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SOME NONINVERTIBLE LINKS

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ABSTRACT. We construct for each $n \geq 2$ infinitely many n-component oriented links that are neither amphicheiral nor invertible; among these examples, infinitely many are Brunnian links.

Amphicheirality and invertibility questions concerning knots and links in the 3-sphere S^3 are among the earliest problems in classical knot theory. These questions are of interest since they add to our knowledge of knot and link groups. The Jones polynomial and its various generalizations detect nonamphicheirality of a wide range of knots and links, but they fail to distinguish a single knot or link from its inverse.

In this paper, we construct for each $n \geq 2$ infinitely many n-component oriented links that are neither amphicheiral nor invertible; among these examples, infinitely many are Brunnian links. Our starting observation is that if an oriented link L in S^3 is amphicheiral or invertible, then the corresponding covering links in the universal abelian cover of the complement of one component of L admit a certain symmetry. The examples are provided by applying the band-summing construction introduced by Cochran [1] to certain based Brunnian links and based homotopy Brunnian links.

It should be noticed that the existence of noninvertible knots was first established by Trotter [7]. Noninvertible links with an arbitrary number of components and invertible proper sublinks were first constructed by Whitten [8], [9]. However, Whitten's examples are not Brunnian links and the noninvertibility was established by difficult arguments from combinatorial group theory. Milnor's $\bar{\mu}$-invariants [4], [5] of weight m have the ability to detect nonamphicheirality or noninvertibility of oriented links for m even or odd respectively. These invariants can be used to show that the n-component Brunnian link in Figure 7 of [4] is nonamphicheiral or noninvertible for n even or odd respectively. By using the algorithm provided by Cochran [1], further nonamphicheiral or noninvertible Brunnian links could be constructed. However, Milnor's $\bar{\mu}$-invariants fail to detect noninvertibility of 2-component oriented links by Traldi [6] and it is not clear whether they detect noninvertibility of n-component oriented links for other even n.

Definitions. An n-component oriented link in an oriented 3-manifold X is an ordered collection of n disjoint oriented circles tamely embedded in X. Two n-component oriented links L_1 and L_2 are *equivalent* or have the same *link type* if there exists an orientation preserving homeomorphism h of X onto itself such that

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$h(L_1) = L_2$, that is, h maps L_1 onto L_2 preserving the orders and the orientations of links. An *amphicheiral link* is one which is equivalent to its mirror image. An *invertible link* is one which is equivalent to its *inverse* obtained by reversing orientations of all the components. A *Brunnian link* is one which is not equivalent to a trivial link, yet each proper sublink is equivalent to a trivial link.

Let K be an oriented knot in S^3 and let $E_K = S^3 - K$ be the complement of K in S^3. By Alexander duality, E_K has the same homology type as a circle, hence its universal abelian cover $\widetilde{E}_K$ admits infinite cyclic covering transformations. We denote by $\lambda \in H_1(E_K, \mathbf{Z}) \cong \mathbf{Z}$ the generator represented by an oriented circle in E_K that links K algebraically one time, and by t the generator of the group of infinite cyclic covering transformations determined in the canonical way by λ.

Suppose we are given an oriented knot L in E_K such that $lk(L, K) = 0$. Then L can be lifted to an oriented circle $\widetilde{L}$ tamely embedded in $\widetilde{E}_K$. Now associated to each sequence $I = (i_1, ..., i_m)$ of m distinct integers, we have an m-component oriented covering link

$$\widetilde{L}(I) = t^{i_1}\widetilde{L} \cup \cdots \cup t^{i_m}\widetilde{L} \subset \widetilde{E}_K$$

whose link type is independent of one's choice of the lift $\widetilde{L}$ of L. This observation has been applied in [2], [3] to construct certain covering invariants for 2-component oriented links with vanishing linking number in S^3 which can be used to detect both nonamphicheirality and noninvertibility of links.

Consider now an n-component oriented link $L_1 \cup \cdots \cup L_n$ in E_K such that $lk(L_r, K) = 0$ for $r = 1, ..., n$ and let $\widetilde{L}_1, ..., \widetilde{L}_n$ be lifts of $L_1, ..., L_n$ to $\widetilde{E}_K$. Then associated to n sequences $I_1, ..., I_n$ of distinct integers, we have again an oriented covering link

$$\widetilde{L}_1(I_1) \cup \cdots \cup \widetilde{L}_n(I_n) \subset \widetilde{E}_K.$$

In this case, it is still possible to construct the various covering invariants as in [2], [3] at least for the case in which K is unknotted. But the results are now Laurent polynomials in several variables, which are well defined only up to multiplication of certain monomials since there exists no canonical way to label the various covering links.

In this paper, instead of making use of these complicated covering invariants, we immediately give the following simple observation which allows us to construct infinitely many concrete examples of oriented links that are neither amphicheiral nor invertible.

Lemma 1. *Let $L = L_1 \cup \cdots \cup L_n \cup K$ be an $(n+1)$-component oriented link in the 3-sphere such that $lk(L_r, K) = 0$ for $r = 1, ..., n$ and let $\widetilde{L}_1, ..., \widetilde{L}_n$ be lifts of $L_1, ..., L_n$ to $\widetilde{E}_K$. If L is amphicheiral (respectively invertible), then there exist lifts $\widetilde{L}_1^*, ..., \widetilde{L}_n^*$ of $L_1, ..., L_n$ to $\widetilde{E}_K$ such that the covering link $\widetilde{L}_1(I_1) \cup \cdots \cup \widetilde{L}_n(I_n)$ is equivalent to the mirror image (respectively the inverse) of the covering link $\widetilde{L}_1^*(-I_1) \cup \cdots \cup \widetilde{L}_n^*(-I_n)$ for any given sequences $I_1, ..., I_n$ of distinct integers.*

Proof. Suppose that L is amphicheiral (respectively invertible). Then there exists an orientation reversing (respectively orientation preserving) homeomorphism h of S^3 onto itself such that $h(L) = L$ (respectively $h(L) = -L$). The homeomorphism h is seen to restrict to a homeomorphism of E_K onto itself, still denoted by h, which reverses the orientation of λ, that is, satisfies $h_*(\lambda) = -\lambda$. Let us denote by $\tilde{h}$ any homeomorphism of $\widetilde{E}_K$ onto itself that covers h. Then $\tilde{h}$ is orientation reversing (respectively orientation preserving) and satisfies $\tilde{h} \circ t = t^{-1} \circ \tilde{h}$. Notice that $\tilde{h}$ is unique only up to the action of the group of infinite cyclic covering transformations. Now let $\widetilde{L}_r^* = \tilde{h}(\widetilde{L}_r)$ where $r = 1, ..., n$. Then $\widetilde{L}_1^*, ..., \widetilde{L}_n^*$ are again lifts of $L_1, ..., L_n$ to $\widetilde{E}_K$. The result then follows since the covering link $\widetilde{L}_1(I_1) \cup \cdots \cup \widetilde{L}_n(I_n)$ is mapped under $\tilde{h}$ onto the covering link (respectively the inverse of the covering link) $\widetilde{L}_1^*(-I_1) \cup \cdots \cup \widetilde{L}_n^*(-I_n)$ for any given sequences $I_1, ..., I_n$ of distinct integers. □

In the remainder of the paper, we will apply the above lemma to construct oriented links that are neither amphicheiral nor invertible.

For our purposes, we need to consider *based links* in a 3-ball D; these are links such that each component intersects ∂D in at least *two base arcs.* A based link B in D is called a *based Brunnian link* if B is not equivalent to a trivial link considered as an ordinary link in D, yet each proper sublink of B is equivalent to a trivial link via an orientation preserving homeomorphism from D onto itself that fixes the boundary 2-sphere. One may also consider a similar notion under a much weaker equivalence relationship of *link homotopy* in which each component of a link is allowed to cross itself during a deformation, but such that no two components are allowed to intersect. A based link B in D is called a *based homotopy Brunnian link* if B is not link homotopic to a trivial link considered as an ordinary link in D, yet each proper sublink of B is link homotopic to a trivial link via a link homotopy that fixes the base arcs. Notice that there exist infinitely many examples of both based Brunnian links and based homotopy Brunnian links.

We call two sequences of integers $I = (i_1, ..., i_m)$ and $J = (j_1, ..., j_m)$ *equivalent* if there exists an integer a such that $a = j_r - i_r$ after a possible change of the order of I or J for each r. A sequence I of integers is called *invertible* if it is equivalent to the sequence $-I$ obtained by changing signs of all elements of I. For instance, every sequence of one or two integers is invertible; while among the sequences of more than two distinct integers, there are many noninvertible sequences.

Construction. To construct the required examples of oriented links, we need an oriented knot K in S^3; an oriented based link B in a 3-ball D lying in the complement E_K of K; a decomposition of B into a union of $n > 0$ oriented links $B = B^1 \cup \cdots \cup B^n$, where $B^r = B_1^r \cup \cdots \cup B_{m_r}^r$, $m_r > 0$, and each B_s^r is a knot; and finally, n sequences $I_1, ..., I_n$ of distinct integers where $I_r = (i_1^r, ..., i_{m_r}^r)$.

Suppose we are given this data. Then an oriented link L in S^3 may be constructed by performing *interior band-summings* via $m_1 + ... + m_n - n$ disjoint bands that avoid the interior of D; see Cochran [1]. To be more precise, for each r in $\{1, ..., n\}$, we let L_r be obtained from B^r by adding $m_r - 1$ disjoint bands $\beta_1^r, ..., \beta_{m_r-1}^r$ in E_K that avoid the interior of D, in such a way that each β_s^r connects a base arc of B_s^r to a base arc of B_{s+1}^r and that the oriented core arc of β_s^r going from B_s^r to B_{s+1}^r links K algebraically $i_{s+1}^r - i_s^r$ times. Then the resulting oriented link $L = L_1 \cup \cdots \cup L_n \cup K$ in S^3 has $n + 1$ components and satisfies $lk(L_r, K) = 0$ for $r = 1, ..., n$. A similar construction in the special case of $n = 1$ could be found in [3].

Remark. In the above construction of L, the bands β_s^r may be twisted, knotted, or even linked with each other, and if a sequence $I_r = (i_1^r, ..., i_{m_r}^r)$ is replaced by one obtained by adding an integer to each element of I_r, then the resulting L remains unchanged since only the differences $i_{s+1}^r - i_s^r$ are needed in constructing L.

In what follows, unless otherwise stated, we shall use the same notations as that used in the above construction. The following result is a generalization of Lemma 3.4 in [3].

Lemma 2. *Let B be a based Brunnian link (respectively a based homotopy Brunnian link) and let $J_1, ..., J_n$ be sequences of distinct integers where $J_r = (j_1^r, ..., j_{m_r}^r)$. Then there exist lifts $\widetilde{L}_1^*, ..., \widetilde{L}_n^*$ of $L_1, ..., L_n$ to $\widetilde{E}_K$ such that the covering link $\widetilde{L}_1^*(J_1) \cup \cdots \cup \widetilde{L}_n^*(J_n)$ is not equivalent (respectively not link homotopic) to a trivial link if and only if J_r is equivalent to I_r for each r.*

Proof. We consider only the case in which B is a based Brunnian link. The case in which B is a based homotopy Brunnian link can be proved in the same way by replacing link equivalence by link homotopy. To begin with, let us fix a lift $\tilde{D}$ of D to the universal abelian cover $\widetilde{E}_K$ so that $B = B^1 \cup \cdots \cup B^n$ lifts to a based Brunnian link $\widetilde{B} = \widetilde{B}^1 \cup \cdots \cup \widetilde{B}^n$ in the 3-ball $\tilde{D}$ where $\widetilde{B}^r = \widetilde{B}_1^r \cup \cdots \cup \widetilde{B}_{m_r}^r$, and let $\widetilde{L}_1, ..., \widetilde{L}_n$ be lifts of $L_1, ..., L_n$ to $\widetilde{E}_K$, where $\widetilde{L}_r$ is obtained from $t^{-i_1^r}\widetilde{B}_1^r, ..., t^{-i_{m_r}^r}\widetilde{B}_{m_r}^r$ by adding $m_r - 1$ suitably chosen bands in $\widetilde{E}_K$ that cover the bands $\beta_1^r, ..., \beta_{m_r-1}^r$ we have used in the construction of L_r.

Let $\widetilde{L}_1^*, ..., \widetilde{L}_n^*$ be lifts of $L_1, ..., L_n$ to $\widetilde{E}_K$. Then there exist integers $a_1, ..., a_n$ such that $t^{a_r}\widetilde{L}_r^* = \widetilde{L}_r$ for each r, hence the covering link $\widetilde{L}_r^*(J_r)$ is obtained from $m_r m_r$ disjoint embedded circles $t^{j_u^r - i_v^r - a_r}\widetilde{B}_v^r$, where u and v range from 1 to m_r, by adding $m_r(m_r - 1)$ disjoint bands. It follows that the covering link $\widetilde{L}_1^*(J_1) \cup \cdots \cup \widetilde{L}_n^*(J_n)$ is nontrivial only if there exists an integer a such that each component of the based Brunnian link $t^a\widetilde{B}$ appears in the $t^{j_u^r - i_v^r - a_r}\widetilde{B}_v^r$ where $1 \leq u, v \leq m_r$ and $1 \leq r \leq n$, since otherwise the $t^{j_u^r - i_v^r - a_r}\widetilde{B}_v^r$ restrict to a trivial link in each lifting 3-ball of D and hence the covering link is equivalent to one obtained by adding $\sum m_r(m_r - 1)$ disjoint bands to a trivial link that bounds disjoint embedded disks missing the bands.

One may check that such an integer a, if it exists, should satisfy $a + a_r = j_s^r - i_s^r$ for $1 \leq s \leq m_r$ and $1 \leq r \leq n$ after possible changes of the orders of the J_r. Hence J_r is equivalent to I_r for each r. This follows since the I_r and the J_r are both sequences of distinct integers, hence $j_p^r - i_q^r$ and $j_u^r - i_v^r$ could be the same only if $p \neq q$ or $u \neq v$. Notice that the integer a, if it exists, should be unique since $m_r(a + a_r) = \sum j_s^r - \sum i_s^r$ for each r. In this case, the $t^{j_u^r - i_v^r - a_r}\widetilde{B}_v^r$ restrict to a trivial link in $t^b\tilde{D}$ for each $b \neq a$. Hence the covering link $\widetilde{L}_1^*(J_1) \cup \cdots \cup \widetilde{L}_n^*(J_n)$ is equivalent to $\widetilde{B}$ after changing the order of $\widetilde{L}_r^*(J_r)$ in exactly the same way as changing the order of J_r for each r.

Now one may conclude that if there exist lifts $\widetilde{L}_1^*, ..., \widetilde{L}_n^*$ of $L_1, ..., L_n$ to $\widetilde{E}_K$ such that the covering link $\widetilde{L}_1^*(J_1) \cup \cdots \cup \widetilde{L}_n^*(J_n)$ is nontrivial, then J_r is equivalent to I_r for each r. On the other hand, if J_r is equivalent to I_r for each r, then there exist integers $a_1, ..., a_n$ such that $a_r = j_s^r - i_s^r$ after a possible change of the orders of the J_r for $1 \leq s \leq m_r$ and $1 \leq r \leq n$. Let $\widetilde{L}_1^*, ..., \widetilde{L}_n^*$ be lifts of $L_1, ..., L_n$ to $\widetilde{E}_K$ defined by $t^{a_r}\widetilde{L}_r^* = \widetilde{L}_r$. Then it follows by the discussion in the last paragraph that the covering link $\widetilde{L}_1^*(J_1) \cup \cdots \cup \widetilde{L}_n^*(J_n)$ is equivalent to the covering link $\widetilde{B}$ ignoring the orders of links, hence is nontrivial. $\square$

We now come to the main result of this paper which asserts that among the $(n+1)$-component oriented links we have constructed, infinitely many are neither amphicheiral nor invertible.

Theorem 1. *The $(n+1)$-component oriented link L in the 3-sphere constructed above is neither amphicheiral nor invertible provided that B is either a based Brunnian link or a based homotopy Brunnian link and that one of the sequences $I_1, ..., I_n$ is noninvertible.*

Proof. Suppose that B is a based Brunnian link. Then it follows by Lemma 2 that if one of the sequences $I_1, ..., I_n$ is nonivertible, then the covering link $\widetilde{L}_1(-I_1) \cup \cdots \cup \widetilde{L}_n(-I_n)$ is trivial for any choice of lifts $\widetilde{L}_1, ..., \widetilde{L}_n$ of $L_1, ..., L_n$ to $\widetilde{E}_K$; while the covering link $\widetilde{L}_1(I_1) \cup \cdots \cup \widetilde{L}_n(I_n)$ is equivalent to the covering link $\widetilde{B}$, which is nontrivial, for a certain choice of lifts $\widetilde{L}_1, ..., \widetilde{L}_n$ of $L_1, ..., L_n$ to $\widetilde{E}_K$. Hence the oriented link L is neither amphicheiral nor invertible by Lemma 1. The case in which B is a based homotopy Brunnian link can be proved in the same way, noticing that link homotopy is a weaker equivalence relation than link equivalence. □

Remark. The oriented link $L = L_1 \cup \cdots \cup L_n \cup K$ we have constructed has in fact a stronger property that if one of the sequences $I_1, ..., I_n$ is noninvertible, and if B is a based Brunnian link or a based homotopy Brunnian link, then there exists no orientation reversing (respectively orientation preserving) homeomorphism h of S^3 onto itself, such that $h(K) = K$ (respectively $h(K) = -K$) and $h(L_1 \cup \cdots \cup L_n) = L_1 \cup \cdots \cup L_n$ ignoring the order and the orientation of each component.

Theorem 2. *There exist for each $n \geq 2$ infinitely many n-component oriented Brunnian links in the 3-sphere that are neither amphicheiral nor invertible.*

Proof. We consider an unknotted oriented circle K in S^3 and an oriented based homotopy Brunnian link $B = B_1 \cup \cdots \cup B_m$ in a 3-ball lying in the complement of K as shown in Figure 1, where $m = 5$ and the base arcs are at the bottom of each component; for simplicity, we have not depicted the 3-ball and the base arcs explicitly there. The link B differs from the Brunnian link shown in Figure 7 of Milnor [4] only on the second component B_2 where a full twist has been added, hence B is not link homotopically trivial since the original link of Milnor is not. Notice that the only proper sublink of B that is not equivalent to a trivial link via an orientation preserving homeomorphism of the 3-ball onto itself fixing the boundary

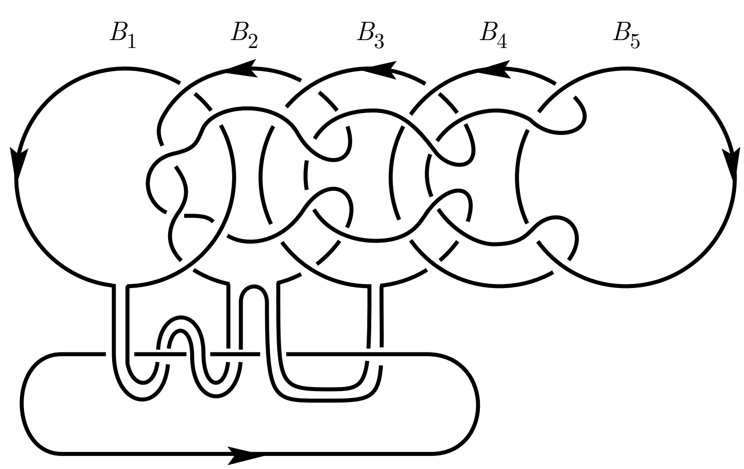

FIGURE 1

2-sphere is the one obtained by removing the first component B_1. However, this sublink is link homotopic to a trivial link via a link homotopy that fixes the base arcs. Hence the based link B is in fact a based homotopy Brunnian link.

Now apply the band-summing construction to the above K and B with further restrictions that the decomposition of B into the union of $B^1, ..., B^{n-1}$ be chosen such that $B^r = B_1 \cup B_2 \cup \cdots$ for some r, and that the band connecting B_1 to B_2 be chosen to be untwisted, unknotted and unlinked with the remaining bands. Then the resulting oriented link L is readily seen to be an n-component Brunnian link (the nontriviality follows since L has a nontrivial m-component covering link by Lemma 2). Moreover, if one of the sequences I_r used in the construction is noninvertible, then the resulting oriented Brunnian link is neither amphicheiral nor invertible. Figure 1 depicts an example of 4-component oriented Brunnian links so obtained, where the decomposition $B = B^1 \cup B^2 \cup B^3$ is given by $B^1 = B_1 \cup B_2 \cup B_3$, $B^2 = B_4$ and $B^3 = B_5$, and the sequences of distinct integers are given by $I_1 = (0, 2, 3)$ and $I_2 = I_3 = (0)$. Since I_1 is noninvertible, this Brunnian link is neither amphicheiral nor invertible.

Our construction in fact gives rise to infinitely many n-component oriented Brunnian links in S^3 that are neither amphicheiral nor invertible for each $n \geq 2$. One way to see this is by noticing that if the two n-component Brunnian links L and L' are constructed as above by using based homotopy Brunnian links in Figure 1 having m and m' components respectively where $m' > m$, then L has an m-component covering link that is not link homotopically trivial by the proof of Lemma 2, while L' has not. It follows that the two Brunnian links are not equivalent to each other. □

Corollary. *There exist for each $n \geq 2$ infinitely many n-component oriented links in the 3-sphere that are neither amphicheiral nor invertible and such that each component has the same knot type as a prescribed one.*

Proof. Let $K_1, ..., K_n$ be oriented knots in S^3 and let $L = L_1 \cup \cdots \cup L_n$ be an oriented Brunnian link which are neither amphicheiral nor invertible. We consider the K_r as oriented knots disjoint from L each of which lies in a 3-ball D_r of S^3, and let the 3-balls be chosen such that each D_r intersects L in a single arc lying on L_r. Connect L_r and K_r via a band lying in D_r for each r. Then the resulting is an oriented link of the form $L^* = L_1^* \cup \cdots \cup L_n^*$ such that each L_r^* has the same knot type as K_r. The nonamphicheirality and noninvertibility of L^* follow from that of L. □

We conclude by noticing that the assertion of Theorem 1 may still be valid even if the sequences $I_1, ..., I_n$ are all invertible. For in that case, although the covering link $\widetilde{L}_1(I_1) \cup \cdots \cup \widetilde{L}_n(I_n)$ in the proof of Lemma 2 is equivalent (respectively link homotopic) to the covering link $\widetilde{L}_1^*(-I_1) \cup \cdots \cup \widetilde{L}_n^*(-I_n)$ for certain choices of lifts $\widetilde{L}_1^*, ..., \widetilde{L}_n^*$ of $L_1, ..., L_n$ to $\widetilde{E}_K$ ignoring the orders of links as shown in the proof of Lemma 2, their link types (respectively link homotopy types) may be quite different by taking account of the orders. Hence Lemma 1 can still be applied.

Consider for example the 4-component link $L_1 \cup L_2 \cup L_3 \cup K$ in Figure 2 obtained by applying the band-summing construction to the same data as that used in constructing the link in Figure 1, except for the sequence I_1 which is now an invertible sequence $(0, 1, 2)$. Let the lifts $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$ of L_1, L_2, L_3 to $\widetilde{E}_K$ be chosen as in the proof of Lemma 2. Then the covering link $\widetilde{L}_1(I_1) \cup \widetilde{L}_2(I_2) \cup \widetilde{L}_3(I_3)$ is link homotopic to the oriented link B. On the other hand, let $\widetilde{L}_1^*$, $\widetilde{L}_2^*$, $\widetilde{L}_3^*$ be lifts of L_1, L_2, L_3. Then

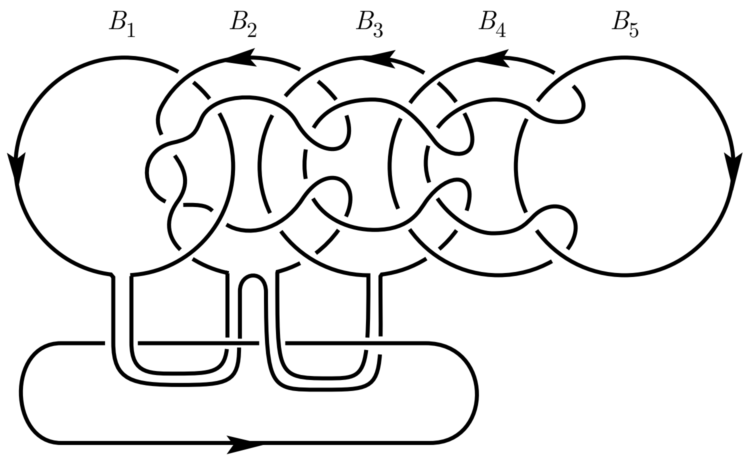

FIGURE 2

by the proof of Lemma 2, the covering link $\widetilde{L}_1^*(-I_1) \cup \widetilde{L}_2^*(-I_2) \cup \widetilde{L}_3^*(-I_3)$ is either link homotopic to a trivial link or to the oriented link $B' = B_3 \cup B_2 \cup B_1 \cup B_4 \cup B_5$. Since B has a nonvanishing link homotopy invariant $\bar{\mu}(12345)$ by Milnor [4], while B' has not, they are even not link homotopic to each other. It follows by Lemma 1 that the oriented Brunnian link in Figure 2 is neither amphicheiral nor invertible.

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