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*-REPRESENTATIONS ON BANACH *-ALGEBRAS

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ABSTRACT. We study notions of g-bounded linear functionals and representable functionals on Banach *-algebras. An equivalence between these two is established for general Banach *-algebras. In particular, we characterize g-bounded linear functionals on Banach *-algebras with approximate identity and isometric involution. In addition, we prove a result on representation of g-bounded positive linear functionals in terms of cyclic vectors for the corresponding *-representation.

1. Introduction

Let A be a complex Banach *-algebra. We assume neither the existence of an identity nor that the involution is continuous. We write $S(A) = \{a \text{ in } A \text{ such that } a^* = a\}$ for the set of all self-adjoint elements of A. A *-ideal of A is an ideal J of A where a in J implies a^* in J.

A B^*-semi-norm on A is a function $\eta : A \to \mathbb{R}$ such that for all a, b in A and α in $\mathbb{C}$,

(1) $\eta(a + b) \leq \eta(a) + \eta(b)$,
(2) $\eta(\alpha a) = |\alpha|\eta(a)$,
(3) $\eta(ab) \leq \eta(a) \cdot \eta(b)$,
(4) $\eta(a^* a) = (\eta(a))^2$.

$P(A)$ denotes the set of all B^*-semi-norms on A. For more on B^*-semi-norms, see [2, 3]. Suppose $g(a) = \sup\{\eta(a) : \eta \text{ in } P(A)\}$. Then g defines a B^*-semi-norm on A; in fact, g is the greatest B^*-semi-norm on A in the pointwise ordering.

A *-representation of A is a mapping $\pi : A \to B(H)$, where $B(H)$ denotes the algebra of all bounded linear operators on a Hilbert space H, such that for all a, b in A and α in $\mathbb{C}$

(1) $\pi(a + b) = \pi(a) + \pi(b)$,
(2) $\pi(\alpha a) = \alpha\pi(a)$,
(3) $\pi(ab) = \pi(a) \cdot \pi(b)$,
(4) $\pi(a^*) = (\pi(a))^*$.

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Note that every *-representation of A is uniformly continuous, since each B^*-semi-norm on A is continuous and the mapping $a \to |\pi(a)|$ is a B^*-semi-norm on A.

A given positive linear functional f on A is representable if there exists a *-representation π of A on H and a vector x in H such that f is the positive linear functional represented by (π, x); that is, $f(a) = (\pi(a)x, x)$ for all $a \in A$.

2. g-BOUNDED FUNCTIONALS

Definition 2.1. A linear functional f on A is g-bounded if there exists a constant $M > 0$ (depending only on f) such that for all a in A, $|f(a)| \leq Mg(a)$.

Since each B^*-semi-norm on A is continuous, it follows that any g-bounded linear functional is continuous with respect to the original norm on A and hence the set of g-bounded linear functionals is a subspace of A^*, the dual space of A. The norm of any g-bounded functional f is defined as follows:

$$|f|_g = \sup\{|f(a)| : g(a) \leq 1\}.$$

The set $D(g)$ consists of all g-bounded positive linear functionals f on A with $|f|_g \leq 1$. A positive g-bounded linear functional f on A will be called a state of A if $|f|_g = 1$.

Lemma 2.1. *Let A be a unital algebra and f be a positive g-bounded linear functional on A. Then f is a state of A.*

Proof. Since $f(1) \leq 1$, it follows that $|f|_g \geq f(1)$, where 1 is the identity element in the algebra A. But for all a in A,

$$|f(a)| \leq |f|_g g(a).$$

Hence for all a in A,

$$|f(a)|^2 \leq f(1)|f|_g g(a^*a) = f(1)|f|_g g(a)^2.$$

Thus $|f|_g^2 \leq f(1)|f|_g$ and consequently f is a state of A. □

Proposition 2.1. *Let u and v be positive g-bounded linear functionals on A. Then for all a in A*

(i) $|u(a)|^2 \leq |u|_g u(a^*a)$,
(ii) $|u + v|_g = |u|_g + |v|_g$,
(iii) $|u|_g = \sup\{u(a^*a) : g(a) \leq 1\}$,
(iv) *u and v are hermitian functionals.*

Proof. Suppose that A has no unit element. If $J_g = \{a \text{ in } A : g(a) = 0\}$, then J_g is a closed two-sided *-ideal of A. In that case A/J_g becomes a quotient *-algebra. Let $a \to \lambda_a$ denote the canonical mapping of A onto A/J_g. We define a B^*-norm $\overline{g}$ on A/J_g as follows: For all a in A/J_g, $\overline{g}(\lambda_a) = g(a)$. The completion A_g of A/J_g with respect to this norm is a B^*-algebra. On A/J_g, define $\overline{u}(\lambda_a) = u(a)$, $\overline{v}(\lambda_a) = v(a)$, for all λ_a in A/J_g. Then $\overline{u}$ and $\overline{v}$ are well defined positive $\overline{g}$-bounded linear functionals on A/J_g. Furthermore, $|\overline{u}|_{\overline{g}} = |u|_g$ and $|\overline{v}|_{\overline{g}} = |v|_g$. Hence $\overline{u}$ and $\overline{v}$ have a unique norm preserving extension to the B^*-algebra A_g. Denote these extensions by U and V, respectively.

Thus U and V are positive linear functionals on A_g, hence (i), (ii), and (iv) follow from 2.1.5 and 2.1.6 in [1]. To prove (iii) we proceed as follows. Let $\sup\{u(a^*a) : g(a) \leq 1\} = \alpha$. Then $\alpha \leq |u|_g$, since if $g(a) \leq 1$, $u(a^*a) \leq |u|_g g(a^*a) \leq |u|_g$. By the definition of $|u|_g$ there exists a sequence $\{a_k\}$ of elements of A with $g(a_k) \leq 1$ and $|u|_g = \lim_{k\to\infty} |u(a_k)|$.

It follows from (i) that

$$|u(a_k)|^2 \leq |u|_g u(a_k^* a_k) \leq |u|_g^2 g(a_k)^2 \leq |u|_g^2.$$

Hence $\lim_{k\to\infty} u(a_k^* a_k) = |u|_g$. The case where A is unital follows from Lemma 2.1 above.

Remark 2.1. Proposition 2.1 is also true for any B^*-semi-norm. The following theorem gives a necessary and a sufficient condition for a positive functional on an algebra to be g-bounded.

Theorem 2.1. *A positive linear functional f on A is g-bounded if and only if there exists a positive constant M, which depends only on f, such that for all a in A, $|f(a)|^2 \leq Mf(a^*a)$.*

Moreover if f is g-bounded, then

$$|f|_g = \sup_{a\in A}\left\{\frac{|f(a)|^2}{f(a^*a)}\right\}.$$

Proof. Let f be a g-bounded positive functional. Then by Proposition 2.1(i) it follows that for all a in A, $|f(a)|^2 \leq Mf(a^*a)$.

Suppose conversely that for all a in A, and for every positive linear functional f on A, there exists a positive constant M such that $|f(a)|^2 \leq Mf(a^*a)$. Then for all x and a in A

$$\beta_f = \sup_{f(x^*x)\leq 1}\left\{\sqrt{f(x^*a^*ax)}\right\} \geq \sup_{f(x^*x)\leq 1}\left\{\frac{|f(ax)|}{\sqrt{M}}\right\}.$$

Let $f(a^*a) > 0$ and $a/\sqrt{f(a^*a)} = x$. Then $f(x^*x) = 1$. Now $\beta_f(a) \geq |f(a^2)|/\sqrt{Mf(a^*a)}$. If a is in $S(A)$, then we have $\sqrt{|f(a)|^2/M^2} \leq \sqrt{f(a^2)/M} \leq \beta_f(a)$; that is, $|f(a)| \leq M\beta_f(a)$. Hence, $|f(a)|^2 \leq Mf(a^*a) \leq M^2\beta_f(a)^2$ so that $|f(a)| \leq Mg(a)$. This proves that f is g-bounded. Futhermore, for all a in A and for any positive linear functional f on A,

$$\sup_{a\in A}\left\{\frac{|f(a)|^2}{f(a^*a)}\right\} \leq |f|_g \leq M,$$

where M is a positive constant (depending only on f). Therefore we may suppose that $M = \sup_{a\in A}\{|f(a)|^2/f(a^*a)\}$ so that we obtain $|f|_g = \sup_{a\in A}\{|f(a)|^2/f(a^*a)\}$.

Remark 2.2. If A has an identity 1, then every positive linear functional f is g-bounded. This follows directly by Theorem 2.1 and the Cauchy-Schwarz inequality. Further, if A is unital with isometric involution, then by Lemma 2.1 it follows that $\|f\| = |f|_g = f(1)$.

We use the fact of Remark 2.2 and Theorem 2.1 to give a characterization of g-bounded linear functionals when the given algebra has approximate identity and isometric involution.

Theorem 2.2. *If A has isometric involution and approximate identity, a positive linear functional f is g-bounded if and only if it is continuous.*

Proof. If f is g-bounded, then for all a in A, $|f(a)| \leq |f|_g g(a) \leq |f|_g\|a\|$ and consequently f is continuous. Suppose conversely that f is continuous, then for all a in A, $|f(a)|^2 \leq \|f\|f(a^*a)$ by 2.1.5 in [1]. Hence by Theorem 2.1 f is g-bounded.

Corollary 2.1. $|f|_g = \|f\|$, *for all f as in Theorem 2.2.*

Proof. Since f is g-bounded we have $|f|_g \leq \|f\|$. Also the involution on A is isometric. Therefore $|f(a)| \leq |f|_g g(a) \leq |f|_g \|a\|$. This implies that $\|f\| \leq |f|_g$ and hence $|f|_g = \|f\|$.

3. g-bounded functionals in terms of representable functionals

In the following theorem we establish a relationship between g-bounded and representable functionals. Here it is shown that the representable functionals are the positive g-bounded linear functionals and these are precisely the functionals generated by cyclic *-representations of the algebra.

Theorem 3.1. *A positive linear functional f on A is representable if and only if it is g-bounded.*

Proof. Suppose that f is representable. Then by the definition of representable functionals there exists *-representation π of A on H and a vector x in H such that for all a in A, $|f(a)| \leq |\pi(a)|$ and $\|x\|^2 \leq g(a)\|x\|^2$. Thus f is g-bounded.

Suppose conversely that f is g-bounded. If the norm on A^+, the unitization of A, is given by $\|(a,\lambda)\| = \|a\| + |\lambda|$, for all a in A and λ in $\mathbb{C}$, then A is isometrically and *-isomorphically embedded in the unital Banach *-algebra A^+. Since f is g-bounded it follows from Proposition 2.1(i) that for all a in A, $|f(a)|^2 \leq |f|_g f(a^*a)$.

Let f^+ be defined on A^+ by $f^+((a,\lambda)) = f(a) + \lambda|f|_g$, where (a,λ) is in A^+. Then f^+ is a linear functional on A^+, which also extends f on A and

$$\begin{aligned} f^+((a,\lambda)^*(a,\lambda)) &= f^+(a^*a + \overline{\lambda}a + \lambda a^*, \lambda\overline{\lambda}) \\ &= f(a^*a) + \overline{\lambda}f(a) + \overline{\lambda f(a)} + |\lambda|^2|f|_g. \end{aligned}$$

Thus,

$$\begin{aligned} f^+((a,\lambda)^*(a,\lambda)) &\geq f(a^*a) - 2|\lambda|(|f|_g^f(a^*a))^{1/2} + |\lambda|^2|f|_g \\ &= ((f(a^*a))^{1/2} - |\lambda|\,|f|_g^{1/2})^2 \geq 0. \end{aligned}$$

Hence f^+ is a positive linear functional on A^+.

Let $\mathcal{L}_f = \{a \text{ in } A^+ \colon f^+(ba) = 0 \text{ for all } b \text{ in } A^+\}$. Then on the quotient space $A^+/\mathcal{L}_f$ we define $(x_a, x_b)_f = f^+(b^*a)$, a in x_a, b in x_b. If H_f is the completion of $A^+/\mathcal{L}_f$, then

$$\|x_f\|_f^2 = (x_f, x_f)_f = f^+((0,1)^*(0,1)) = |f|_g.$$

For each a in A^+ let $\pi(a)$ be defined by $\pi(a)x_b = x_{ab}$, $x_b \in A^+/\mathcal{L}_f$. Then $\pi(a)$ is a well-defined linear operator on $A^+/\mathcal{L}_f$ and for a in A^+

$$\|\pi(a)x_b\|_f^2 = (x_{ab}, x_{ab})_f = f^+(b^*a^*ab).$$

It is easy to see that $\pi(a)$ is a bounded linear operator on $A^+/\mathcal{L}_f$ and so it has a unique extension to a bounded linear operator $\pi^+(a)$ on H_f. Also the mapping $a \to \pi^+(a)$ is an algebra homomorphism from A^+ into $B(H_f)$ and for all a, b and c in A^+,

$$(\pi(a)x_b, x_c)_f = (x_b, \pi(a^*)x_c)_f.$$

Hence for all x, y in H_f, $(\pi(a)x, y)_f = (y, \pi(a^*)y)_f$ so that for all a in A^+, $(\pi^+(a))^* = \pi^+(a^*)$. Thus π^+ is a *-representation of A^+ on H_f and hence the restriction map $\pi_f = \pi^+|_A$ is a *-representation of A on H_f. Next, for all a in A

$$(\pi(a)x_f, x_f)_f = (x_{(a,0)}, x_{(0,1)})_f = f^+((a,0)) = f(a).$$

Thus f is a positive linear functional represented by the pair (π_f, x_f), and the proof is complete.

The following proposition shows that our definition of representability of positive linear functionals is equivalent to the definition given by Rickart ([5], 4.5.5). The proof uses the construction of the proof of Theorem 3.1.

Proposition 3.1. *If f is a g-bounded positive linear functional on A then f can be represented by a pair (π, x), where x is a cyclic vector for the *-representation π, and moreover $|f|_g = \|x_f\|_f^2$.*

Proof. By the definition of cyclic vector and Theorem 3.1 it is obvious that x_f is a cyclic vector for the *-representation π^+ of A^+ on H_f. We claim that x_f is a cyclic vector for the *-representation π_f of A.

Since f is g-bounded, there exists a sequence $\{a_k\}$ of elements of A with $g(a_k) \leq 1$ such that $|f|_g = \lim_{k\to\infty} f(a_k^* a_k)$. Let $b_k = a_k^* a_k$. Then b_k is in $S(A)$ and $g(b_k) \leq 1$, and since $|f(b_k)|^2 \leq |f|_g f(b_k b_k^*) \leq |f|_g^2$, it follows that $\lim_{k\to\infty} f(b_k^* b_k) = |f|_g$.

Consider $\|x_{b_k} - x_f\|_f$. Then $\|x_{b_k} - x_f\|_f^2 = f(b_k^* b_k) - 2f(b_k) + |f|_g \to 0$ as $k \to \infty$. Hence for any a in A^+ we have $\|x_{ab_k} - x_a\|_f = \|\pi^+(a)(x_{b_k} - x_f)\|_f \to 0$ as $k \to \infty$. However, ab_k is in A and hence $\pi_f(A)x_f$ is dense in $A^+/\mathcal{L}_f$. It follows that it is also dense in H_f. Thus x_f is a cyclic vector for π_f. The equality $|f|_g = \|x_f\|_f^2$ follows from the proof of Theorem 3.1.

4.

In this section we question why g-bounded functionals may be interesting and supply an answer to this question as well. During the course of this research, we observed that in the case of B^*-algebras, g-bounded functionals coincide with the original norm. A g-bounded functional is more general in defining the state space of A since the set $D(g)$ of g-bounded functionals is a subspace of A^*, the dual space of A.

Proposition 2.1 holds for any B^*-semi-norm. A g-bounded functional can be represented by a pair of cyclic vectors for the *-representations.

Let π be a *-representation of A on the Hilbert space H and let x be in H. Then any positive linear functional f, which is represented by (π, x), defines a g-bounded positive functional f_T on A with $f_T \leq f$, where T is a self-adjoint operator on H such that $T\pi(A) = \pi(A)T$ and $0 \leq T \leq I_H$ (see Lemma 1.1 in [4]). Lemma 1.1 in [4] is an extension of a result by Dixmier ([1], 2.5.1).

Another application of g-bounded functionals to the representation theory can be seen in characterizing representable functionals which can be represented by a topologically irreducible representation.

A positive linear functional f is a pure state of A (see Definition 2.1, [4]) if it is non-zero and g-bounded and if any g-bounded positive linear functional dominated by f is of the form βf with β in $[0, 1]$. The following result, which has been submitted to another journal, gives an application of g-bounded functionals.

Theorem 4.1 ([4])**.** *Let (π, x) be a cyclic representation of a positive linear functional f. Then π is topologically irreducible and non-zero if and only if f is a pure state of A.*

An intriguing development in the representation theory of g-bounded functionals is Theorem 3.1 in [4] which states that the extreme points of $D(g)$ are the zero functional and the pure states of A.

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