). By definition, \( \delta = f - g = R_h - L_h = \delta_h \) on \( J \). From this we infer that
\[
(\delta y)x = \delta(yx) - y(\delta x) \\
= f(yx) - g(yx) - yf(x) + yg(x) \\
= yg(x) - g(yx) \\
= yhx - hyx = [y, h]x
\]
for all \( x \in J \) and \( y \in A \). Since \( J \) is essential, this yields that \( \delta = \delta_h \) on \( A \).

The identity
\[
a(yuh - hyu) = a\delta(yu) = 0 \quad (y \in A, u \in L)
\]
implies that
\[
M_{a,uh} = M_{ah,u} \quad (u \in L).
\]
Therefore, the mapping
\[
\sum_i x_iay_i + v \mapsto \sum_i x_iahy_i \quad (x_i, y_i \in A, \ v \in (AaA)^\perp)
\]
is a well-defined \( A \)-bimodule map from the essential ideal \( AaA + (AaA)^\perp \) into \( A \) which gives rise to an element \( \lambda \in C \) with the property \( \lambda a = ah \). This together with (4) entails that
\[
M_{a,uh} = M_{a,uh} - \lambda M_{a,u} = 0 \quad (u \in L),
\]
whence \( 0 = e_a(u(h - \lambda) = u(h - \lambda) \) as \( e_a h = h \) and \( e_a \lambda = \lambda \). Replacing \( h \) by \( h - \lambda \), we thus obtain \( \delta = \delta_h \) as well as \( ah = 0 \) and \( Lh = 0 \). In particular, \( xhu = -x\delta u \) for all \( x \) in the domain of \( h \) and \( u \in L \) (that is, (2)); thus the same reasoning as before shows that \( h \) still is bounded with \( \|h\| \leq \|\delta\| \).
A more symmetric version of the condition appearing in Theorem A is obtained in our first corollary.

**Corollary 1.** Let $\delta$ be a derivation on a $C^*$-algebra $A$. For each pair of elements $a, b \in A$ such that

$$M_{a, b} + M_{a, b} \circ \delta = 0$$

there is $h \in Q_b(A)$ with the properties

$$e_a e_b \delta = \delta_h, \quad ah = bh = 0 \quad \text{and} \quad \|h\| \leq \|\delta\|.$$  

**Proof.** We first follow the proof of Theorem A. Put $J = \delta L$ and $L = Ab$. Note that the left annihilators of $L$ and $Ab$ coincide. By assumption,

$$ay\delta(xb) + a(\delta y)xb = ay(\delta x)b + ayxb + ab(\delta y)b - ay(\delta x)b = 0$$

for all $x, y \in A$. Therefore (1) holds for all $u \in L$. Defining $f : J \to A$ as above, we thus obtain a well-defined bounded left $A$-module map $e_b f$ which we may extend to $J + J^\perp$ as before. Note that (2) changes to

$$(2) \quad e_b f(x) e_b u = -x \delta (e_a e_b u) \quad (x \in J + J^\perp, \ u \in L).$$

Letting $e_b g = e_b f - e_b \delta$, we obtain a bounded right $A$-module map on $J + J^\perp$ such that $e_b f(x) y = e_b xg(y)$ for all $x, y \in J$. Let $h$ be the element in $Q_b(A)$ corresponding to the local double centralizer $(e_b f, e_b g)$ of $A$. Then, $e_b \delta = \delta_h$ on $J$. As above, this entails that $(e_b \delta - \delta_h)A \subseteq J^\perp$, so that $e_a e_b \delta = \delta_{h'}$ with $h' = e_a h$. Since we still have (4) with $h'$ instead of $h$, we find $\lambda' = \lambda e_a \in C$ such that $\lambda' a = ah'$ as well as $\lambda' u = uh'$ for all $u \in L$. Now define $h$ anew by $h = h' - e_b \lambda'$. Then, $e_a e_b h = h, ah = 0, Lh = 0$, and $e_a e_b \delta = \delta_h$. A final application of the Lemma yields $\|h\| \leq \|\delta\|$. \hfill $\square$

**Corollary 2.** For every derivation $\delta$ on a prime $C^*$-algebra $A$ the following conditions are equivalent.

(a) There are a non-zero element $a \in A$ and a non-zero left ideal $L$ of $A$ such that $a \delta L = 0$.

(b) There is an element $h \in Q_b(A)$ such that $\delta = \delta_h$ and $yh = 0$ for some non-zero $y \in Q_b(A)$.

**Proof.** (a) $\Rightarrow$ (b) As $L$ is essential and $e_a = 1$, the assertion follows immediately from Theorem A.

(b) $\Rightarrow$ (a) Let $I$ be a non-zero closed ideal of $A$ such that $y \in M(I)$, and put $L = Iy$. Then, for each non-zero $a \in L$ and all $u \in L$ we have $a \delta u = ahu - ahu = 0$, i.e. $a \delta L = 0$. \hfill $\square$

Our arrangement of the proof of Theorem A reveals that its algebraic part carries over verbatim to the setting of semiprime rings. Suppose that the element $a$ in Theorem B below is zero. Then $\delta L = 0$, wherefore $(\delta y) u = \delta(y u) - y \delta u = 0$ for all $y \in R, u \in L$ implies that $\delta y = 0$ for all $y$. Thus, $a = 0$ entails that $\delta = 0$ and Theorem B extends the corresponding statement for prime rings in [5, Theorem], replacing the Martindale ring of quotients by its symmetric version.

**Theorem B.** Let $\delta$ be a derivation of a semiprime ring $R$. Suppose there exist an essential left ideal $L$ of $R$ and an element $a \in R$ satisfying $a \delta L = 0$ and $(1 - e_a) \delta L = 0$. Then there is $h \in Q_{a}(R)$ such that $\delta = \delta_h$, $ah = 0$, and $Lh = 0$.  

REFERENCES


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