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CHARACTERIZATIONS OF W-TYPE SPACES

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ABSTRACT. In this paper we obtain new characterizations of certain spaces of W-type.

1. Introduction

The spaces of W-type were studied by B.L. Gurevich [5] and I.M. Gelfand and G.E. Shilov [4]. They investigated the behaviour of the Fourier transformation on the W-spaces. Also W-spaces are applied to the theory of partial differential equations. These spaces are generalizations of spaces of S-type [3].

R.S. Pathak [6] and S.J.L. van Eijndhoven and M.J. Kerkhof [2] introduced new spaces of W-type and investigated the behaviour of the Hankel transformation over them.

In this paper, motivated by the work of R.S. Pathak and S.K. Upadhyay [7], we give new characterizations of the spaces of W-type introduced in [2].

In our investigation the Hankel integral transformation defined by

$$h_\mu(\phi)(x) = \int_0^\infty y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\phi(y)dy \ , \ x \in (0,\infty) \ ,$$

plays an important role, where as usual J_μ denotes the Bessel function of the first kind and order μ. Throughout this paper μ will always represent a real number greater than $-1/2$.

It is known (Corollary 4.8, [1]) that h_μ is an automorphism of the space Se constituted by all those complex valued even smooth functions $\phi = \phi(x)$, $x \in \mathbb{R}$, such that

$$\gamma_{m,n}(\phi) = \sup_{x\in\mathbb{R}}|x^m D^n \phi(x)| < \infty \ , \text{ for every } m, n \in \mathbb{N} \ .$$

Moreover h_μ^{-1}, the inverse of h_μ, coincides with h_μ on Se.

Throughout this paper we will denote by K the following set of functions:

$$K = \left\{ M \in C^2([0,\infty)) : M(0) = M'(0) = 0, M'(\infty) = \infty \text{ and } M''(x) > 0, x \in (0,\infty) \right\} .$$

For every $M \in K$ we will represent by $M^\times$ the Young dual function of M ([4], p.19). Interesting and useful properties of the functions in K can be found in [2] and [4].

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In [4] the W-spaces were defined as follows. Let $M, \Omega \in K$ and $a, b > 0$.

The space $W_{M,a}$ consists of all those complex valued and smooth functions ϕ on $\mathbb{R}$ such that for every $m \in \mathbb{N} - \{0\}$ and $k \in \mathbb{N}$ there exists $C_{m,k} > 0$ for which

$$|D^k \phi(x)| \leq C_{m,k} \exp\Big(-M(a(1-\frac{1}{m})|x|)\Big)\ ,\ x \in \mathbb{R}\ .$$

The space $W^{\Omega,b}$ consists of all entire functions ϕ such that for every $m \in \mathbb{N} - \{0\}$ and $k \in \mathbb{N}$ there exists $C_{m,k} > 0$ for which

$$|z^k \phi(z)| \leq C_{m,k} \exp\Big(\Omega(b(1+\frac{1}{m})|\Im z|)\Big), z \in \mathbb{C}\ .$$

An entire function ϕ is in $W^{\Omega,b}_{M,a}$ if, and only if, for each $m, k \in \mathbb{N} - \{0\}$ there exists $C_{m,k}$ for which

$$|\phi(z)| \leq C_{m,k} \exp\Big(-M(a(1-\frac{1}{m})|\Re z|) + \Omega(b(1+\frac{1}{k})|\Im z|)\Big)\ ,\ z \in \mathbb{C}\ .$$

S.J.L. van Eijndhoven and M.J. Kerkhof [2] investigated the behaviour of the transformation h_μ on the subspaces of the W-spaces defined as follows.

A function ϕ is in $We_{M,a}$ (respectively, $We^{\Omega,b}$ and $We^{\Omega,b}_{M,a}$) when ϕ is even and ϕ is in $W_{M,a}$ (respectively, $W^{\Omega,b}$ and $W^{\Omega,b}_{M,a}$).

We now introduce new spaces of W-type.

Let $M, \Omega \in K$, $a, b > 0$ and $1 \leq p \leq \infty$. A complex valued and smooth function $\phi = \phi(x)\ ,\ x \in I = (0, \infty)$, is in $We^p_{\mu,M,a}$ if, and only if, ϕ belongs to Se and

$$\Big\|\exp\Big(M[a(1-\frac{1}{m})x]\Big)\Delta^k_\mu \phi(x)\Big\|_p < \infty \text{ for every } m \in \mathbb{N} - \{0\} \text{ and } k \in \mathbb{N}\ .$$

Here and in the sequel $\|\ \|_p$ denotes the norm in the Lebesgue space $L_p(0, \infty)$. By Δ_μ we denote the Bessel operator $x^{-2\mu-1} D x^{2\mu+1} D$.

The space $We^{p,\Omega,b}$ consists of $\phi \in Se$ that admit a holomorphic extension to the whole complex plane and that satisfy the following two conditions:

(i) there exists $\epsilon > 0$ such that for every $k \in \mathbb{N}$ we can find $C_k > 0$ for which

$$|z^k \phi(z)| \leq C_k \exp\Big(\Omega(b\epsilon|\Im z|)\Big)\ ,\ z \in \mathbb{C}\ ,$$

(ii) $\sup_{y\in\mathbb{R}} \Big\|\exp\Big(-\Omega[b(1+\frac{1}{n})|y|]\Big)(x+iy)^m \phi(x+iy)\Big\|_p < \infty$, for every $n \in \mathbb{N} - \{0\}$ and $m \in \mathbb{N}$.

A complex valued and smooth function $\phi = \phi(x), x \in I$, is in $We^{p,\Omega,b}_{M,a}$ if, and only if, ϕ is in Se admitting a holomorphic extension to the whole complex plane and ϕ satisfies (i) and

(iii) $\sup_{y\in\mathbb{R}} \Big\|\exp\Big(M[a(1-\frac{1}{m})x] - \Omega[b(1+\frac{1}{n})|y|]\Big)\phi(x+iy)\Big\|_p < \infty$ for every $m, n \in \mathbb{N} - \{0\}$.

In Section 2 we establish that $We^p_{\mu,M,a} = We_{M,a}$, $We^{p,\Omega,b} = We^{\Omega,b}$ and $We^{p,\Omega,b}_{M,a} = We^{\Omega,b}_{M,a}$, for every $\mu > -1/2$ and $1 \leq p \leq \infty$.

Throughout this paper for every $1 < p < \infty$ we denote by p' the conjugate of p $\Big($i.e., $p' = \dfrac{p}{p-1}\Big)$. Also by C we always represent a suitable positive constant, not necessarily the same in each occurrence.

2. Characterizations of We-spaces

In this Section we prove, by using the Hankel transformation h_μ, that $We^p_{\mu,M,a} = We_{M,a}$, $We^{p,\Omega,b} = We^{\Omega,b}$ and $We^{p,\Omega,b}_{M,a} = We^{\Omega,b}_{M,a}$, for every $\mu > -1/2$ and $1 \le p \le \infty$.

Lemma 2.1. *Let $1 \le p \le \infty$ and $\mu > -1/2$. Then $We^p_{\mu,M,a}$ is contained in $We_{M,a}$.*

Proof. Assume first that $1 < p < \infty$. Let ϕ be in $We^p_{\mu,M,a}$. Define

$$\psi(y) = h_\mu(\phi)(y) = \int_0^\infty (xy)^{-\mu} J_\mu(xy)\phi(x)x^{2\mu+1}dx\ ,\ y \in \mathbb{C}\ . \tag{1}$$

According to Corollary 4.8 in [1], ψ is in Se. Moreover, the last integral is defined for every $y \in \mathbb{C}$. In fact, for every $y \in \mathbb{C}$ and $n \in \mathbb{N} - \{0\}$, by virtue of (5.3.b) of [2] and Hölder's inequality we have

$$\int_0^\infty |(xy)^{-\mu} J_\mu(xy)||\phi(x)|x^{2\mu+1}dx \le C\int_0^\infty \exp\Big(x|\Im y|\Big)|\phi(x)|x^{2\mu+1}dx$$

$$\le C\int_0^\infty \exp\Big[x|\Im y| - M\Big(a(1-\frac{1}{n})x\Big)\Big]\exp\Big[M\Big(a(1-\frac{1}{n})x\Big)\Big]|\phi(x)|x^{2\mu+1}dx$$

$$\le C\left(\int_0^\infty \Big|\exp\Big[x|\Im y| - M\Big(a(1-\frac{1}{n})x\Big)\Big]x^{2\mu+1}\Big|^{p'}dx\right)^{1/p'}$$

$$\cdot\left(\int_0^\infty \Big|\exp\Big[M\Big(a(1-\frac{1}{n})x\Big)\Big]\phi(x)\Big|^{p}dx\right)^{1/p}$$

$$\le C\left(\int_0^\infty \Big|\exp\Big[x|\Im y| - M\Big(a(1-\frac{1}{n})x\Big)\Big]x^{2\mu+1}\Big|^{p'}dx\right)^{1/p'}\ .$$

Moreover, denoting as usual by $M^\times$ the Young dual of M, according to well-known properties of $M^\times$ ([4]) we obtain for every $x \in I$, $y \in \mathbb{C}$, $n, m \in \mathbb{N} - \{0\}$, where $1 < m < n$,

$$x|\Im y| - M\Big(a(1-\frac{1}{n})x\Big) = \frac{x|\Im y|}{a(1-1/m)}a(1-\frac{1}{m}) - M\Big(a(1-\frac{1}{n})x\Big)$$

$$\le M\Big(a(1-\frac{1}{m})x\Big) - M\Big(a(1-\frac{1}{n})x\Big) + M^\times\left(\frac{|\Im y|}{a(1-1/m)}\right)$$

$$\le -M\Big(a(\frac{1}{m}-\frac{1}{n})x\Big) + M^\times\left(\frac{|\Im y|}{a(1-1/m)}\right)\ .$$

Hence for every $m, n \in \mathbb{N} - \{0\}$ with $1 < m < n$ we can write

$$\int_0^\infty |(xy)^{-\mu} J_\mu(xy)||\phi(x)|x^{2\mu+1}dx$$
$$\le C\left(\int_0^\infty \left(\exp\Big[-M\Big(a(\frac{1}{m}-\frac{1}{n})x\Big)\Big]x^{2\mu+1}\right)^{p'}dx\right)^{1/p'}\exp\Big[M^\times\Big(\frac{|\Im y|}{a(1-1/m)}\Big)\Big]$$
$$\le C\exp\Big[M^\times\Big(\frac{|\Im y|}{a(1-1/m)}\Big)\Big]\ ,\ y\in\mathbb{C}\ ,$$

because $\lim_{x\to\infty} M'(x) = \infty$.

If $p = 1$ or $p = \infty$ we can argue in a similar way.

Thus we conclude that the integral in the right hand side of (1) is a continuous extension of ψ to the whole complex plane. Moreover, by proceeding in a similar way we can see that it also is entire. Such an extension will be denoted again by ψ. Note that ψ is an even function.

We prove that $\psi \in We^{M^\times,1/a}$.

It is not difficult to deduce from Lemma 5.4-1 of [9] that for every $k \in \mathbb{N}$

$$y^{2k}\psi(y) = (-1)^k \int_0^\infty (xy)^{-\mu}J_\mu(xy)\Delta_\mu^k[\phi(x)]x^{2\mu+1}dx\ ,\quad y\in\mathbb{C}\ .$$

Then, proceeding as above, we get for every $k, m \in \mathbb{N}$, $m > 1$,

$$|y^{2k}\psi(y)| \le \int_0^\infty \left|(xy)^{-\mu}J_\mu(xy)\right| |\Delta_\mu^k[\phi(x)]|x^{2\mu+1}dx$$
$$\le C\int_0^\infty \exp\Big(x|\Im y|\Big)x^{2\mu+1}|\Delta_\mu^k[\phi(x)]|dx$$
$$\le C\exp\Big[M^\times\Big(\frac{|\Im y|}{a(1-1/m)}\Big)\Big]\ ,\quad y\in\mathbb{C}\ . \tag{2}$$

Hence ψ is in $We^{M^\times,1/a}$.

Since $h_\mu = h_\mu^{-1}$ on *Se*, according to Lemma 7.4 of [2], we conclude that $We^p_{\mu,M,a}$ is contained in $We_{M,a}$. □

Lemma 2.2. *Let $1 \le p \le \infty$ and $\mu > -1/2$. Then $We_{M,a}$ is contained in $We^p_{\mu,M,a}$.*

Proof. By virtue of Lemma 7.3 of [2], $h_\mu(We_{M,a}) \subset We^{M^\times,1/a}$. Let $\phi \in We^{M^\times,1/a}$. Since $h_\mu = h_\mu^{-1}$ on *Se*, our result will be established when we see that $h_\mu(\phi)$ is in $We^p_{\mu,M,a}$.

Note first that according to Corollary 4.8 of [1], $h_\mu\phi$ is in *Se*.

Let $k \in \mathbb{N}$. By invoking Lemma 5.4-1 of [9] we can obtain that

$$\Delta_\mu^k h_\mu(\phi)(x) = (-1)^k h_\mu(z^{2k}\phi(z))(x)\ ,\quad x\in I\ . \tag{3}$$

A procedure similar to the one developed in the proof of Lemma 6.1 of [2] allows us to write, for every $x > 1$ and $\tau > 0$,

$$\Delta_\mu^k h_\mu(\phi)(x) = \frac{1}{2}\int_{-\infty}^\infty (x(\sigma+i\tau))^{-\mu}H_\mu^{(1)}(x(\sigma+i\tau))\phi(\sigma+i\tau)(\sigma+i\tau)^{2\mu+2k+1}d\sigma,$$

where $H_\mu^{(1)}$ denotes the Hankel function ([8], p. 73).

Now for every $x > 1$ and $\tau > 0$ we divide the last integral as follows:

$$\int_{-\infty}^{\infty} (x(\sigma+i\tau))^{-\mu} H_\mu^{(1)}(x(\sigma+i\tau))\phi(\sigma+i\tau)(\sigma+i\tau)^{2\mu+2k+1} d\sigma$$

$$= \left(\int_{|x(\sigma+i\tau)|\le 1} + \int_{|x(\sigma+i\tau)|>1}\right) (x(\sigma+i\tau))^{-\mu} H_\mu^{(1)} \cdot \cdot (x(\sigma+i\tau))\phi(\sigma+i\tau)(\sigma+i\tau)^{2\mu+2k+1} d\sigma \,.$$

We will analyze each of the integrals separately.

Assume first that $\mu \ge 1/2$. On the one hand, by using (5.3.c) of [2], we get for every $n \in \mathbb{N} - \{0\}$

$$\begin{aligned} &\int_{|x(\sigma+i\tau)|\le 1} \left|(x(\sigma+i\tau))^{-\mu} H_\mu^{(1)}(x(\sigma+i\tau))\phi(\sigma+i\tau)(\sigma+i\tau)^{2\mu+2k+1}\right| d\sigma \\ &\le C\exp(-x\tau)\int_{-\infty}^{\infty} |\phi(\sigma+i\tau)| d\sigma \\ &\le C\exp\Big(-x\tau + M^{\times}\Big[\frac{1}{a}(1+\frac{1}{n})\tau\Big]\Big), x > 1 \text{ and } \tau > 0; \end{aligned} \tag{4}$$

on the other hand, by using again (5.3.c) of [2], for every $n \in \mathbb{N} - \{0\}$

$$\begin{aligned} &\int_{|x(\sigma+i\tau)|>1} \left|(x(\sigma+i\tau))^{-\mu} H_\mu^{(1)}(x(\sigma+i\tau))\phi(\sigma+i\tau)(\sigma+i\tau)^{2\mu+2k+1}\right| d\sigma \\ &\le C\exp(-x\tau)\int_{-\infty}^{\infty} |\phi(\sigma+i\tau)(\sigma+i\tau)^{2\mu+2k+1}| d\sigma \\ &\le C\exp\Big(-x\tau + M^{\times}\Big[\frac{1}{a}(1+\frac{1}{n})\tau\Big]\Big) \,,\; x > 1 \text{ and } \tau > 0 \,. \end{aligned} \tag{5}$$

For fixed $n \in \mathbb{N} - \{0\}$ we choose $\tau > 0$ such that

$$M^{\times\prime}\Big(\frac{1}{a}(1+\frac{1}{n})\tau\Big) = \frac{ax}{(1+1/n)}.$$

Then from Lemma 2.4 of [2] we have

$$-x\tau + M^{\times}\Big(\frac{1}{a}(1+\frac{1}{n})\tau\Big) = -M\left(\frac{ax}{(1+1/n)}\right) \,. \tag{6}$$

Hence by combining (4), (5) and (6) it follows that

$$\left|\Delta_\mu^k h_\mu(\phi)(x)\right| \le C\exp\Big(-M\Big[ax(1-\frac{1}{n+1})\Big]\Big) \,,\; x > 1 \text{ and } n \in \mathbb{N} \,.$$

Note also that, if $-1/2 < \mu < 1/2$, by invoking (5.3.d) of [2] one has

$$\left|\Delta_\mu^k h_\mu(\phi)(x)\right| \le C\exp(-x\tau)\int_{-\infty}^{\infty} \left|\phi(\sigma+i\tau)(\sigma+i\tau)^{\mu+2k+1/2}\right| d\sigma, \tau > 0 \text{ and } x > 1.$$

Proceeding as above, we conclude that

$$\left|\Delta_\mu^k h_\mu(\phi)(x)\right| \le C\exp\Big(-M\Big[ax(1-\frac{1}{m})\Big]\Big) \,,\; x > 1 \text{ and } m \in \mathbb{N} - \{0\} \,.$$

Now let $x \in (0,1)$ and $m \in \mathbb{N} - \{0\}$. According to (5.3.b) of [2] we have

$$\Big|\exp\Big(M\Big[ax(1-\frac{1}{m})\Big]\Big)\Delta_\mu^k[h_\mu(\phi)(x)]\Big| = \Big|\exp\Big(M\Big[ax(1-\frac{1}{m})\Big]\Big)h_\mu(z^{2k}\phi(z))(x)\Big|$$

$$\leq C\int_0^\infty \sigma^{2\mu+2k+1}|\phi(\sigma)|d\sigma$$

because M is an increasing function on $(0,\infty)$.

Hence, for every $k \in \mathbb{N}$ and $m \in \mathbb{N} - \{0\}$,

$$\Big|\exp\Big(M\Big[ax(1-\frac{1}{m})\Big]\Big)\Delta_\mu^k h_\mu(\phi)(x)\Big| \leq C\ ,\ x > 0\ ,$$

and, if $m \in \mathbb{N} - \{0\}$, $k \in \mathbb{N}$ and $1 \leq p < \infty$, then

$$\Big\{\int_0^\infty \Big|\exp\Big(M\Big[ax(1-\frac{1}{m})\Big]\Big)\Delta_\mu^k h_\mu(\phi)(x)\Big|^p dx\Big\}^{1/p} \leq C$$

because $\displaystyle\int_0^\infty \exp\Big(-pM\Big[ax(\frac{1}{m}-\frac{1}{m+1})\Big]\Big)dx < \infty$.

Thus we establish that $h_\mu\phi \in We^p_{\mu,M,a}$, $1 \leq p \leq \infty$, and the proof is finished. □

From Lemmas 2.1 and 2.2 we deduce

Theorem 2.1. *For every* $1 \leq p \leq \infty$ *and* $\mu > -1/2$, $We^p_{\mu,M,a} = We_{M,a}$.

Lemma 2.3. *Let* $1 \leq p \leq \infty$. *Then* $We^{p,\Omega,b}$ *is contained in* $We^{\Omega,b}$.

Proof. Let ϕ be in $We^{p,\Omega,b}$. Assume that $\mu > -1/2$. Proceeding as in the proof of Lemma 2.2, we can establish that for every $k \in \mathbb{N}$ there exists $l = l(k)$ such that

$$|\Delta_\mu^k h_\mu(\phi)(x)| \leq C\exp(-x\tau)\int_{-\infty}^\infty |\phi(\sigma+i\tau)|(|\sigma+i\tau|^l+1)d\sigma\ ,\ \tau, x \in (0,\infty)\ .$$

Hence, according to Hölder's inequality and (6), we obtain for each $k \in \mathbb{N}$, $m \in \mathbb{N} - \{0\}$ and suitable $\tau > 0$

$$\exp\Big(\Omega^\times\Big[\frac{1}{b}(1-\frac{1}{m})x\Big]\Big)|\Delta_\mu^k h_\mu(\phi)(x)|$$

$$\leq C\exp\Big(\Omega^\times\Big[\frac{1}{b}(1-\frac{1}{m})x\Big] - \Omega^\times\Big[\frac{1}{b}(1-\frac{1}{m+1})x\Big]\Big)\Big\{\int_{-\infty}^\infty \frac{d\sigma}{(1+\sigma^2)^{p'}}\Big\}^{1/p'}$$

$$\cdot\Big\{\int_{-\infty}^\infty\Big(\exp\Big[-\Omega[b(1+\frac{1}{m})\tau]\Big](|\sigma+i\tau|+1)(|\sigma+i\tau|^l+1)|\phi(\sigma+i\tau)|\Big)^p d\sigma\Big\}^{1/p}$$

$$\leq C\ ,\ x \in (0,\infty)\ ,$$

provided that $1 < p < \infty$. When $p = 1$ or $p = \infty$ we can proceed in a similar way. Thus we prove that $h_\mu(\phi) \in We^\infty_{\mu,\Omega^\times,1/b}$. Therefore Theorem 2.1 shows that $h_\mu(\phi) \in We_{\Omega^\times,1/b}$.

Since $h_\mu = h_\mu^{-1}$ on Se, it is sufficient to take into account Lemma 7.3 of [2] to see that $\phi \in We^{\Omega,b}$, and the proof of this lemma is complete. □

The next result is not hard to see.

Lemma 2.4. *Let* $1 \le p \le \infty$. *Then* $We^{\Omega,b}$ *is contained in* $We^{p,\Omega,b}$.

As an immediate consequence of Lemmas 2.3 and 2.4 we obtain the following

Theorem 2.2. *Let* $1 \le p \le \infty$. *Then* $We^{p,\Omega,b} = We^{\Omega,b}$.

Lemma 2.5. *Let* $1 \le p \le \infty$. *Then* $We^{p,\Omega,b}_{M,a}$ *is contained in* $We^{\Omega,b}_{M,a}$.

Proof. Let ϕ be in $We^{p,\Omega,b}_{M,a}$. Choose $\mu \ge 1/2$. Since $h_\mu = h_\mu^{-1}$ on *Se*, by virtue of Lemma 7.5 of [2], to prove this lemma it is sufficient to see that $h_\mu\phi$ is in $We^{M^\times,1/a}_{\Omega^\times,1/b}$. The Hankel transformation $h_\mu\phi$ of ϕ is in *Se* (Corollary 4.8 [1]). Moreover, proceeding as in the proof of Lemma 2.1, we can see that $h_\mu\phi$ can be holomorphically extended to the whole complex plane.

Let $\tau > 0$. An argument similar to the one developed in Lemma 6.1 of [2] allows us to write

$$(h_\mu\phi)(x) = \frac{1}{2}\int_{-\infty}^{\infty}(x(\sigma+i\tau))^{-\mu}H_\mu^{(1)}(x(\sigma+i\tau))\phi(\sigma+i\tau)(\sigma+i\tau)^{2\mu+1}d\sigma, |x|>1.$$

As in the proof of Lemma 2.2,

$$(h_\mu\phi)(x) = \frac{1}{2}\left(\int_{|x(\sigma+i\tau)|\le 1} + \int_{|x(\sigma+i\tau)|>1}\right)(x(\sigma+i\tau))^{-\mu}H_\mu^{(1)}(x(\sigma+i\tau)) \cdot \phi(\sigma+i\tau)(\sigma+i\tau)^{2\mu+1}d\sigma\ ,\ \ |x|>1\ .$$

We must analyze each of the two integrals.

According to (5.3.c) of [2] we have, for every $n, m \in \mathbb{N} - \{0\}$,

$$\begin{aligned}
&\int_{|x(\sigma+i\tau)|>1}\Big|(x(\sigma+i\tau))^{-\mu}H_\mu^{(1)}(x(\sigma+i\tau))\phi(\sigma+i\tau)(\sigma+i\tau)^{2\mu+1}\Big|d\sigma\\
&\le C|x|^{-\mu-1/2}\int_{-\infty}^{\infty}\exp\Big(-(\Re x)\tau-(\Im x)\sigma\Big)\Big|\phi(\sigma+i\tau)(\sigma+i\tau)^{\mu+1/2}\Big|d\sigma\\
&\le C|x|^{-\mu-1/2}\\
&\quad\cdot\left\{\int_{-\infty}^{\infty}\left(\exp\Big[-(\Re x)\tau+|\Im x||\sigma|-M\Big(a(1-\frac{1}{n})\sigma\Big)+\Omega\Big(b(1+\frac{1}{m})\tau\Big)\Big]\right.\right.\\
&\qquad\qquad\left.\left.\cdot\,|\sigma+i\tau|^{\mu+1/2}\right)^{p'}d\sigma\right\}^{1/p'},
\end{aligned}$$

where $|x|>1$, provided that $1<p<\infty$. By Lemma 2.4 of [2],

$$|\Im x||\sigma| \le M^\times\left(\frac{|\Im x|}{a(1-1/l)}\right)+M\Big(a(1-\frac{1}{l})|\sigma|\Big)\ ,\sigma\in\mathbb{R}, x\in\mathbb{C} \text{ and } l\in\mathbb{N}, l>1\ .$$

Then

$$|\Im x||\sigma| - M\Big(a(1-\frac{1}{n})|\sigma|\Big) \le M^\times\left(\frac{|\Im x|}{a(1-1/l)}\right) - M\Big(a(\frac{1}{l}-\frac{1}{n})|\sigma|\Big)\ ,$$

where $\sigma\in\mathbb{R},\ x\in\mathbb{C}$ and $l,n\in\mathbb{N}\ ,\ n>l>1$.

We assume now that $\Re x > 0$, and we choose $\tau > 0$ such that

$$\Omega'\Big(b(1+\frac{1}{m})\tau\Big) = \frac{\Re x}{b(1+1/m)}.$$

Then, again by Lemma 2.4 of [2],

$$\tau \Re x = \Omega\Big(b(1+\frac{1}{m})\tau\Big) + \Omega^{\times}\left(\frac{\Re x}{b(1+1/m)}\right) .$$

Hence, since $-\mu - \dfrac{1}{2} \leq 0$ and $1 < p < \infty$, we obtain for every $|x| \geq 1$ and $\Re x > 0$

$$\begin{aligned}
&\int_{|x(\sigma+i\tau)|>1} \Big|(x(\sigma+i\tau))^{-\mu} H_\mu^{(1)}(x(\sigma+i\tau))\phi(\sigma+i\tau)(\sigma+i\tau)^{2\mu+1}\Big| d\sigma \\
&\quad \leq C \exp\Big[M^{\times}\Big(\frac{|\Im x|}{a(1-1/l)}\Big) - \Omega^{\times}\Big(\frac{\Re x}{b(1+1/m)}\Big)\Big] \\
&\quad\quad \cdot \left(\int_{-\infty}^{\infty}\Big(\exp\Big[-M\Big(a(\frac{1}{l}-\frac{1}{n})|\sigma|\Big)\Big]|\sigma+i\tau|^{\mu+1/2}\Big)^{p'} d\sigma\right)^{1/p'} \\
&\quad \leq C \exp\Big[M^{\times}\Big(\frac{|\Im x|}{a(1-1/l)}\Big) - \Omega^{\times}\Big(\frac{\Re x}{b(1+1/m)}\Big)\Big] \; n,m,l \in \mathbb{N}-\{0\}, 1<l<n ,
\end{aligned} \tag{7}$$

because $\displaystyle\int_{-\infty}^{\infty}\Big(\exp\Big[-M\Big(a(\frac{1}{l}-\frac{1}{n})|\sigma|\Big)\Big]|\sigma+i\tau|^{\mu+1/2}\Big)^{p'} d\sigma < \infty$.

If $p = 1$ or $p = \infty$, we can proceed in a similar way.

On the other hand, by (5.3.c) of [2]

$$\begin{aligned}
&\int_{|x(\sigma+i\tau)|\leq 1} \Big|(x(\sigma+i\tau))^{-\mu} H_\mu^{(1)}(x(\sigma+i\tau))\phi(\sigma+i\tau)(\sigma+i\tau)^{2\mu+1}\Big| d\sigma \\
&\quad \leq C|x|^{-2\mu}\int_{-\infty}^{\infty}\exp\Big(-(\Re x)\tau + |\Im x||\sigma|\Big)\Big|\phi(\sigma+i\tau)(\sigma+i\tau)\Big| d\sigma \\
&\quad \leq C\exp\Big[M^{\times}\Big(\frac{|\Im x|}{a(1-1/l)}\Big) - \Omega^{\times}\Big(\frac{\Re x}{b(1+1/m)}\Big)\Big], |x| \geq 1 \text{ and } \Re x > 0,
\end{aligned} \tag{8}$$

for $m, l \in \mathbb{N} - \{0\}, 1 < l$.

Hence from (7) and (8) we conclude that

$$|h_\mu\phi(x)| \leq C\exp\Big[M^{\times}\Big(\frac{1}{a}[1+\frac{1}{l-1}]|\Im x|\Big) - \Omega^{\times}\Big(\frac{1}{b}[1-\frac{1}{m+1}]\Re x\Big)\Big] \tag{9}$$

for every $|x| \geq 1$ and $\Re x > 0$, $m, l \in \mathbb{N}$, where $1 < l$.

Since $h_\mu\phi$ is even, the corresponding inequality (9) also holds when $\Re x < 0$.

Now let $|x| < 1$. By using (5.3.b) of [2] we deduce that

$$|h_\mu\phi(x)| \leq C\int_0^\infty \exp(t|\Im x|)|\phi(t)|t^{2\mu+1}dt .$$

Proceeding as in the above case, we conclude that $h_\mu\phi \in We_{\Omega^{\times},1/b}^{M^{\times},1/a}$. □

The following result can be proved without difficulty.

Lemma 2.6. *Let $1 \leq p \leq \infty$. Then $We_{M,a}^{\Omega,b}$ is contained in $We_{M,a}^{p,\Omega,b}$.*

From Lemmas 2.5 and 2.6 we obtain

Theorem 2.3. *Let* $1 \leq p \leq \infty$. *Then* $We^{p,\Omega,b}_{M,a} = We^{\Omega,b}_{M,a}$.

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