PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 5, May 1998, Pages 1415–1423
S 0002-9939(98)04227-0

ON THE GROWTH OF POLYNOMIALS

D. DRYANOV AND Q. I. RAHMAN

(Communicated by Albert Baernstein II)

Dedicated to the memory of Professor Paul Erdös

Abstract. Let f be a polynomial of degree n having only real zeros and none in $(-1,1)$. We look for a sharp upper bound for $|f(z)|$ at an arbitrary point of the complex plane $\mathbb{C}$ in terms of the supremum norm on $[-1,1]$.

1. Introduction

Let $\mathcal{P}_n$ be the class of all polynomials of degree at most n. As usual, we denote by T_n the n-th Chebyshev polynomial of the first kind. According to a classical result of P.L. Chebyshev (see [5] or [6]), if $f \in \mathcal{P}_n$ and $|f(x)| \le 1$ for $-1 \le x \le 1$, then

$$|f(x)| \le |T_n(x)| \quad \text{for} \quad x \in \mathbb{R} \setminus [-1,1]\,. \tag{1}$$

In (1) equality holds at some point $x_0 \in \mathbb{R} \setminus [-1,1]$ if and only if $f(z) \equiv e^{i\gamma} T_n(z)$ for some real γ. It was noted by S. Bernstein [2] that if $f(z)$ is real for real z, then

$$|f(z)| \le |T_n(z)| \quad \text{for} \quad |z| \ge 1\,, \tag{2}$$

even if $|f|$ is bounded by 1 only at the $n+1$ points $\cos \frac{\nu\pi}{n}$, $\nu = 0, 1, \ldots, n$, which are the extrema of T_n in $[-1,1]$. Bernstein's paper went unnoticed, and his result was rediscovered by P. Erdös [4].

The polynomial T_n is extremal for several other problems. For example, it was proved by A. Markov (see [5] or [6]) that if $f \in \mathcal{P}_n$ and $|f(x)| \le 1$ for $-1 \le x \le 1$, then

$$\max_{-1 \le x \le 1} |f'(x)| \le n^2 = \max_{-1 \le x \le 1} |T'{}_n(x)|\,. \tag{3}$$

All the zeros of T_n are real and lie in the open interval $(-1,1)$. This suggests that the above inequalities can be considerably improved if the zeros of f are all real but none of them lies in $(-1,1)$. As regards (3), P. Erdös [3]; see in particular the second half of p. 311 proved the following result.

Theorem A. *Let $f \in \mathcal{P}_n$ be such that $|f(x)| \le 1$ for $-1 \le x \le 1$. If the zeros of f are all real and lie on $\mathbb{R} \setminus (-1,1)$, then*

$$\max_{-1 \le x \le 1} |f'(x)| \le \frac{1}{2}\left(1 - \frac{1}{n}\right)^{-n+1} n\,. \tag{4}$$

Received by the editors October 16, 1996.
1991 *Mathematics Subject Classification.* Primary 30A10, 30C10, 30D15, 41A17.

The inequality becomes an equality for polynomials of the form

$$\frac{e^{i\gamma}n^n(1+x)^{n-1}(1-x)}{2^n(n-1)^{n-1}}\,,\ \frac{e^{i\gamma}n^n(1-x)^{n-1}(1+x)}{2^n(n-1)^{n-1}},\ \textit{where } \gamma\in\mathbb{R}\,.$$

Here we obtain a result which may be seen as an analogue of (2) for polynomials with zeros restricted as in Theorem A.

For each positive integer n and $k = 0, 1, \ldots, n$ let $\eta_{n,k} := -1 + \frac{2k}{n}$. Denote by $\mathcal{P}_{n,\mathbb{R},1}$ the family of all polynomials f in $\mathcal{P}_n$ which have only real zeros, none of which lies in $(-1, 1)$, and satisfy $|f(x)| \le 1$ for all $x \in F_n := \{\eta_{n,k} : 0 \le k \le n\}$. Furthermore, for $k = 0, 1, \ldots, n$, let

$$q_{n,k}(z) := \frac{n^n}{2^n k^k (n-k)^{n-k}}(1+z)^k(1-z)^{n-k}\,.$$

Note that $\max_{-1\le x\le 1}|q_{n,k}(x)| = q_{n,k}(\eta_{n,k}) = 1$ for each k. We prove

Theorem 1. *Let Ω be the union of the open disks*

$$\left\{z\in\mathbb{C} : \left|z\pm\frac{i}{\sqrt{3}}\right| < \frac{2}{\sqrt{3}}\right\},$$

and $\mathbb{S} := \mathbb{C}\setminus\Omega$. If $f\in\mathcal{P}_{n,\mathbb{R},1}$, then for all $z\in\mathbb{S}$

$$|f(z)| \le \max_{0\le k\le n}|q_{n,k}(z)|\,. \tag{5}$$

Remark 1. Our proof of the theorem will show that except when $z = \pm i\sqrt{3}$ and $n = 1$, strict inequality holds in (5) unless $f(z) \equiv e^{i\gamma}\,q_{n,k}(z)$ for some k and $\gamma\in\mathbb{R}$. If $n = 1$, then $|f(\pm i\sqrt{3})|$ is also maximized by each constant of modulus 1.

2. Proof of Theorem 1

There is nothing to prove for $z = \pm 1$. Besides, for reasons of symmetry, it is enough to prove (5) for all

$$z\in\mathbb{E} := \mathbb{S}\cap\{x+iy : x\ge 0\}\setminus\{1\}\,.$$

The result is obtained in several steps.

Step 1. Given any point $\zeta\in\mathbb{E}$, let $\sigma_\zeta := \sup_{f\in\mathcal{P}_{n,\mathbb{R},1}}|f(\zeta)|$. There exists a polynomial $g\in\mathcal{P}_{n,\mathbb{R},1}$ such that $|g(\zeta)| = \sigma_\zeta$. For this we observe that if

$$\psi(z) := \prod_{k=0}^{n}(z-\eta_{n,k})\,,$$

then for all $z\in\mathbb{C}$ we have

$$f(z) = \sum_{k=0}^{n} f(\eta_{n,k})\frac{\psi(z)}{\psi'(\eta_{n,k})(z-\eta_{n,k})}\,.$$

From this it follows that the polynomials in $\mathcal{P}_{n,\mathbb{R},1}$ are uniformly bounded on every compact subset of $\mathbb{C}$, and so they form a normal family [1, p.216]. Consequently, there exists a polynomial $g\in\mathcal{P}_n$ such that $|g(\eta_{n,k})| \le 1$ for $k = 0, 1, \ldots, n$ and $|g(\zeta)| = \sigma_\zeta$. Obviously, g cannot be identically zero. So by a well-known theorem of Hurwitz [1, p.176] it must belong to $\mathcal{P}_{n,\mathbb{R},1}$. We call such a polynomial **extremal**.

Step 2. Let

$$\mathcal{E}_\zeta \;:=\; \{g \in \mathcal{P}_{n,\mathbb{R},1} \;:\; |g(\zeta)| = \sigma_\zeta\}\;.$$

It is clear that if $g \in \mathcal{E}_\zeta$, then $\max_{1\le k\le n} |g(\eta_{n,k})|$ must be equal to 1. Besides, *it must be of degree n unless $n = 1$ and $\zeta = i\sqrt{3}$*. Indeed, if a polynomial $g \in \mathcal{P}_{n,\mathbb{R},1}$ is of degree $n - j < n$, then

$$g_+(z) \;:=\; \left(\frac{1+z}{2}\right)^j g(z)$$

also belongs to $\mathcal{P}_{n,\mathbb{R},1}$, and $|g_+(\zeta)| > |g(\zeta)|$ if $|\zeta + 1| > 2$. In particular, $|g_+(\zeta)| > |g(\zeta)|$ for all $\zeta \in \mathbb{E}$, $\zeta \neq i\sqrt{3}$, and so g cannot be extremal for any $\zeta \in \mathbb{E}$ except possibly for $\zeta = i\sqrt{3}$. Now let $n \ge 2$ and suppose that a polynomial $g \in \mathcal{P}_{n,\mathbb{R},1}$ of degree $n - j < n$ is extremal for $\zeta = i\sqrt{3}$. Without loss of generality we may assume g to be positive on $(-1, 1)$. Since $f(z) := 1 - z^2$ belongs to $\mathcal{P}_{n,\mathbb{R},1}$ for $n \ge 2$ and $|f(i\sqrt{3})| = 4 > 1$, it follows that $g(z)$ cannot be identically equal to 1, i.e. g cannot be of degree 0. Next we note that g cannot attain its maximum on the set $\{\eta_{n,k} : k = 0, 1, \ldots, n\}$ at both -1 and $+1$. If it did, it would have at least three critical points between the largest zero in $(-\infty, -1]$ and the smallest zero in $[1, \infty)$. But that is not possible since the zeros of g are all real. Hence if $g_-(z) := ((1 - z)/2)^j\, g(z)$, then either $\max_{0\le k\le n} |g_+(\eta_{n,k})| < 1$ or $\max_{0\le k\le n} |g_-(\eta_{n,k})| < 1$. Since $|g_+(i\sqrt{3})| = |g_-(i\sqrt{3})| = |g(i\sqrt{3})|$, we see that $g \notin \mathcal{E}_{i\sqrt{3}}$.

Step 3. In order to prove the theorem *we need to show that if $g \in \mathcal{E}_\zeta$ and $g(\alpha) = 0$, then α is either $+1$ or -1*. First of all we show that α cannot lie in $(-\infty, -1)$.

Let us agree to denote by A, B, P and X the points of the complex plane which represent the numbers $-1, +1, \zeta$ and α, respectively. Assume that $\alpha \in (-\infty, -1)$. It is easily seen that in this situation

$$\left|\frac{1-\alpha}{2}\,\frac{z+1}{z-\alpha}\right| \le 1 \quad \text{for} \;\; z \in [-1, 1]\,.$$

Furthermore,

$$\left|\frac{1-\alpha}{2}\,\frac{\zeta+1}{\zeta-\alpha}\right| = \left|\frac{1-\alpha}{\zeta-\alpha}\right|\left|\frac{\zeta+1}{2}\right| = \frac{\sin\widehat{XPB}}{\sin\widehat{XBP}}\,\frac{\sin\widehat{ABP}}{\sin\widehat{APB}} = \frac{\sin\widehat{XPB}}{\sin\widehat{APB}} \;>\; 1\,.$$

Hence, the polynomial

$$g_1(z) \;:=\; \frac{1-\alpha}{2}\,\frac{z+1}{z-\alpha}\,g(z)$$

belongs to $\mathcal{P}_{n,\mathbb{R},1}$, but $|g_1(\zeta)| > |g(\zeta)|$, contradicting the assumption that $g \in \mathcal{E}_\zeta$.

It remains to show that if $g \in \mathcal{E}_\zeta$ and $g(\alpha) = 0$, then α cannot lie in $(1, \infty)$.

Step 4. For each $\varphi \in (0, \frac{\pi}{2})$, we denote by $\mathcal{L}_\varphi$ the line

$$z \;=\; -1 + t\,e^{i\varphi} \;\;(-\infty \;<\; t \;<\; \infty\,)$$

and by $\mathbb{H}_\varphi$ that half-plane bounded by $\mathcal{L}_\varphi$ which contains the infinite interval $[-1, \infty)$. Let

$$t_1 = t_1(\varphi) \;:=\; \inf\left\{t \;:\; -1 + t\,e^{i\varphi} \;\in\; \mathbb{E}\right\} \quad (0 < \varphi < \frac{\pi}{2})$$

and consider the half-lines

$$\mathcal{L}^+_\varphi \;:\; -1 + t\,e^{i\varphi} \;\;(t_1 \;\le\; t \;<\; \infty)\,.$$

Take any point P on $\mathcal{L}_\varphi^+$, i.e. P represents $\zeta = \zeta(t) = -1 + te^{i\varphi}$, where $t \geq t_1$. Take another point Q on the same half-line such that P is an interior point of the line segment AQ.

Before we go on, we need to take a point C on the positive real axis and draw three half-lines Λ_1, Λ_2, Λ_3 contained in $\mathbb{H}_\varphi$ with P as initial point. The half-lines are drawn so that Λ_1 makes with $\vec{PB}$ an angle equal to $\widehat{APB}$, Λ_2 makes with $\vec{PB}$ an angle equal to $\widehat{PBC}$, and Λ_3 makes with $\vec{PQ}$ an angle equal to $\widehat{APB}$.

At this stage the reader will find it useful to note that the chord AB subtends an angle $\frac{\pi}{3}$ at each point of the circle $\left|z - \frac{i}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$ which lies in the upper half-plane. It should be clear that except when $\zeta = i\sqrt{3}$, the half-line Λ_1 intersects the positive real axis. Denote the point of intersection by $L(\zeta)$, or by L for brevity. We may say that $L(i\sqrt{3}) = \infty$. Let $a = a(\zeta)$ be the distance of L from the origin. Thus $a(\zeta) < +\infty$ unless $\zeta = i\sqrt{3}$.

The half-line Λ_2 intersects the positive real axis if and only if $\widehat{APB} + \widehat{PBC} < \pi - \widehat{PAB}$, which is equivalent to

$$\widehat{APB} + \widehat{PAB} < \frac{\pi}{2}$$

since $\widehat{PBC} = \widehat{PAB} + \widehat{APB}$. Thus Λ_2 intersects the positive real axis if and only if $\zeta \in \mathbb{E} \cap \{x + iy : x > 1, y > 0\}$. Denote the point of intersection by $M(\zeta)$, or by M if there is no ambiguity. We say that M is the point at infinity for all $\zeta \in \mathbb{E} \setminus \{x + iy : x > 1, y > 0\}$. Let $b = b(\zeta)$ be the distance of M from the origin. Note that $b\left(-1 + t\,e^{i\varphi}\right) < +\infty$ for each $t \geq t_1$ when $0 < \varphi < \frac{\pi}{6}$, and only for $t > 2\sec\varphi$ when $\varphi \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right)$.

The half-line Λ_3 intersects the positive real axis if and only if $\widehat{APB} > \widehat{PAB}$. This cannot be the case if P lies on $\mathcal{L}_\varphi^+$ for any $\varphi \in \left[\frac{\pi}{3}, \frac{\pi}{2}\right)$. If $\varphi \in \left(0, \frac{\pi}{3}\right)$ and $\zeta = -1 + t\,e^{i\varphi}$, then $\widehat{APB} > \widehat{PAB}$ if and only if $t_1(\varphi) = \frac{2}{\sqrt{3}}\sin\varphi + 2\cos\varphi \leq t < 4\cos\varphi$. If Λ_3 intersects the positive real axis, we denote the point of intersection by $N(\zeta)$ or simply by N. Let $c = c(\zeta)$ be the distance of N from the origin.

Step 5. We need to compare the quantities $a(\zeta)$, $b(\zeta)$ and $c(\zeta)$. Note that $c(\zeta) = +\infty$ if $\zeta \in \mathcal{L}_\varphi^+$, where $\frac{\pi}{3} \leq \varphi < \frac{\pi}{2}$, and also when $\zeta = \zeta(t) = -1 + t\,e^{i\varphi}$, where $\varphi \in \left(0, \frac{\pi}{3}\right)$ but $t \geq 4\cos\varphi$. For each $\varphi \in \left(0, \frac{\pi}{3}\right)$ the quantity $c\left(-1 + t\,e^{i\varphi}\right)$ increases as t increases from $t_1(\varphi)$ to $4\cos\varphi$.

It follows that

$$b(\zeta) \leq c(\zeta)$$

except *possibly* when ζ is of the form $-1 + t\,e^{i\varphi}$, where $0 < \varphi < \frac{\pi}{3}$ and $t_1 \leq t < 4\cos\varphi$.

Since $b(\zeta) = +\infty$ for $\zeta \in \mathbb{E} \setminus \{x + iy : x > 1, y > 0\}$, we trivially have

$$c(\zeta) < b(\zeta)$$

if $\zeta = -1 + t\,e^{i\varphi}$, where $\frac{\pi}{4} \leq \varphi < \frac{\pi}{3}$ and $t_1(\varphi) \leq t < 4\cos\varphi$. At least one of $c(\zeta)$ and $b(\zeta)$ will be $+\infty$ for $\varphi \in \left[\frac{\pi}{4}, \frac{\pi}{3}\right)$, because $2\sec\varphi \geq 4\cos\varphi$ for such values of φ.

For each $\varphi \in \left(0, \frac{\pi}{4}\right)$ there exists a number

$$t_2(\varphi) \in \left(\max\{t_1(\varphi), 2\sec\varphi\}, 4\cos\varphi\right)$$

such that

$$c\left(-1+t_2(\varphi)e^{i\varphi}\right)=b\left(-1+t_2(\varphi)e^{i\varphi}\right),$$

whereas $c\left(-1+t\,e^{i\varphi}\right)\,<\,b\left(-1+t\,e^{i\varphi}\right)$ or $c\left(-1+t\,e^{i\varphi}\right)>b\left(-1+t\,e^{i\varphi}\right)$ according as $t_1\,(\varphi)\le t<t_2\,(\varphi)$ or $t\,>\,t_2\,(\varphi)$, respectively. Indeed, for $\zeta=-1+t\,e^{i\varphi}\in\mathcal{L}_\varphi^+$ the angles $\widehat{APB}$, $\widehat{PBC}$ decrease as t increases; so there is one and only one $\zeta\in\mathcal{L}_\varphi^+$ corresponding to which $\widehat{APB}\,+\,\widehat{PBC}\,=\,\pi-\widehat{APB}$, and this occurs for $\frac{\pi}{3}<\widehat{PBC}<\frac{\pi}{2}$. For such ζ the point $M(\zeta)$ coincides with $N(\zeta)$; $N(\zeta)$ lies to the left of $M(\zeta)$ or to the right of $M(\zeta)$ according as $\widehat{APB}\,+\,\widehat{PBC}\,>\,\pi-\widehat{APB}$ or $\widehat{APB}\,+\,\widehat{PBC}\,<\,\pi-\widehat{APB}$, respectively.

Since $\widehat{PBC}\,=\,\widehat{APB}\,+\,\widehat{PAB}\,>\,\widehat{APB}$, it follows that $a(\zeta)\,<\,b(\zeta)$ for all $\zeta\in\mathbb{E}\setminus\left\{i\sqrt{3}\right\}$, whereas $a(i\sqrt{3})\,=\,b(i\sqrt{3})=+\infty$. If $\zeta=-1+t\,e^{i\varphi}$, where $0\,<\,\varphi\,<\,\frac{\pi}{2}$ and $t\,>\,t_1\,(\varphi)$, then $\widehat{APB}\,<\,\frac{\pi}{3}$; so $\widehat{APL}\,<\,\frac{2\pi}{3}$, whereas $\widehat{APN}\,>\,\frac{2\pi}{3}$, which implies that $a(\zeta)\,<\,c(\zeta)$. The same can be said if $t=t_1\,(\varphi)$ and $\varphi\in\left(\frac{\pi}{3},\frac{\pi}{2}\right)$. However, if $t=t_1\,(\varphi)$ and $\varphi\in\left(0,\frac{\pi}{3}\right)$, then $a(\zeta)\,=\,c(\zeta)\,<\,+\infty$, and for $\varphi\,=\,\frac{\pi}{3}$ we have $a(i\sqrt{3})\,=\,c(i\sqrt{3})\,=\,+\infty$.

The above observations about $a(\zeta)$, $b(\zeta)$ *and* $c(\zeta)$ *allow us to conclude that the interval* $(1,+\infty)$ *can be the same as* $(1,c(\zeta))$. *But in the case* $c(\zeta)\,<\,+\infty$, *we can express* $(1,+\infty)$ *as* $(1,c(\zeta)]\cup[a(\zeta),b(\zeta))\cup[b(\zeta),+\infty)$ *if* $b(\zeta)\,<\,+\infty$ *and as* $(1,c(\zeta)]\cup[a(\zeta),b(\zeta))$ *if* $b(\zeta)\,=\,+\infty$.

Step 6. We are finally ready to prove that *if* $g\in\mathcal{E}_\zeta$ *and* $g(\alpha)=0$, *then* α *cannot lie in* $(1,+\infty)$.

I. Let $\alpha\in(1,c]$ if $c<+\infty$; otherwise let $\alpha\in(1,+\infty)$. It is clear that

$$\left|\frac{z-1}{z-\alpha}\frac{\alpha+1}{2}\right|\,\le\,1\quad\text{for}\quad z\in[-1,1]\,.$$

Since

$$\left|\frac{\zeta-1}{2}\frac{\alpha+1}{\zeta-\alpha}\right|\,=\,\frac{\sin\widehat{PAB}}{\sin\widehat{APB}}\frac{\sin\widehat{APX}}{\sin\widehat{PAX}}=\frac{\sin\widehat{APX}}{\sin\widehat{APB}}\,,$$

and

$$\widehat{APB}\,<\,\widehat{APX}\,\le\,\widehat{APN}\,=\,\pi-\widehat{APB},$$

it follows that $\sin\widehat{APB}\,\le\,\sin\widehat{APX}$, i.e. $\left|\frac{\zeta-1}{\zeta-\alpha}\frac{\alpha+1}{2}\right|\,\ge\,1$, where the inequalities are strict unless X coincides with $N(\zeta)$. Hence the polynomial

$$g_2(z)\,:=\,\frac{z-1}{z-\alpha}\frac{\alpha+1}{2}\,g(z)$$

belongs to $\mathcal{P}_{n,\mathbb{R},1}$ and $|g_2(\zeta)|\,\ge\,|g(\zeta)|$, where the inequality is strict unless $c(\zeta)<+\infty$ and $\alpha\,=\,c(\zeta)$. So the assumption $g\in\mathcal{E}_\zeta$ is contradicted except in such a situation. Since $c(i\sqrt{3})=+\infty$, we may hereafter assume $\zeta\neq i\sqrt{3}$.

II. Now let $\alpha\in[a,b)$. There exists a point X_1 in $(-\infty,-1)$ such that $\widehat{X_1PB}=\widehat{BPX}$. Denote by $-\alpha_1$ the corresponding real number. The function

$$g_3(z)\,:=\,\frac{z+\alpha_1}{z-\alpha}\frac{\alpha-1}{\alpha_1+1}\,g(z)$$

is easily seen to belong to $\mathcal{P}_{n,\mathbb{R},1}$. Besides,

$$\left|\frac{\zeta+\alpha_1}{1+\alpha_1}\right|\left|\frac{\alpha-1}{\zeta-\alpha}\right| = \frac{\sin\widehat{X_1BP}}{\sin\widehat{X_1PB}}\frac{\sin\widehat{BPX}}{\sin\widehat{PBX}},$$

i.e. $|g_3(\zeta)| = |g(\zeta)|$. Thus the function

$$h_3(z) := \frac{z+1}{2}\frac{1+\alpha_1}{z+\alpha_1}\,g_3(z)$$

belongs to $\mathcal{P}_{n,\mathbb{R},1}$ and

$$|h_3(z)| = \left|\frac{\zeta+1}{2}\right|\left|\frac{1+\alpha_1}{\zeta+\alpha_1}\right|\,|g(\zeta)| = \frac{\sin\widehat{ABP}}{\sin\widehat{APB}}\frac{\sin\widehat{X_1PB}}{\sin\widehat{X_1BP}}\,|g(\zeta)| \geq |g(\zeta)|,$$

where the inequality is strict unless $\alpha = a(\zeta)$. So, we get a contradiction except when $\alpha = a(\zeta)$.

III. Next, let $b < +\infty$ and $\alpha \geq b$. The polynomial

$$g_4(z) := \frac{\alpha-1}{z-\alpha}\,g(z)$$

belongs to $\mathcal{P}_{n-1,\mathbb{R},1}$, and

$$|g_4(\zeta)| = \frac{\sin\widehat{BPX}}{\sin\widehat{PBX}}\,|g(\zeta)| \geq |g(\zeta)|\,,$$

where the inequality is strict unless $\alpha = b(\zeta)$. But there is really no extremal polynomial of degree less than n, since $\zeta \neq i\sqrt{3}$. This is a contradiction.

Summarizing the above conclusions, we see that we have got a contradiction except when $c(\zeta) < +\infty$ and $\alpha = c(\zeta)$ or when $\alpha = a(\zeta)$. Comparing parts **I**, **II** and **III**, we see that $\alpha = c(\zeta)$ and $\alpha = a(\zeta)$ are covered if $a(\zeta) < c(\zeta)$. All that remains is the case $\alpha = a(\zeta) = c(\zeta) < +\infty$. But this occurs only when $\zeta = \zeta_1 := -1+t_1\,e^{i\varphi}$, where $0 < \varphi < \frac{\pi}{3}$, i.e. ζ lies on the circle $\left|z-\frac{i}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$. In that case we introduce the polynomial

$$g_5(z) := \frac{a(\zeta)+1}{z-a(\zeta)}\frac{z-1}{2}\,g(z).$$

First we note that g must vanish at -1. If not, the polynomial $g(-z)$ would have all its zeros in $(-\infty,-1]$, and $|g(-\zeta)|$ would be larger than $|g(\zeta)|$, contrary to the assumption that $g \in \mathcal{E}_\zeta$. Next we note that $\frac{a(\zeta)+1}{x-a(\zeta)}\frac{x-1}{2}$ decreases from 1 to 0 as x increases from -1 to $+1$. This means that $\max_{x\in F_n}|g(x)|$ and $\max_{x\in F_n}\left|\frac{a(\zeta)+1}{x-a(\zeta)}\frac{x-1}{2}\right|$ are not attained at the same point of F_n. Hence

$$\max_{x\in F_n}|g_5(x)| = \mu\max_{x\in F_n}|g(x)|\,,$$

where $\mu \in (0,1)$. We get a contradiction with the fact that if $g_5 \in \mathcal{E}_\zeta$ then $\max\{|g_5(\eta_{n,k})| : k = 0,1,\dots,n\}$ must be equal to 1.

It remains to consider points $\zeta \in (1,+\infty)$. Clearly $g(\zeta)$ cannot be zero. So we have to consider two possibilities, namely, $\alpha \in (\zeta,+\infty)$ and $\alpha \in (1,\zeta)$. In the first case, we may consider $g_4(z) := \frac{\alpha-1}{\alpha-z}\,g(z)$ to see that g cannot belong to $\mathcal{E}_\zeta$ if $g(\alpha) = 0$. In the second case, $g_2(z) = \frac{1-z}{\alpha-z}\frac{\alpha+1}{2}\,g(z)$ shows the same.

3. Determination of $\max_{0\le k\le n} |q_{n,k}(z)|$ for a given z

First let $Re(z) \ge 0$, i.e. $\left|\frac{1+z}{1-z}\right| \ge 1$. Then, for $\left[\frac{n+1}{2}\right] \le k \le n$, we have $|q_{n,k}(z)/q_{n,n-k}(z)| \ge 1$. Hence

$$\Theta(z) := \max_{0\le k\le n} |q_{n,k}(z)| = \max_{\left[\frac{n+1}{2}\right]\le k\le n} |q_{n,k}(z)| .$$

For $0 \le t \le n$, let

$$\rho(t) := \frac{(t+1)^{t+1}(n-t-1)^{n-t-1}}{t^t(n-t)^{n-t}},$$

where, as usual, $0^0 = 1$. Examining its logaritmic derivative, we see that $\rho(t)$ is strictly increasing on $[0, n-1]$. Hence

$$\rho\left(\left[\frac{n-1}{2}\right]\right) < \rho\left(\left[\frac{n+1}{2}\right]\right) < \cdots < \rho\,(n-1)\,. \tag{7}$$

Furthermore, it is easily checked that

$$\rho\left(\left[\frac{n-1}{2}\right]\right) < 1 < \rho\left(\left[\frac{n+1}{2}\right]\right) \quad \text{if } n \text{ is even}, \tag{8}$$

whereas

$$\rho\left(\left[\frac{n-1}{2}\right]\right) = 1 < \rho\left(\left[\frac{n+1}{2}\right]\right) \quad \text{if } n \text{ is odd} . \tag{9}$$

It is clear that if $z \ne 1$ and $w(z) := \frac{1+z}{1-z}$, then

$$|q_{n,k+1}(z)\,/\,q_{n,k}(z)| \ge 1 \quad \text{for } 0 \le k \le n-1 \tag{10}$$

if and only if

$$|w(z)| = \left|\frac{1+z}{1-z}\right| \ge \rho(k)\,, \tag{11}$$

where equality holds in (10) if and only if it does in (11). Hence for $\left[\frac{n-1}{2}\right] \le k \le n-1$,

$$|q_{n,k+1}(z)| \ge |q_{n,k}(z)| \quad \text{if and only if } \rho(k) \le |w(z)| < +\infty\,.$$

Besides, $|q_{n,k+1}(z)| = |q_{n,k}(z)|$ only if $|w(z)| = \rho(k)$. Because of (7), it follows that for each integer j such that $n \ge j \ge \left[\frac{n-1}{2}\right]$, we have

$$|q_{n,j}(z)| \ge |q_{n,k}(z)| \quad \text{for all } k \le j-1 \text{ if } \rho(j-1) \le |w(z)|\,,$$

where $|q_{n,j}(z)| = |q_{n,k}(z)|$ only if $k = j-1$ and $|w(z)| = \rho(j-1)$. In addition, for $n > j \ge \left[\frac{n-1}{2}\right]$,

$$|q_{n,j}(z)| \ge |q_{n,k}(z)| \quad \text{for } j < k \le n \quad \text{if } |w(z)| \le \rho(j)\,,$$

where $|q_{n,j}(z)| = |q_{n,k}(z)|$ only if $k = j+1$ and $|w(z)| = \rho(j)$. Thus, setting $\rho(n) = +\infty$, we see that for each integer j such that $n \ge j \ge \left[\frac{n+1}{2}\right]$,

$$\max_{0\le k\le n,\, k\ne j} |q_{n,k}(z)| < |q_{n,j}(z)| \quad \text{if } \rho(j-1) < |w(z)| < \rho(j)\,.$$

It may be added that if $|w(z)| = \rho(j-1)$, then $|q_{n,j}(z)| = |q_{n,j-1}(z)|$. Thus, for any given z belonging to the closed right half-plane, $\Theta(z)$ is attained by $|q_{n,j}(z)|$ *alone* if $\rho(j-1) < |w(z)| < \rho(j)$; it is also attained by $|q_{n,j-1}(z)|$ if $|w(z)| = \rho(j-1)$. Each point z belonging to the closed right half-plane $\mathbb{H}^+$ is covered, since for each such z there exists, in view of (8) and (9), an integer j in $\left[\left[\frac{n+1}{2}\right], n\right]$ such that

$\rho(j-1) \le |w(z)| = \left|\frac{1+z}{1-z}\right| < \rho(j)$. To determine $\Theta(z)$ for points belonging to the left half-plane, it suffices to observe that $\Theta(z) = \Theta(-z)$ for all z.

For $\left[\frac{n+1}{2}\right] \le k \le n-1$ let

$$c_k := \frac{(\rho(k))^2+1}{(\rho(k))^2-1}, \quad r_k := \frac{2\rho(k)}{(\rho(k))^2-1}.$$

It is easily checked that $c_k - r_k$ increases with k, whereas $c_k + r_k$ decreases. So if $\mathbb{D}_k := \{z \in \mathbb{C} : |z-c_k| < r_k\}$, then

$$\mathbb{D}_l \supset \bar{\mathbb{D}}_m \quad \text{if} \quad \left[\frac{n+1}{2}\right] \le l < m \le n-1.$$

Now note that $|w(z)| \ge \rho(k)$ for some integer k such that $\left[\frac{n+1}{2}\right] \le k \le n-1$ if and only if $z \in \bar{\mathbb{D}}_k$. We therefore have the following:

Theorem 2. *Let $\Theta(z), \mathbb{H}^+$ and Ω_k be as above. If $z \in \bar{\mathbb{D}}_{n-1}$, then $\Theta(z) = |q_{n,n}(z)|$; if $z \in \bar{\mathbb{D}}_{n-2} \setminus \mathbb{D}_{n-1}$, then $\Theta(z) = |q_{n,n-1}(z)|$. More generally, if $z \in \bar{\mathbb{D}}_k \setminus \mathbb{D}_{k+1}$ for some k such that $\left[\frac{n+1}{2}\right] \le k \le n-2$, then $\Theta(z) = |q_{n,k+1}(z)|$. If $z \in \mathbb{H}^+ \setminus \mathbb{D}_{\left[\frac{n+1}{2}\right]}$, then $\Theta(z) = \left|q_{n,\left[\frac{n+1}{2}\right]}(z)\right|$.*

Remark 2. It may be noted that if $|z-c_k| = r_k$ for some k such that $\left[\frac{n+1}{2}\right] \le k \le n-1$, then $\Theta(z) = |q_{n,k+1}(z)| = |q_{n,k}(z)|$. Furthermore, if n is odd and $Re(z) = 0$, then $\Theta(z) = \left|q_{n,\left[\frac{n+1}{2}\right]}(z)\right| = \left|q_{n,\left[\frac{n-1}{2}\right]}(z)\right|$.

4. Some additional remarks

Remark 3. Even if we assume $|f(x)|$ to be bounded by 1 for all $x \in [-1,1]$, it is clearly not possible to improve upon (5). But for (5) to remain true for all $z \in \mathbb{S}$, do we have to assume that $|f(\eta_{n,k})| \le 1$ for $0 \le k \le n$? The answer is yes. In fact, if we require $|f(x)|$ to be bounded above by 1 on any closed subset F of $[-1,1]$ which does not contain one of the above $n+1$ points, say $\eta_{n,j}$, then (5) will fail at least for all $z \in \mathbb{C}$, where $\Theta(z) = |q_{n,j}(z)|$. This is because there exists $\delta > 0$ such that $|(1+\delta)q_{n,j}(x)| \le 1$ for all $x \in F$.

Remark 4. What can we say about $|f(z)|$ when $z \notin \mathbb{S}$, i.e. $z \in \Omega$? First of all we wish to point out that $\mathbb{S}$ *is independent of* n. *The answer to the question depends on* n. Note that inequality (5) does not hold for any z belonging to the intersection Δ of the two disks $|1+z| < 2$ and $|1-z| < 2$ in case $n = 1$. This follows from the definition of Δ. Now consider the polynomial $f(z) := \frac{1}{2}(1+z)(2-z)$, which satisfies the conditions of Theorem 1 in case $n = 2$. Comparing $|f(1+iy)|$ with $|q_{2,0}(1+iy)|$, $|q_{2,1}(1+iy)|$ and $|q_{2,2}(1+iy)|$, we see that

$$|f(z)| > \max_{0 \le k \le 2} |q_{2,k}(z)|$$

if $z = 1+iy$, $0 < |y| < \frac{1}{\sqrt{3}}$. Note that these points lie in $\Omega \setminus \Delta$. On the other hand, not only for $n = 2$ but for all even n, we have $|f(iy)| = |f(0)| \prod_{\nu=1}^{n} |1 - iyx_\nu|$, where $-1 \le x_\nu \le 1$ for $\nu = 1, \ldots, n$. Hence,

$$|f(iy)| \le |f(0)|\,(1+y^2)^{\frac{n}{2}} = \left|q_{n,\frac{n}{2}}(iy)\right|,$$

i.e. (5) holds at all points of the imaginary axis in case n is even.

Remark 5. Although a polynomial f of degree n having only real zeros none of which lies in $(-1,1)$ is not completely determined by the value it takes at any point α in $(-1,1)$, its modulus at any point $z \in \mathbb{C}$ can be estimated in terms of n and $|f(\alpha)|$. To see this note that $f(z) = f(0)\prod_{\nu=1}^{n}(1-zx_\nu)$, where $-1 \le x_\nu \le 1$ for $\nu = 1, \ldots, n$. For all z,

$$|1 - zx_\nu| \le \max\{|1+z|, |1-z|\}.$$

Hence, if $|f(0)| \le 1$, then

$$|f(z)| \le |f(0)| \max\{|1+z|^n, |1-z|^n\} = \begin{cases} |1+z|^n & \text{if } \ Re(z) \ge 0, \\ |1-z|^n & \text{if } \ Re(z) \le 0. \end{cases} \tag{12}$$

If $|f(\alpha)| \le 1$, where $\alpha \in (-1,0) \cup (0,1)$, then

$$F(w) := (\alpha w + 1)^n f\left(\frac{w+\alpha}{\alpha w + 1}\right)$$

is a polynomial of degree at most n having only real zeros none of which lies in $(-1,1)$. Furthermore, $|F(0)| = |f(\alpha)| \le 1$, and so by (12)

$$|\alpha w + 1|^n f\left(\frac{w+\alpha}{\alpha w+1}\right) = |F(w)| \le \max\{|1+w|^n, |1-w|^n\}.$$

Replacing $\frac{w+\alpha}{\alpha w+1}$ by z, we easily conclude that if $|f(\alpha)| \le 1$, where $\alpha \in (-1,0) \cup (0,1)$, then for all $z \in \mathbb{C}$,

$$f(z) \le \begin{cases} \left|\frac{1+z}{1+\alpha}\right|^n & \text{if } \ \left|z - \frac{1+\alpha^2}{2\alpha}\right| \le \frac{1-\alpha^2}{2\alpha}, \\ \\ \left|\frac{1-z}{1-\alpha}\right|^n & \text{if } \ \left|z - \frac{1+\alpha^2}{2\alpha}\right| \ge \frac{1-\alpha^2}{2\alpha}. \end{cases}$$

References

1. L.V. Ahlfors, *Complex Analysis*, 2nd ed., McGraw-Hill Book Company, New York, 1966. MR **32:**5844
2. S.N. Bernstein, *Sur une propriété des polynômes*, Comm. Soc. Math. Kharkow Sér. 2 **14** (1913), pp. 1–6.
3. P. Erdös, *On extremal properties of the derivatives of polynomials*, Ann. of Math. **41** (1940), pp. 310–313. MR **1:**323g
4. P. Erdös, *Some remarks on polynomials*, Bull. Amer. Math. Soc. **53** (1947), pp. 1169–1176. MR **9:**281g
5. I.P. Natanson, *Constructive Function Theory*, vol. **I**, Frederick Ungar Publishing Co., Inc., New York, 1964. MR **33:**4529a
6. T.J. Rivlin, *Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory*, 2nd ed., Wiley, New York, 1990. MR **92a:**41016

Department of Mathematics, University of Sofia, James Boucher 5, 1126 Sofia, Bulgaria

Current address: Département de Mathématiques et de Statistique, Université de Montréal, Montréal, Canada H3C 3J7

E-mail address: dryanovd@ere.UMontreal.CA

Département de Mathématiques et de Statistique, Université de Montréal, Montréal, Canada H3C 3J7

E-mail address: rahmanqi@ere.UMontreal.CA