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THE DETERMINATION OF THE PAIRS OF TWO-BRIDGE KNOTS OR LINKS WITH GORDIAN DISTANCE ONE

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Abstract. We thoroughly determine the pairs of two-bridge knots or links with Gordian distance one. In addition, we examine the Gordian distance between a Montesinos knot (or link) and a two-bridge knot (or link).

1. Introduction

For any two knots or links K, K' in S^3, we can define the Gordian distance from K to K', denoted by $d_G(K, K')$, to be the minimal number of crossing changes needed to deform a diagram of K into that of K', where the minimum is taken over all diagrams of K from which one can obtain a diagram of K'.

Then d_G defines a metric on the space of the equivalence classes of knots or links. If O is a trivial knot or link, then $d_G(K, O)$ is the unknotting or unlinking number of K, denoted by $u(K)$ (see [15]).

In this paper we determine the pairs of two-bridge knots or links with Gordian distance one. This result can be thought of as a generalization of those of Kanenobu-Murakami [7] and Kohn [8].

After having done this work, the author heard that J. Berge and I. Dazey-D. W. Sumners had independently obtained a result similar to the main theorem, respectively (see [4]).

Throughout this paper we say that K and K' are *equivalent*, denoted by $K = K'$, if and only if there exists an orientation preserving homeomorphism of S^3 which maps K to K'.

2. Main theorem

Let $S(p, q)$ be the two-bridge knot or link whose two-fold branched cover is the lens space $L(p, q)$, where p and q are relatively prime. When p is even, $S(p, q)$ is a two-component link, for p odd, $S(p, q)$ is a knot.

$S(p, q)$ and $S(p', q')$ are equivalent if and only if $p = p'$ and (I) $q \equiv q' \pmod p$ or (II) $qq' \equiv 1 \pmod p$ [2, Theorem 12.6 (b)].

Our main theorem is then the following.

Theorem 1. *Let $S(p, q)$ and $S(r, s)$ be two-bridge knots or links. Then the following conditions are equivalent:*

(i) $d_G(S(p, q), S(r, s)) = 1$.

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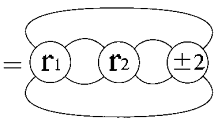
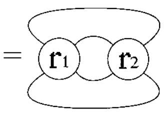
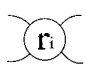
(ii) *There exist pairs of relatively prime integers* (m,n) *and* (a,b) *such that* $n \neq 0, rm+an \neq 0, rb-sa=1$ *and* $S(p,q)$ *is equivalent to*

$$S(2an^2 + r(2mn \pm 1), 2bn^2 + s(2mn \pm 1)).$$

(iii) *There exist rational numbers* r_1 *and* r_2 *such that*

$S(p,q) =$ 

$S(r,s) =$ 

where 

is a rational tangle *of slope* r_i *(for the definition of a rational tangle, see* [2, Chapter 12]*).*

Remark 2. (i) $S(1,0)$ is a trivial knot. So the condition (ii) of Theorem 1 says that $d_G(S(p,q), S(1,0)) = 1$ if and only if

$$\begin{aligned} S(p,q) &= S(2an^2 + 2mn \pm 1, 2n^2) \\ &= S(2m'n \pm 1, 2n^2) \end{aligned}$$

where $m' = m + an$. Therefore Theorem 1 is a generalization of Kanenobu-Murakami's theorem [7].

(ii) $S(0,1)$ is a trivial link. So the condition (ii) of Theorem 1 also says that $d_G(S(p,q), S(0,1)) = 1$ if and only if

$$\begin{aligned} S(p,q) &= S(-2n^2, 2bn^2 + 2mn \pm 1) \\ &= S(-2n^2, 2m'n \pm 1) \end{aligned}$$

where $m' = m+bn$. Therefore Theorem 1 is also a generalization of Kohn's theorem [8].

3. Preliminaries

Let $N(k)$ be a regular neighborhood of a knot k in a closed orientable 3-manifold M, with μ a meridian of $N(k)$. Let $E(k)$ be the exterior of k in M, that is, $E(k) = M - intN(k)$. Now, let $k(\gamma)$ denote the closed manifold obtained by attaching a solid torus V to $E(k)$ so that a curve of slope γ on $\partial E(k)$ bounds a disk in V. Here the slope indicates the isotopy class of a nontrivial simple closed curve in $\partial E(k)$. We shall say that $k(\gamma)$ is the result of γ-surgery on k in M. For two slopes γ and δ in $\partial E(k)$, let $\Delta(\gamma, \delta)$ be their minimal geometric intersection number.

For oriented manifolds M and N, $M \cong N$ means M and N are homeomorphic by an orientation preserving homeomorphism.

Lemma 3. *If* $d_G(S(p,q), S(r,s)) = 1$*, then* $L(p,q)$ *is obtained by* γ*-surgery on some knot in* $L(r,s)$*, where* $\Delta(\gamma, \mu) = 2$*.*

Proof. This is obtained by an argument similar to that of the proof of [9, Lemma 1] (cf. [7], [8]). In fact this follows from Montesinos' technique [11] and the fact that the double branched covering of S^3 along $S(r,s)$ is $L(r,s)$. □

Let $m_g^c((\alpha_1,\beta_1),\cdots,(\alpha_t,\beta_t))$ be a Seifert fibred space, where g is the genus of the orbit surface F, c is the number of boundary components, and t is the number of surgery instructions used to obtain the Seifert fibred space from the genuine (orientable) S^1-bundle over F. Each pair (α_i,β_i) specifies a particular surgery (for example, see [10, Chapter Four]). Our convention shall be that when g is nonnegative, F is orientable, while g negative implies that F is nonorientable ($F = \sharp_{i=1}^{|g|} RP^2$). For later use, we note that $m_g^c((\alpha_1,\beta_1),\cdots,(\alpha_t,\beta_t),(1,e)) \cong m_g^c((\alpha_1,\beta_1),\cdots,(\alpha_i,\beta_i+e\alpha_i),\cdots,(\alpha_t,\beta_t))$ for any integer e and any number i $(1 \le i \le t)$.

Lemma 4. *Let k be a knot in $L(r,s)$. If $E(k)$ is a Seifert fibred space, then k is a (possibly singular) fiber in some Seifert fibration of $L(r,s)$.*

Proof. Let V be a solid torus with meridian μ. By hypothesis $E(k)$ is Seifert fibred and $L(r,s)$ is the union of $E(k)$ and V along the boundary. The core of V is isotopic to k in $L(r,s)$. If $E(k) = m_g^1((\alpha_1,\beta_1),\cdots,(\alpha_t,\beta_t))$, then by using the fact that $H_1(k(\mu))$ is a cyclic group (cf. [8]), we see $g = 0$ or -1. There are also two cases to consider: μ is either identified with a fiber of $E(k)$ or not. If μ is not a fiber, then the Seifert fibration extends on the resulting manifold and k is a fiber. Therefore hereafter we assume μ is a fiber. First we assume $g = 0$, then $t \le 1$ because if otherwise $L(r,s)$ has a separating essential 2-sphere by a standard argument (see [6]), a contradiction. Therefore in this case $E(k)$ is a solid torus. Hence the statement follows immediately by re-fibering $E(k)$. Second we assume $g = -1$. Then $t \le 0$ by a standard argument as above. Then $E(k)$ is a twisted S^1-bundle over a Möbius band and it is homeomorphic to $S^2 \times S^1$ minus a regular neighborhood of a "(2,1)-torus knot" (see [8, Lemma 4]), and this is homeomorphic to a twisted annulus bundle over a circle. And the fibers are parallel circles on the annuli. Therefore as in [8, Lemma 4], the statement of Lemma 4 follows by re-fibering $E(k)$. □

Lemma 5 (The classification of the Seifert fibration for a lens space). *Suppose $m_g^0((\alpha_1,\beta_1),\cdots,(\alpha_t,\beta_t))$ $(\alpha_i \ge 2)$ is a lens space. Then one of the following conditions holds.*

(1) $g = 0$ *and* $t \le 2$,

(2) $g = -1$ *and* $t \le 1$.

Proof. This can be proved by a standard argument as in the proof of Lemma 4. □

Lemma 6. *If $d_G(S(p,q),S(r,s)) = 1$, then for some Seifert fibering of $L(r,s)$, $L(p,q)$ is obtained by γ-surgery along a fiber, where $\Delta(\gamma,\mu) = 2$.*

Proof. From Lemma 3 we know that $L(p,q)$ is obtained by γ-surgery on some knot k in $L(r,s)$, where $\Delta(\gamma,\mu) = 2$. So both γ and μ are cyclic slopes; that is, γ and μ-surgery yield the manifold with cyclic fundamental groups. By the Cyclic Surgery Theorem [3], $E(k)$ is reducible or Seifert fibred. If $E(k)$ is reducible, then $k(\gamma)$ has $L(r,s)$ as a connected summand. Hence $L(p,q) = k(\gamma) = L(r,s)$. This contradicts our assumption. Therefore $E(k)$ is a Seifert fibred space. Lemma 4 implies that k is a fiber in some Seifert fibration of $L(r,s)$. □

Now we describe surgeries on fibers of Seifert fibrations of lens spaces. For $i = 1, 2$, let V_i be a solid torus standardly embedded in S^3 and let μ_i and λ_i be a meridian and a longitude of V_i respectively. Let h be an orientation-reversing homeomorphism from ∂V_1 to ∂V_2 such that $h(\mu_1) = s\mu_2 + r\lambda_2$. Then the space $V_1 \cup_h V_2$ obtained from V_1 and V_2 by identifying their boundaries by h is the lens space $L(r, s)$.

Let $C_{m,n}$ be a (m, n)-curve on ∂V_1, that is, a simple loop on ∂V_1 which is isotopic to $m\mu_1 + n\lambda_1$. Let a and b be integers such that $rb - sa = 1$. Then we may assume $h(\lambda_1) = b\mu_2 + a\lambda_2$ and $C_{m,n}$ is isotopic to $(sm+bn)\mu_2 + (rm+an)\lambda_2$ on $\partial V_2 = \partial V_1$. $L(r, s)$ has a Seifert fibration in which $C_{m,n}$ is a fiber if and only if $n \neq 0$ and $rm + an \neq 0$. In fact, such a Seifert fibration is given by $m_0^0((n, x), (rm + an, y))$, where x and y are some integers (cf. [8, p.1137]). We may push $C_{m,n}$ into $intV_1$. Then, for a slope γ on $\partial N(C_{m,n})$, using the usual meridian-longitude coordinates of $\partial N(C_{m,n})$ in $V_1 \subset S^3$, we identify γ with $c/d \in \mathbb{Q} \cup \{\infty\}$, with c and d relatively prime. Now we perform c/d-surgery on $C_{m,n}$ in $V_1 \subset L(r, s)$.

Lemma 7. *Let $L(r, s) \supset C_{m,n}$ be as above. Then*

(i) *$C_{m,n}(c/d) \cong m_0^0((n, x), (rm + an, y), (c - dmn, d))$, where $L(r, s) \cong m_0^0((n, x), (rm + an, y))$ and a, x, y are as above.*

(ii) *If $c = dmn \pm 1$, then $C_{m,n}(c/d) \cong L(dan^2 + r(dmn \pm 1), dbn^2 + s(dmn \pm 1))$, where a and b are as above.*

Proof. (i) This is obtained by using the fact that an ordinary fiber on $\partial N(C_{m,n}) \subset m_0^0((n, x), (rm + an, y))$ has slope mn (cf. [12]).

(ii) By [5, Lemma 7.2 and its proof], if $c = dmn \pm 1$, then after surgery V_1 changes into another solid torus V_1' with meridian $(dmn \pm 1)\mu_1 + dn^2\lambda_1$. Since the image of the meridian of V_1' by h is

$$\begin{aligned}\begin{pmatrix} h(\mu_1) & h(\lambda_1) \end{pmatrix} \begin{pmatrix} dmn \pm 1 \\ dn^2 \end{pmatrix} &= \begin{pmatrix} \mu_2 & \lambda_2 \end{pmatrix} \begin{pmatrix} s & b \\ r & a \end{pmatrix} \begin{pmatrix} dmn \pm 1 \\ dn^2 \end{pmatrix} \\ &= \begin{pmatrix} \mu_2 & \lambda_2 \end{pmatrix} \begin{pmatrix} dbn^2 + s(dmn \pm 1) \\ dan^2 + r(dmn \pm 1) \end{pmatrix},\end{aligned}$$

we obtain the desired result. □

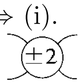
4. Proof of Theorem 1

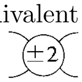
(i) ⇒ (ii). By Lemma 6, there is a fiber k of $L(r, s)$ such that γ-surgery along k results in $L(p, q)$, where $\Delta(\gamma, \mu) = 2$. By Lemma 5, the Seifert fibration is over S^2 or RP^2. First we consider the S^2 case. Then k is isotopic to some $C_{m,n} \subset \partial V_1 = \partial V_2 \subset V_1 \cup_h V_2 = L(r, s)$, where $V_1 \cup_h V_2$ is as in Section 3. Since $k = C_{m,n}$ is a fiber, n and $rm + an$ are nonzero. Suppose $|n|$ and $|rm + an| \geq 2$. Then $\gamma = c/2$ for some odd integer c, because $\Delta(\gamma, \mu) = 2$. By Lemma 7 (i), $L(p, q) \cong m_0^0((n, x), (rm + an, y), (c - 2mn, 2))$, where $L(r, s) \cong m_0^0((n, x), (rm + an, y))$. Then, since $|n|, |rm + an| \geq 2$, we have $c - 2mn = \epsilon$ where $\epsilon = \pm 1$ by Lemma 5. So by Lemma 7 (ii), $L(p, q) \cong L(2an^2 + r(2mn \pm 1), 2bn^2 + s(2mn \pm 1))$. Therefore $S(p, q)$ is equivalent to $S(2an^2 + r(2mn \pm 1), 2bn^2 + s(2mn \pm 1))$, where (m, n) and (a, b) satisfy the condition of the statement (ii). Secondly we consider the case where $|n| = 1$ or $|rm + an| = 1$. Here, without loss of generality, we can assume $|n| = 1$. Then if $c = 2m' + 1$ for some integer m', we can put $k = C_{m',1}$. Therefore as in the case $n \geq 2$, by Lemma 7 (ii), the statement follows.

Next we consider the RP^2 case. Then $L(r,s) = E(k) \cup_{id} N(k)$ and $L(p,q) = E(k) \cup_h N(k)$, where $\Delta(h(\mu),\mu) = 2$. If k is an ordinary fiber, then the core of $N(k) \subset E(k) \cup_h N(k)$ is a singular fiber, since $\Delta(h(\mu),\mu) = 2$. Therefore by Lemma 5, $E(k)$ is a twisted S^1-bundle over a Möbius band whether k is a singular fiber or not. As in the proof of Lemma 4, $L(r,s)$ and $L(p,q)$ are obtained by surgery along a "(2,1)-torus knot" in $S^2 \times S^1$. Therefore, by re-fibering $E(k)$, we see $L(r,s) \cong m_0^0((2,1),(2,-1),(1,d))$ and $L(p,q) \cong m_0^0((2,1),(2,-1),(1,d \pm 2))$ for some d (cf. [8, p.1139]). Hence, by changing invariants $L(r,s) \cong m_0^0((2,1),(2,2d-1))$ and $L(p,q) \cong m_0^0((2,1),(2,2d-1),(1,\pm 2))$. Then $L(p,q)$ is obtained by $c/2$-surgery along $C_{2,z} \subset V_1 \cup V_2 \subset L(r,s)$ for some z, where $h(\mu_1) = r\lambda_2 + s\mu_2$, $h(\lambda_1) = a\lambda_2 + b\mu_2$, $rb - sa = 1$ and $c = 4z \pm 1$. Therefore as in the S^2 case, (p,q) has the desired description.

(ii) $\Rightarrow$ (iii). As above $L(p,q)$ can be obtained by $c/2$-surgerying along $C_{m,n} \subset V_1 \subset V_1 \cup_h V_2 = L(r,s)$, where $V_1 \cup_h V_2$ is as in Section 3. Then $L(p,q) \cong m_0^0((n,x),(rm+an,y),(\pm 1,2))$, where $L(r,s) \cong m_0^0((n,x),(rm+an,y))$ similarly. Hence $L(p,q)$ and $L(r,s)$ are the double covers of the *Montesinos* knots or links (see [2, Chapter 12]) as in the statement (iii) of Theorem 1, where $r_1 = x/n$ and $r_2 = y/(rm+an)$. Therefore $S(p,q)$ and $S(r,s)$ have the desired description.

(iii) $\Rightarrow$ (i).

Since  =  or , we can easily see $d_G(S(p,q),S(r,s)) \leq 1$.

It remains to prove $d_G(S(p,q),S(r,s)) \neq 0$. Suppose $S(p,q)$ is not equivalent to $S(r,s)$. Then applying the above arguments to a crossing change in  by (ii), there exist pairs of relatively prime integers (m,n) and (a,b) such that $n \neq 0, rm+an \neq 0, rb-sa = 1$ $S(2an^2 + r(2mn \pm 1), 2bn^2 + s(2mn \pm 1))$ is equivalent to $S(r,s)$. Therefore it follows that $r = 2an^2 + r(2mn \pm 1)$ and (I) $s \equiv 2bn^2 + s(2mn \pm 1) \pmod r$ or (II) $s(2bn^2 + s(2mn \pm 1)) \equiv 1 \pmod r$. But under the above conditions, elementary number theory for the congruences easily proves these cases never occur. This makes a contradiction.

This completes the proof of Theorem 1.

5. Addendum

In this section we measure the Gordian distance between a Montesinos knot (or link) and a two-bridge knot (or link). Though the proof of Theorem 1 depends on the Cyclic Surgery Theorem, that of the following theorem will depend on the recent Boyer-Zhang's theorem [1].

Theorem 8. *Let $K = M((\alpha_1,\beta_1),\cdots,(\alpha_t,\beta_t))$ be a Montesinos knot or link with $\alpha_i \geq 2$, $t \geq 4$. Then $d_G(K,S(p,q)) \geq 2$, for any two-bridge knot or link $S(p,q)$.*

Remark 9. In [14] Motegi independently proved Theorem 8 for the knot case.

Proof of Theorem 8 (cf. [13], [17]). We assume that $d_G(K,S(p,q)) = 1$. Then by analogy of Lemma 3, the double cover of M_K of K is obtained by γ-surgery on some knot k in $L(p,q)$, where $\Delta(\gamma,\mu) = 2$. Here $M_K = m_0^0((\alpha_1,\beta_1),\cdots,(\alpha_t,\beta_t))$. We remark that $E(k)$ is irreducible. Because if otherwise, then $k(\gamma)$ must be a reducible Seifert fibred space. This contradicts our assumption. Hence by Boyer-Zhang's result [1, Theorem I] $E(k)$ is a Seifert fibred space or a cable on a Seifert fibred space.

If $E(k)$ is the former, then, by Lemma 4, k is a fiber for some Seifert fibration of $L(p,q)$. Then, by an argument as in Section 3, we can prove $k(\gamma)$ can never be such a Seifert fibred space as the number of singular fibers ≥ 4. If $E(k)$ is the latter, then there exists a cable space C such that $\partial C = T_1 \amalg T_2$, where $\partial E(k) = T_1$ and

$$E(k) = m_g^1((\alpha_1,\beta_1),\cdots,(\alpha_u,\beta_u)) \cup_{T_2} C$$

for some integers g and u. For any slope δ on $T_1 = \partial E(k)$, let $C(\delta)$ denote the manifold obtained by γ-surgery on k in C. Since $k(\mu) = L(p,q)$, $C(\mu)$ must be a solid torus. Let k' be a core of $C(\mu)$. We can regard k' as a knot in $L(p,q)$. Then

$$E(k') = m_g^1((\alpha_1,\beta_1),\cdots,(\alpha_u,\beta_u)).$$

Again by Lemma 4, k' is a fiber for some Seifert fibration of $L(p,q)$. Hence we can also prove that $k(\gamma) = E(k') \cup C(\mu)$ cannot be such a Seifert fibred space.

This makes a contradiction. □

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