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VOLUME OF INTERSECTIONS AND SECTIONS OF THE UNIT BALL OF ℓ_p^n

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ABSTRACT. An asymptotic formula for the volume of the intersection of a suitable multiple of the unit ball of ℓ_p^n and the cube $[-1/2, 1/2]^n$ will be proved. We also show that the isotropic constant of the unit ball of ℓ_n^p, $1 \le p \le 2$, is bounded by $1/\sqrt{12}$.

1. INTRODUCTION AND NOTATION

Let $r(n,p)B_p^n$ be the multiple of the unit ball B_p^n of ℓ_p^n such that

$$\mathrm{Vol}_n(r(n,p)B_p^n) = 1.$$

In [SS] the following theorem is proved:

Theorem 1.1. *For all $0 < p \le \infty$ and all $0 < q < \infty$ we have*

$$\lim_{n\to\infty} \mathrm{Vol}_n(r(n,p)B_p^n \cap t.r(n,q)B_q^n) = \begin{cases} 0 & \text{if } tA(p,q) < 1, \\ 1 & \text{if } tA(p,q) > 1, \end{cases}$$

where

$$A(p,q) = \begin{cases} \frac{e^{1/p}\Gamma(1+\frac{1}{p})^{1+1/q}p^{1/q}}{e^{1/q}\Gamma(1+\frac{1}{q})\Gamma(\frac{1+q}{p})^{1/q}q^{1/q}} & \text{if } p < \infty, \\ \frac{(1+q)^{1/q}}{e^{1/q}\Gamma(1+\frac{1}{q})q^{1/q}} & \text{if } p = \infty. \end{cases}$$

Also, the following problem was posed: What is the asymptotic behavior of

$$\mathrm{Vol}_n(r(n,p)B_p^n \cap t.r(n,q)B_q^n)$$

for $t = A(p,q)^{-1}$? In section 2 it will be proved that in the case $p = \infty$ this limit equals $\frac{1}{2}$—the case $p = \infty$ and $q = 1$ has also been solved by B. Weißbach.

In section 3 we consider a different problem: Let E be a subspace of $\mathbf{R}^n$ of codimension k. M. Meyer and A. Pajor (cf. [MeP]) proved that for all $p \ge 2$ and $p = 1$: $\mathrm{Vol}_{n-k}(E \cap r(n,p)B_p^n) \ge 1$. We will prove this inequality in the case $k = 1$ and $1 < p < 2$.

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2. Intersection with the cube

Let $0 < q < \infty$ and $t = A(\infty, q)^{-1}$. Define $a(n,q)$ by the equation

$$a(n,q)\frac{1}{2(1+q)^{1/q}} = \frac{r(n,q)}{n^{1/q}A(\infty,q)}.$$

Using Stirling's formula it is easily checked that

$$a(n,q) = 1 + \frac{q\log n}{2n} + O\left(\frac{1}{n}\right). \tag{1}$$

Now, let X be uniformly distributed on the interval $[-\frac{1}{2}, \frac{1}{2}]$ and let X_j, $j = 1, \dots, n$, be independent copies of X. Then

(2)

$$\mathrm{Vol}_n\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^n \cap t.r(n,p)B_p^n\right) = 1 - \mathbf{P}\left(\left(\frac{1}{n}\sum_{j=1}^n |X_j|^q\right)^{1/q} > \|X\|_q a(n,q)\right).$$

Theorem 2.1. *For all* $0 < q < \infty$ *we have*:

$$\lim_{n\to\infty} \mathrm{Vol}_n\left(\left[-\frac{1}{2}, \frac{1}{2}\right]^n \cap \frac{r(n,q)}{A(\infty,q)}B_q^n\right) = \frac{1}{2}.$$

Proof. By (1) and simple algebra we conclude that the probability in (2) is given by

$$\mathbf{P}\left(\frac{1}{\sqrt{n}}\left(\sum_{j=1}^n |X_j|^q - \mathbf{E}|x|^q\right) > \|X\|_q^q\left(\frac{q^2\log n}{2n} + O\left(\frac{1}{n}\right)\right)\sqrt{n}\right). \tag{3}$$

A version of the Berry-Esseen Theorem (cf. e.g. [C, p. 225]) states that if Y_j, $j = 1, \dots, n$, is an i.i.d. sequence of random variables such that $\mathbf{E}Y = 0$ and $\|Y\|_3 < \infty$, then there exists an absolute constant C such that for all $s \in \mathbf{R}$:

$$\left|\mathbf{P}\left(\frac{1}{\|Y\|_2\sqrt{n}}\sum_{j=1}^n Y_j < s\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^s e^{-1/2\,x^2}\,dx\right| \le C\left(\frac{\|Y\|_3}{\|Y\|_2}\right)^2\frac{1}{\sqrt{n}}.$$

Applying this theorem to $Y = |X|^q - \mathbf{E}|X|^q$ and

$$s = \frac{\|X\|_q^q}{\|Y\|_2}\left(\frac{q^2\log n}{2\sqrt{n}} + O\left(\frac{1}{\sqrt{n}}\right)\right) = O\left(\frac{\log n}{\sqrt{n}}\right)$$

we get:

$$\left|\mathbf{P}\left(\left(\frac{1}{n}\sum_{j=1}^n |X_j|^q\right)^{1/q} > \|X\|_q a(n,q)\right) - \frac{1}{2}\right| \le C_1\frac{\log n}{\sqrt{n}} + C\frac{1}{\sqrt{n}},$$

which proves the theorem. □

3. Central sections of B_p^n

Suppose B is a convex symmetric body in $\mathbf{R}^n$ with $\mathrm{Vol}_n(B) = 1$. It is well-known that there exists an affine image $\widetilde{B}$ of B such that the function

$$x \mapsto \int_{\widetilde{B}} \langle x, y\rangle^2 \, dy$$

is constant on S^{n-1}. This constant is called the isotropic constant of B and is denoted by L_B. We also say that $\widetilde{B}$ is in isotropic position. It is easy to see that if the standard basis of $\mathbf{R}^n$ is a 1-symmetric basis of B, then B is in isotropic position. Let $1 \le p \le 2$. Then

$$L^2_{B_p^n} = \int_{r(n,p)B_p^n} x_1^2 \, dx$$

where as above $r(n,p) = \mathrm{Vol}_n(B_p^n)^{-1/n}$. A direct computation yields:

$$L^2_{B_p^n} = \frac{\Gamma(1+\frac{3}{p})\Gamma(1+\frac{n}{p})^{1+2/n}}{12\Gamma(1+\frac{n+2}{p})\Gamma(1+\frac{1}{p})^3}.$$

Let H be a hyperplane containing the origin. A well known result (cf. e.g. [B1]) states that:

$$\mathrm{Vol}_{n-1}(r(n,p)B_p^n \cap H) L_{B_p^n} \ge \frac{1}{\sqrt{12}}.$$

In order to prove $\mathrm{Vol}_{n-1}(\alpha B_p^n \cap H) \ge 1$ it is enough to prove the following inequality:

$$\frac{\Gamma(1+\frac{n}{p})^{1+2/n}}{\Gamma(1+\frac{n+2}{p})} \le \frac{\Gamma(1+\frac{1}{p})^3}{\Gamma(1+\frac{3}{p})}. \tag{4}$$

By Stirling's formula we have:

$$\Gamma(1+z) = \sqrt{2\pi}(1+z)^{z+\frac{1}{2}} e^{-1-z} \exp(\gamma(z))$$

where γ is a decreasing function on the interval $[0, \infty)$ satisfying $0 < \gamma(z) < (12(z+1))^{-1}$. Putting $x = p^{-1}$ the inequality (4) can be written equivalently:

$$\begin{aligned}(2\pi(1+nx))^{1/n} &\left(\frac{1+nx}{1+(n+2)x}\right)^{1/2+(n+2)x} e^{1/n+(1+2/n)\gamma(nx)-\gamma((n+2)x)} \\ &\le 2\pi(1+x)\left(\frac{1+x}{1+3x}\right)^{1/2+3x} e^{1+3\gamma(x)-\gamma(3x)}.\end{aligned}$$

Putting $z_n = \frac{2x}{1+(n+2)x}$ we get

$$\begin{aligned}\log &\left(\frac{1+nx}{1+(n+2)x}\right)^{1/2+(n+2)x} - \log\left(\frac{1+x}{1+3x}\right)^{1/2+3x} \\ &= \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{2x}{k+1}\right)(z_n^k - z_1^k) \le (2x-1)\log\left(\frac{1}{1-z_1}\right).\end{aligned}$$

On the other hand we have for all $n \ge 2$ and all $x > 0$:

$$(1+nx)^{1/n} \le 1+x \quad \text{and} \quad 3\gamma(x) - \gamma(3x) - \left(1+\frac{2}{n}\right)\gamma(nx) + \gamma((n+2)x) \ge 0.$$

Therefore it suffices to prove that for all $\frac{1}{2} \leq x \leq 1$:

$$\left(\frac{1+3x}{1+x}\right)^{2x-1} \leq \sqrt{2\pi e}$$

which follows from the fact that the left hand side is bounded by 2. With some modifications the proof given above also yields for all $n \geq 3$:

$$\frac{\Gamma(1+\frac{n}{p})^{1+2/n}}{\Gamma(1+\frac{n+2}{p})} \leq \frac{\Gamma(1+\frac{2}{p})^2}{\Gamma(1+\frac{4}{p})}.$$

Hence we have the following

Proposition 3.1. *For all hyperplanes H containing the origin, all $1 < p < 2$ and all $n \geq 2$ we have*:

$$\mathrm{Vol}_{n-1}(r(n,p)B_p^n \cap H) \geq \sqrt{\frac{\Gamma(1+\frac{4}{p})\Gamma(1+\frac{1}{p})^3}{\Gamma(1+\frac{2}{p})^2\Gamma(1+\frac{3}{p})}} \geq 1.$$

References

[B1] K. Ball, *Logarithmically Concave Functions and Sections of Convex Sets in $\mathbf{R}^n$*, Studia Math. **88** (1988), 69–84. MR **89e:**52002

[B2] K. Ball, *Normed Spaces with a weak Gordon Lewis Property*, Proc. Funct. Anal., Univ. of Texas and Austin, 1987–1989, Springer LNM 1470, 36–47. MR **93e:**46013

[C] K. L. Chung, *A Course in Probability Theory*, Academic Press, 1974. MR **49:**11579

[MeP] M. Meyer and A. Pajor, *Sections of the Unit Ball of ℓ_p^n*, J. Funct. Anal. **80** (1988), 109–123. MR **89h:**52010

[MiP] V. D. Milman and A. Pajor, *Isotropic Position, Inertia Ellipsoids and Zonoids of the Unit Ball of an n-dimensional Normed Space*, GAFA Seminar 87–89, Springer LNM 1376, 64–104. MR **90g:**52003

[SS] G. Schechtman and M. Schmuckenschläger, *Another Remark on the Volume of the Intersection of Two L_p^n Balls*, GAFA 89/90, 174–178, Springer 1991. MR **92j:**52008

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