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A RELATION BETWEEN HOCHSCHILD HOMOLOGY AND COHOMOLOGY FOR GORENSTEIN RINGS

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Abstract. Let "HH" stand for Hochschild (co)homology. In this note we show that for many rings A there exists $d \in \mathbb{N}$ such that for an arbitrary A-bimodule N we have $HH^i(N) = HH_{d-i}(N)$. Such a result may be viewed as an analog of Poincaré duality.

Combining this equality with a computation of Soergel allows one to compute the Hochschild homology of a regular minimal primitive quotient of an enveloping algebra of a semisimple Lie algebra, answering a question of Polo.

In the sequel the base field will be denoted by k. Let $\mathfrak{g}$ be a semisimple Lie algebra and let A be a regular minimal primitive quotient of $U(\mathfrak{g})$. The Hochschild cohomology of A was computed by Soergel in [6] and shown to be equal to the cohomology of the corresponding flag variety. Soergel's computation is rather ingenious and makes use of the Bernstein-Beilinson theorem together with the Riemann-Hilbert correspondence. The case of singular A is still open.

After Soergel's result Patrick Polo asked whether perhaps the Hochschild *homology* of A also coincided with the *homology* of the underlying flag manifold. It is indeed rather likely that Soergel's techniques can be adapted to this end.

In this note we answer Polo's question with a different method. Indeed we show, using elementary homological algebra, that for many algebras A (in particular those considered above) we actually have

$$HH^i(N) = HH_{d-i}(N) \tag{1}$$

for some $d \in \mathbb{N}$ and for an arbitrary A-bimodule N (see Corollary 6). This can be considered a kind of Poincaré duality. In more general situations (1) may remain true provided we twist N on the right hand side by an automorphism or an invertible bimodule (see Theorem 1).

(1) may be used to complete the results in [7] where the Hochschild homology of a generic three dimensional type A regular algebra was computed (see [1] for terminology). Since we show below that (1) holds for such a algebras, we now also obtain the Hochschild cohomology without any extra work.

Our basic result is the following theorem, which is easily proved.

Theorem 1. *Assume that A is an algebra such that there exists a $d \in \mathbb{N}$ with the property that* $\operatorname{Ext}^i_{A^e}(A, A^e) = 0$ *unless $i = d$. Put* $U = \operatorname{Ext}_A(A, A^e)$*, and assume*

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in addition that U is an invertible A-bimodule. Then for every A-bimodule N we have

$$HH^i(N) = HH_{d-i}(U \otimes_A N). \tag{2}$$

Proof. To prove this it is convenient to make use of the derived category. We have

$$\begin{aligned} HH^i(N) &= H^i(\mathrm{RHom}_{A^e}({}_{A^e}A, {}_{A^e}N)) = H^i(\mathrm{RHom}_{A^e}({}_{A^e}A, A^e) \overset{L}{\otimes}_{A^e} N) \\ &= H^i(U[-d] \overset{L}{\otimes}_{A^e} N) = H^{i-d}(U \overset{L}{\otimes}_{A^e} N) \\ &= H^{i-d}(A \overset{L}{\otimes}_{A^e} (U \overset{L}{\otimes}_A A)) = HH_{d-i}(U \otimes_A N). \end{aligned}$$

□

We now give several situations in which the theorem can be applied. First from [8] we recall some results concerning Koszul algebras.

Proposition 2. *Let A be a (graded) Koszul algebra of global dimension d such that $A^!$ is Frobenius. Then the hypotheses of Theorem 1 are satisfied, and furthermore we have the following precise description of U.*

Let $\phi^!$ be the graded automorphism of $A^!$ such that $(A^!)^* = A^!_{\phi^!}$. Furthermore let ϕ be the automorphism of A which is adjoint to $\phi^!$. Finally let ϵ be the automorphism of A which is multiplication by $(-)^m$ on A_m. Then

$$U = A_{\phi^{-1}\epsilon^{d+1}}.$$

In particular, (2) *specializes to*

$$HH^i(N) = HH_{d-i}({}_{\phi\epsilon^{d+1}}N).$$

The simplest non-trivial Koszul algebras of finite global dimension are those of global dimension three. These have been studied extensively in [1, 2, 3]. Recall that they have a presentation $k\langle x_1, x_2, x_3\rangle/(f_1, f_2, f_3)$ in which $x = (x_1, x_2, x_3)^t$, $f = (f_1, f_2, f_3)^t$ can be chosen in such a way that $f = Mx$, $x^t M = (Qf)^t$ for 3×3 matrices M and Q, respectively with linear an scalar entries. It is shown in [8] that in this case the automorphism ϕ is given in terms of the generators by $x \mapsto Q^t x$. In particular if A is a type A algebra (see [1]), then (1) holds with $d = 3$.

Let us now exhibit another situation in which Theorem 1 applies and which occurs frequently in practice.

Proposition 3. *Let A be a k-algebra equipped with a k-linear ascending filtration $(F_n A)_{n\geq 0}$ such that $\mathrm{gr}_F A$ is commutative of finite type over k. Assume that $\mathrm{gr}\, A$ is Gorenstein of dimension d. Then the hypotheses of Theorem 1 hold, and furthermore U carries an A-bimodule filtration such that $\mathrm{gr}\, U$ is equal to $\omega^{-1}_{\mathrm{gr}\, A/k}$, the inverse of the dualizing module of $\mathrm{gr}\, A$.*

Proof. We use the standard spectral sequence

$$\mathrm{Ext}^i_{(\mathrm{gr}\, A)^e}(\mathrm{gr}\, A, (\mathrm{gr}\, A)^e) \Rightarrow \mathrm{Ext}^i_{A^e}(A, A^e), \tag{3}$$

which converges because we have positive filtrations.

Let C is a commutative Cohen-Macaulay ring of dimension d. From the adjunction formula one has

$$\mathrm{RHom}_{C^e}(C, \omega_{C^e}) = \omega_C[-d].$$

Assume now that C is Gorenstein. Then $\omega_{C^e} = \omega_C \otimes_k \omega_C$ and ω_C are invertible, and hence

$$\operatorname{RHom}_{C^e}(C, \omega_{C^e}) = \operatorname{RHom}_{C^e}(C, C^e) \otimes_{C^e} \omega_{C^e}.$$

Combining this yields

$$\operatorname{RHom}_{C^e}(C, C^e) = \omega_C[-d] \otimes_{C^e} \omega_{C^e}^{-1} = \omega_C^{-1}[-d].$$

Applying this with $C = \operatorname{gr} A$ shows that (3) degenerates, and we obtain

$$\operatorname{RHom}_{A^e}(A, A^e) = U[-d]$$

where U lives in one degree and where, from a functoriality argument using (3), one obtains that the filtration on U is an A^e filtration, hence an A bimodule filtration. Furthermore for this filtration, $\operatorname{gr} U = \omega_C^{-1}$. It now suffices to apply lemma 4 below to deduce that U is invertible. □

Lemma 4. *Assume that A is a ring equipped with a positive ascending filtration, and let U be an A-bimodule, equipped with a left limited A-bimodule filtration. If $\operatorname{gr} U$ is an invertible $\operatorname{gr} A$-bimodule, then U is an invertible A-bimodule.*

Proof. We have to show [4, §1] that U is left and right projective and that the canonical maps

$$A \to \operatorname{End}_A({}_AU), \tag{4}$$

$$A \to \operatorname{End}_A(A_U) \tag{5}$$

are surjective. The fact that U is projective is standard (see [5, Cor. D.VII.6]). Therefore let us concentrate on (4). (5) is similar. We use the standard spectral sequence

$$\operatorname{Ext}^i(\operatorname{gr}({}_AU), \operatorname{gr}({}_AU)) \Rightarrow \operatorname{Ext}^i_A(U, U)$$

which is compatible with (4) and which obviously degenerates. From this we deduce that

$$\operatorname{gr}\operatorname{End}_A({}_AU) = \operatorname{End}_A(\operatorname{gr}({}_AU)).$$

Combining this with the hypotheses that $\operatorname{gr} U$ is invertible yields that

$$\operatorname{gr} A \to \operatorname{gr}\operatorname{End}_A({}_AU)$$

is surjective. Hence (4) is also surjective. □

We use the following lemma below.

Lemma 5. *Let the notation be as in the previous lemma. Assume that $\operatorname{gr} U$ is left free of rank one and let θ be the automorphism of $\operatorname{gr} A$ such that $\operatorname{gr} U = (\operatorname{gr} A)_\theta$. Then there exists a filtered automorphism ψ of A such that $U = A_\psi$ and $\operatorname{gr}\psi = \theta$.*

Proof. Let $\bar{u}$ be a generator of $\operatorname{gr} U$ such that

$$\bar{u}y = \theta(y)\bar{u} \tag{6}$$

and let u be a lifting to U. It is easy to see that u is a basis for U. Thus there is an automorphism ψ of A such that $ux = \psi(x)u$. If we take the image of this equality in $\operatorname{gr} A$, we find that $\bar{u}\bar{x} = \overline{\psi(x)}\bar{u}$. Comparing with (6) yields $\theta(\bar{x}) = \overline{\psi(x)}$. From this we easily deduce that ψ is filtered, and $\operatorname{gr}\psi = \theta$. □

Corollary 6. *Assume that A is equipped with a filtration $(F_nA)_{n\in\mathbb{N}}$ such that $F_0A = k$ and such that $\operatorname{gr} A$ is commutative and generated by $\operatorname{gr}_1 A$. Put $\mathfrak{g} = \operatorname{gr}_1 A$ with Lie algebra structure deduced from the commutator on A. Assume furthermore that $\operatorname{gr} A$ is Gorenstein and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Then*

$$HH^i(N) = HH_{d-i}(N).$$

Proof. Since $\operatorname{gr} A$ is graded local, we have $\omega_A = A$, as A-bimodules, and hence we know from lemma 5 that $H^i(N) = H_{d-i}({}_{\psi^{-1}}N)$ for some filtered automorphism of A such that $\operatorname{gr}\psi = \mathrm{id}_{\operatorname{gr} A}$. Let us consider $\mathfrak{g}$ as a subspace of F_1A. For $x, y \in \mathfrak{g}$ we have

$$yx - xy = [y, x] + \alpha \tag{7}$$

for some $\alpha \in k$. We also have $\psi(x) = x + \beta(x)$, where $\beta \in \mathfrak{g}^*$. Applying ψ to (7) yields

$$yx - xy = [y, x] + \beta([y, x]) + \alpha$$

and thus $\beta \mid [\mathfrak{g}, \mathfrak{g}] = 0$. Hence $\beta = 0$, and we are done. □

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