HIGH ORDER MOMENTS OF CHARACTER SUMS

TODD COCHRANE AND ZHIYONG ZHENG

(Communicated by Dennis A. Hejhal)

Abstract. We establish the upper bound
\[ \left| \frac{1}{p-1} \sum_{\chi \neq \chi_0} \sum_{x=a+1}^{a+B} \chi(x)^{2k} \right| \leq \epsilon k p^{k-1+\epsilon} + B^k p^\epsilon, \]
with \( p \) a prime and \( k \) any positive integer, the sum being over all nonprincipal multiplicative characters (mod \( p \)).

1.

In this paper we obtain upper bounds on the character sum
\[ \frac{1}{p-1} \sum_{\chi \neq \chi_0} \left| \sum_{x=a+1}^{a+B} \chi(x)^{2k} \right|, \]
where \( a, B \) and \( k \) are positive integers, \( p \) is a prime, \( \chi \) runs through the set of multiplicative characters (mod \( p \)), and \( \chi_0 \) is the principal character. We shall assume that \( B < p \) and that the interval \( a + 1 \leq x \leq a + B \) does not contain a multiple of \( p \). A trivial bound for the sum in (1) that follows directly from the Polya-Vinogradov inequality is
\[ \left| \frac{1}{p-1} \sum_{\chi \neq \chi_0} \sum_{x=a+1}^{a+B} \chi(x)^{2k} \right| \ll p^k (\log p)^{2k}. \]

Let \( B \) be the cube
\[ B = \{ x \in \mathbb{Z}^{2k} : a + 1 \leq x_i \leq a + B, 1 \leq i \leq 2k \} \]
of cardinality \( |B| = B^{2k} \), and let \( V \) be the set of integer solutions of the congruence
\[ x_1 x_2 \ldots x_k \equiv x_{k+1} x_{k+2} \ldots x_{2k} \pmod{p}. \]
Then
\[ |B \cap V| = \frac{1}{p-1} \sum_{x_1=a+1}^{a+B} \cdots \sum_{x_{2k}=a+1}^{a+B} \sum_{\chi} \chi(x_1 x_2 \ldots x_k x_{k+1}^{-1} \ldots x_{2k}^{-1}). \]
\[ \frac{|B|}{p-1} + \frac{1}{p-1} \sum_{x \neq x_0} \sum_{x_1=a+1}^{a+B} \cdots \sum_{x_{2k}=a+1}^{a+B} \chi(x_1x_2 \ldots x_kx_{k+1} \ldots x_{2k}) \]

(3)

Thus, we have

\[ \frac{1}{p-1} \sum_{x \neq x_0} \sum_{x=a+1}^{a+B} \chi(x)^2 = |B \cap V| - \frac{|B|}{p-1}. \]

(4)

For \( k = 1 \) it is plain that \( |B \cap V| = B \) and so (4) is just

\[ \frac{1}{p-1} \sum_{x \neq x_0} \sum_{x=a+1}^{a+B} \chi(x)^2 = B - \frac{B^2}{p-1}. \]

(5)

In particular, if \( B < (p-1)/2 \), then

\[ \frac{B}{2} \leq \frac{1}{p-1} \sum_{x \neq x_0} \sum_{x=a+1}^{a+B} \chi(x)^2 < B, \]

whence

\[ \min_{x \neq x_0} \sum_{x=a+1}^{a+B} \chi(x) \leq \sqrt{B} \]

and

\[ \max_{x \neq x_0} \sum_{x=a+1}^{a+B} \chi(x) \geq \sqrt{B/2}. \]

(7)

(8)

For \( k = 2 \) it was shown in the paper of Ayyad, Cochrane and Zheng ([1], Theorem 2) that

\[ \frac{1}{p-1} \sum_{x \neq x_0} \sum_{x=a+1}^{a+B} \chi(x)^4 \ll B^2 \log^2 p, \]

(9)

and that, for \( B < \sqrt{p} \),

\[ \frac{1}{p-1} \sum_{x \neq x_0} \sum_{x=a+1}^{a+B} \chi(x)^4 \gg B^2 \log B, \]

(10)

whence

\[ \max_{x \neq x_0} \sum_{x=a+1}^{a+B} \chi(x) \gg \sqrt{B (\log B)^{1/4}}. \]

(11)

For higher moments Montgomery and Vaughan ([4], Theorem 1) established

\[ \frac{1}{p-1} \sum_{x \neq x_0} \max_B \sum_{x=a+1}^{a+B} \chi(x)^{2k} \ll p^k, \]

(12)

which is sharper, by a power of \( \log p \), than what one obtains trivially from the Polya-Vinogradov inequality. The main result of this paper is the following
Theorem. For positive integers $k$, 

$$
\frac{1}{p-1} \sum_{\chi \neq \chi_0} \left| \sum_{x=a+1}^{a+B} \chi(x) \right|^{2k} \ll_{\varepsilon,k} p^{k-1+\varepsilon} + B^k p^\varepsilon.
$$

(13)

In particular, for intervals of length $B \gg p^{1-\frac{1}{k}}$ we have 

$$
\frac{1}{p-1} \sum_{\chi \neq \chi_0} \left| \sum_{x=a+1}^{a+B} \chi(x) \right|^{2k} \ll_{\varepsilon,k} B^k p^\varepsilon.
$$

(14)

It is significant to note that the validity of (14) for arbitrary $k$ and $B < p$ is equivalent to the upper bound 

$$
| \sum_{x=a+1}^{a+B} \chi(x) | \ll_{\varepsilon} B^{1/2} p^{\varepsilon},
$$

(15)

for nonprincipal $\chi$, which on the assumption of the Grand Riemann Hypothesis is known to be true; see Montgomery and Vaughan ([3]). We note that for $k=1$ and $k=2$ the upper bounds in (5) and (9) are sharper than (13).

For intervals of short length we may improve on (13) by writing 

$$
\frac{1}{p-1} \sum_{\chi \neq \chi_0} \left| \sum_{x=a+1}^{a+B} \chi(x) \right|^{2k} \leq \max_{\chi \neq \chi_0} \left| \sum_{x=a+1}^{a+B} \chi(x) \right|^{2k-4} \frac{1}{p-1} \sum_{\chi \neq \chi_0} \left| \sum_{x=a+1}^{a+B} \chi(x) \right|^{4}
$$

(16)

$$
\ll B^2 \log^2 p \max_{\chi \neq \chi_0} \left| \sum_{x=a+1}^{a+B} \chi(x) \right|^{2k-4},
$$

and then inserting the upper bound of Burgess ([2]), 

$$
| \sum_{x=a+1}^{a+B} \chi(x) | \ll_{\varepsilon} B^{1-\frac{1}{r}} p^{\frac{r+1}{r^2}} \log p,
$$

where $r$ is any positive integer $\geq 2$. For intervals of length $< p^{1/4}$ we just insert the trivial upper bound $| \sum_{x=a+1}^{a+B} \chi(x) | \leq B$. In summary, we have for $k \geq 3$, 

(17)

$$
\frac{1}{p-1} \sum_{\chi \neq \chi_0} \left| \sum_{x=a+1}^{a+B} \chi(x) \right|^{2k} \ll_{\varepsilon,k} \begin{cases} 
B^k p^\varepsilon, & \text{if } p^{1-\frac{1}{k}} \leq B < p, \\
p^{k-1+\varepsilon}, & \text{if } p^{\frac{r}{r-1}} B \leq p^{1-\frac{1}{k}}, \\
B^{\frac{r(r-1)}{2}+\varepsilon}, & \text{if } p^{\frac{r}{2}} B \leq p^{\frac{r}{r-1}} \ll_{\varepsilon,k} \begin{cases} 
B^{k-2}, & \text{if } B \leq p^{1/4}, 
\end{cases}
\end{cases}
$$

We have indicated (roughly speaking) the best upper bound available on each of the intervals in (17).
2. The Fundamental Identity and a Key Lemma

View $B$ and $V$ as subsets of $\mathbb{F}_p^{2k}$, and let $\alpha$ denote the characteristic function of $B$ with finite Fourier expansion

\[(18) \quad \alpha(x) = \sum_y a(y)e_p(x \cdot y),\]

where as usual $e_p(*) = e^{\pi i * / p}$, $x \cdot y = \sum_{i=1}^{2k} x_i y_i$, $\sum_y = \sum_{y \in \mathbb{F}_p^{2k}}$. The Fourier coefficients are given by

\[(19) \quad a(y) = p^{-2k} \prod_{i=1}^{2k} e_p\left(-(a + \frac{1}{2} + \frac{B}{2}) y_i\right) \frac{\sin(\pi B y_i / p)}{\sin(\pi y_i / p)},\]

where a term in the product is taken to be $B$ if $y_i = 0$. We have

\[(20) \quad \sum_{\chi \neq \chi_o} \sum_{x = a+1}^{a+B} \chi(x) = \sum_{\chi \neq \chi_o} \sum_{x \neq 0} \cdot \sum_{x_{2k} \neq 0} \alpha(x) \chi(x_1 x_2 \ldots x_k x_k^{-1} \ldots x_{2k}^{-1})
\]

\[= \sum_y a(y) \sum_{\chi \neq \chi_o} \sum_{x \neq 0} \cdot \sum_{x_{2k} \neq 0} \chi(x_1 x_2 \ldots x_k x_k^{-1} \ldots x_{2k}^{-1}) e_p(x \cdot y)
\]

\[= \sum_y a(y) \prod_{i=1}^{k} \sum_{\chi \neq \chi_o} \chi(x_i) e_p(x_i y_i) \prod_{i=k+1}^{2k} \sum_{x \neq 0} \chi(x_i^{-1}) e_p(x_i y_i).
\]

Now if $y_i = 0$ for some $i$, then the sum over $x_i$ is zero, since $\chi$ is nonprincipal. If all of the $y_i$ are nonzero, then the sum over $x$ is just

\[(21) \quad \chi \left( \prod_{i=1}^{k} y_i^{-1} y_{i+k} \right) G(\chi)^k G(\chi^{-1})^k = p^k \chi((-1)^k y_1^{-1} \ldots y_k^{-1} y_{k+1} \ldots y_{2k}),
\]

where $G(\chi)$ denotes the Gaussian sum $G(\chi) = \sum_{x \neq 0} \chi(x) e_p(x)$. Here we have used the identities $G(\chi^{-1}) = \chi(-1) G(\chi)$ and $|G(\chi)|^2 = p$ for $\chi \neq \chi_o$. Summing over $\chi$ and using the identity,

\[(22) \quad \sum_{y_i \neq 0} a(y) = p^{-2k} \sum_{x \in B} \sum_{y_i \neq 0} e_p(x_i y_i) = \frac{B^{2k}}{p^{2k}},
\]

we obtain the

**Fundamental Identity.**

\[(23) \quad \frac{1}{p - 1} \sum_{\chi \neq \chi_o} \sum_{x = a+1}^{a+B} \chi(x) \cdot \sum_{y \neq 0} a(y) = \frac{B^{2k}}{p^k(p-1)}.
\]

**Lemma.** Let $V^{\pm} \subset \mathbb{F}_p^{2k}$ be the set of integer solutions of

\[(24) \quad y_1 \ldots y_k \equiv \pm y_{k+1} \ldots y_{2k} \pmod{p},
\]
and let $B$ be the box of points $0 < \left| y_i \right| < B_i$, $1 \leq i \leq 2k$, with the $B_i$ positive integers. Then

\begin{equation}
|B \cap V^\pm| \ll_{\epsilon, k} \left( \frac{|B|}{p} + \sqrt{|B|} \right) p^\epsilon.
\end{equation}

**Proof.** We may suppose without loss of generality that $\prod_{i=1}^{k} B_i \geq \prod_{i=k+1}^{2k} B_i$ and that all of the $y_i$ are positive. Let $y_{k+1}, \ldots, y_{2k}$ be any fixed values with $0 < y_i < B_i$, $1 \leq i \leq k$, and put $c \equiv y_{k+1} \cdots y_{2k} \pmod{p}$ with $0 < c < p$. Then any integer solution $y_1, \ldots, y_k$ of (24) with $0 < y_i < B_i$, $1 \leq i \leq k$, must satisfy

\begin{equation}
y_1 \cdots y_k = c + \ell p \quad \text{or} \quad y_1 \cdots y_k = (p - c) + \ell p
\end{equation}

for some integer $\ell$ with $0 \leq \ell \leq \prod_{i=1}^{k} B_i / p$. For each such value $\ell$ the number of solutions of (26) is $\ll_{\epsilon, k} p^{k+\epsilon}$, where $\tau$ is the divisor function. Thus, the total number of solutions of (26) with $\ell$ in the specified range is

\[ \ll_{\epsilon, k} \left( \prod_{i=1}^{k} B_i / p + 1 \right) p^\epsilon. \]

We obtain the upper bound in (25) on multiplying by the number of choices for $y_{k+1}, \ldots, y_{2k}$.

\[ \square \]

3. **Proof of the theorem**

We start by noting that the Fourier coefficients (19) of the characteristic function $\alpha$ admit the upper bound

\begin{equation}
|a(y)| \ll \prod_{i=1}^{2k} \min \left( \frac{B}{p}, \frac{1}{|y_i|} \right) \quad (|y_i| < p/2).
\end{equation}

Letting the $y_i$ run through the intervals $0 < \left| y_i \right| \leq p/B_i$ and $2^{r_i}p/B_i < \left| y_i \right| \leq 2^{r_i+1}p/B_i$ for $r_i = 0, 1, 2, \ldots$, stopping when $2^{r_i} > B_i/2$, we have

\[ \sum_{y_1 \cdots y_k = (-1)^k y_{k+1} \cdots y_{2k}} |a(y)| \ll B^{2k} p^{-2k} \sum_{r_1=0}^{2k} \cdots \sum_{r_{2k}=0}^{2k} \prod_{i=1}^{2k} 2^{-r_i} \sum_{0 < |y_i| \leq 2^{r_i}p/B} 1, \]

where $V^\pm$ is as defined in the Lemma. Inserting the upper bound in (25), the above is

\[ \ll_{\epsilon, k} B^{2k} p^{-2k} \sum_{r_1=0}^{2k} \cdots \sum_{r_{2k}=0}^{2k} \prod_{i=1}^{2k} 2^{-r_i} \left( \frac{p^{2k-1}}{B^{2k}} \prod_{i=1}^{2k} 2^{r_i} + \frac{p^k}{B^k} \prod_{i=1}^{2k} 2^{r_i/2} \right) p^\epsilon \]

\[ \ll_{\epsilon, k} p^{-1+\epsilon} + B^k p^{-k+\epsilon}. \]

The theorem now follows immediately from the Fundamental Identity (23).
References

[1] A. Ayyad, T. Cochrane and Z. Zheng, The congruence $x_1 x_2 \equiv x_3 x_4 \pmod{p}$, the equation $x_1 x_2 = x_3 x_4$ and mean values of character sums, J. Number Theory 59 (2) (1996), 398–413. MR 97i:11091


Department of Mathematics, Kansas State University, Manhattan, Kansas 66506
E-mail address: cochrane@math.ksu.edu

Department of Mathematics, Zhongshan University, Guangzhou 510275, People’s Republic of China