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WHEN MUST PURE EXTENSIONS OF COUNTABLE ABELIAN GROUPS NECESSARILY SPLIT?

H. PATRICK GOETERS AND PATRICK KEEF

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Abstract. Necessary and sufficient conditions are given for the group of pure extensions of a countable abelian group by a countable abelian group to equal zero.

Introduction

In this work, by the term "group" we will mean "abelian group". Any notation and terminology not specifically defined can be found in [1].

If A and B are groups, then a short exact sequence $0 \to B \to X \to A \to 0$ is *pure* if for every integer $n, nB = B \cap nX$. The collection of pure extensions is denoted by $\mathrm{Pext}(A,B)$. This can be shown to agree with the first Ulm subgroup, $\mathrm{Ext}(A,B)^1 = \bigcap n\,\mathrm{Ext}(A,B)$. Many different properties of this bifunctor are established in Chapter IX of [1], and we will assume the results contained there.

We will be interested in describing when $\mathrm{Pext}(A,B) = 0$, where A and B are countable groups. This question was suggested to the first author by C. Schochet, who noted its importance to subjects of KK-theory (see [2]). In particular, the graded Kasparov group $KK_*(M,N)$ of certain C^*-algebras M and N possesses a direct factor of the form $\mathrm{Pext}(K_*(M), K_*(N))$. It is important to know when this latter group vanishes, and it should also be noted that every countable abelian group is $K_0(M)$ for some separable C^*-algebra M. In addition, as also noted in [2], by results of Jensen, if $M = \varinjlim M_i$, then

$$\mathrm{Pext}(M,N) = \varprojlim{}^1 \mathrm{Hom}(M_i, N),$$

so that the collection of pure extensions is of some importance to topologists.

Our main result is Theorem 1, where the above problem is completely solved. It is perhaps surprising that the question of when $\mathrm{Pext}(A,B)$ vanishes can be separated into two distinct cases; when B is torsion and when it is torsion-free (Corollary 2). In addition, we show that whenever A and B are countable, then $\mathrm{Pext}(A,B)$ is either 0 or has cardinality c (Corollary 3).

If A is a group, then tA will denote the torsion subgroup of A and $fA \overset{\mathrm{def}}{=} A/tA$. More generally, if $\mathcal{P}$ is a collection of primes, then $t_{\mathcal{P}}A$ will consist of those elements of A whose order is a product of elements of $\mathcal{P}$, and $t^*_{\mathcal{P}}A$ will consist of those elements whose order is not divisible by any element of $\mathcal{P}$. We let $f_{\mathcal{P}}A = A/t_{\mathcal{P}}A$

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and $f^*_{\mathcal{P}}A = A/t^*_{\mathcal{P}}A$. The integers localized at $\mathcal{P}$ will be denoted by $\mathbf{Z}_{(\mathcal{P})}$. A group A will be said to be $\mathcal{P}$-*free* if $A_{(\mathcal{P})} \stackrel{\text{def}}{=} A \otimes \mathbf{Z}_{(\mathcal{P})}$ is a free $\mathbf{Z}_{(\mathcal{P})}$-module, and $\mathcal{P}$-Σ-*cyclic* if $A_{(\mathcal{P})}$ is a direct sum of cyclic $\mathbf{Z}_{(\mathcal{P})}$-modules. If $\mathcal{P} = \{p\}$ is a single prime, we drop the parentheses in all of the above notation and terminology, e.g., $t_pA = t_{\{p\}}A$.

There are several standard identities and properties of localizations that we will utilize without comment. For example, if A is a group, then there is a natural homomorphism $A \cong A \otimes \mathbf{Z} \to A_{(\mathcal{P})}$, whose kernel and cokernel are torsion groups with no p torsion for any $p \in \mathcal{P}$. Another property which we use is the existence of a natural isomorphism $(fA)_{(\mathcal{P})} \cong fA_{(\mathcal{P})}$. Finally, if $\mathcal{P}$ and $\mathcal{Q}$ are collections of primes, $\mathcal{P} \cup \mathcal{Q} = \mathcal{R}$, and A is a group with finite torsion-free rank, then A is $\mathcal{R}$-free (or $\mathcal{R}$-Σ-cyclic) iff it is both $\mathcal{P}$- and $\mathcal{Q}$-free (respectively, $\mathcal{P}$- and $\mathcal{Q}$-Σ-cyclic).

The $\mathbf{Z}$-adic completion of a group A will be denoted by LA, and if p is a prime, the p-adic completion will be denoted by L_pA. There are natural homomorphisms $A \to LA$, $A \to L_pA$ and $LA \to L_pA$, which determine a natural isomorphism $LA \cong \prod_p L_pA$.

Preliminaries

If A and B are countable, and $0 \to Q \to P \to A \to 0$ is a free resolution of A, then there is a surjection $\mathrm{Hom}(Q, B) \to \mathrm{Ext}(A, B)$. Since $\mathrm{Hom}(Q, B)$ is the direct product of at most a countable number of copies of B, $\mathrm{Hom}(Q, B)$ has cardinality at most c, and so the same can be said of $\mathrm{Ext}(A, B)$. In many of our arguments, our fundamental strategy is to show that $\mathrm{Pext}(A, B)$ is non-zero by showing that its cardinality is at least c. This will, in turn, imply that its cardinality is exactly c. In other words, though we will primarily be concerned with when $\mathrm{Pext}(A, B) = 0$, our arguments actually show that this group either vanishes or it has cardinality c.

Lemma 1. *Suppose A and B are countable groups and B is torsion-free. Let*

$$\mathcal{P} = \{p : pB \neq B\}.$$

Then $\mathrm{Pext}(A, B) = 0$ *iff fA is $\mathcal{P}$-free.*

Proof. If X is a torsion group, then since B is torsion-free, we have $\mathrm{Hom}(X, B) = 0$. If, in addition, $X = t^*_{\mathcal{P}}X$, then $\mathrm{Ext}(X, B)$ is p-divisible for every $p \in \mathcal{P}$ (because multiplication by p is an automorphism of X) and also p-divisible for every $p \notin \mathcal{P}$ (since multiplication by p is an automorphism of B). This means that $\mathrm{Ext}(X, B)$ is divisible; but since X is torsion, it must also be reduced (see 55.3 of [1]), proving that it is, in fact, zero.

The kernel and cokernel of the homomorphism $A \to A_{(\mathcal{P})}$ are torsion groups without p-torsion for any $p \in \mathcal{P}$. It follows from the last paragraph that the homomorphism $\mathrm{Ext}(A, B) \to \mathrm{Ext}(A_{(\mathcal{P})}, B)$ is an isomorphism, so that we may assume A is a $\mathbf{Z}_{(\mathcal{P})}$-module. If $0 \to B \to X \to A \to 0$ is a short exact sequence, then since multiplication by any prime $q \notin \mathcal{P}$ is an automorphism of A and B, it is also an automorphism of X. It follows that X is also a $\mathbf{Z}_{(\mathcal{P})}$-module, so that the entire computation can be regarded as taking place over the ring $\mathbf{Z}_{(\mathcal{P})}$.

For our next reduction, we begin with the observation that $\mathrm{Pext}(tA, B) = 0$. To verify this, suppose $0 \to B \to X \to tA \to 0$ is a pure short-exact sequence. Purity implies that tX maps onto tA, and since B is torsion-free, this mapping is also injective, so that $tX \cong tA$ and the sequence splits. Considering the sequence

$$\mathrm{Pext}(tA, B) \to \mathrm{Pext}(A, B) \to \mathrm{Pext}(fA, B) \to 0,$$

it follows that $\mathrm{Pext}(A, B) \cong \mathrm{Pext}(fA, B)$, so that we may assume A is torsion-free. This implies that $\mathrm{Ext}(A, B)$ is divisible, so that $\mathrm{Pext}(A, B) = \mathrm{Ext}(A, B)$.

To begin the proof, note that if A is $\mathcal{P}$-free, then clearly $\mathrm{Pext}(A, B) = 0$. Conversely, if A is not $\mathcal{P}$-free, then by Pontryagin's Theorem, it has a finite-rank pure submodule A' which fails to be $\mathcal{P}$-free. Since there is a surjection $\mathrm{Pext}(A, B) \to \mathrm{Pext}(A', B)$, if we can prove $\mathrm{Pext}(A', B) \neq 0$, the result follows. We may assume, therefore, that A has finite torsion-free rank.

Let F be a $\mathcal{P}$-free subgroup of A of maximal rank. There is, therefore, an exact sequence

$$\mathrm{Hom}(F, B) \to \mathrm{Ext}(A/F, B) \to \mathrm{Ext}(A, B).$$

Observe $\mathrm{Hom}(F, B)$ is countable so the result will follow if we can show that $\mathrm{Ext}(A/F, B)$ is uncountable.

Since A is not $\mathcal{P}$-free, A/F must be infinite. There are two possibilities: either A/F has a summand isomorphic to $\mathbf{Z}_{p^\infty}$ for some $p \in \mathcal{P}$, or summands isomorphic to $\mathbf{Z}_{p^{k_p}}$ for an infinite set of primes $p \in \mathcal{P}$. In the first case, $\mathrm{Ext}(A/F, B)$ will have a summand isomorphic to $\mathrm{Ext}(\mathbf{Z}_{p^\infty}, B) \cong \mathrm{Hom}(\mathbf{Z}_{p^\infty}, \mathbf{Z}_{p^\infty} \otimes B)$ which is uncountable, and in the second, it will have a summand isomorphic to $\prod \mathrm{Ext}(\mathbf{Z}_{p^{k_p}}, B) \cong \prod B/p^{k_p}B$, which is also uncountable. In either case, the result has been established.

Lemma 2. *Suppose A and B are countable groups, B is reduced and p is a prime such that t_pB is unbounded. If $\mathrm{Pext}(A, B) = 0$, then A is p-Σ-cyclic.*

Proof. Again, we prove that if A is not p-Σ-cyclic, then $\mathrm{Pext}(A, B)$ has cardinality at least c. By 56.1 of [1] there is a surjection

$$\mathrm{Pext}(A, B) \to \mathrm{Pext}(A, B_0),$$

so that if $\mathrm{Pext}(A, B_0)$ has cardinality c, then so does $\mathrm{Pext}(A, B)$. It follows that we may assume that $B^1 = 0$.

Since $0 \to t_p^*B \to B \to f_p^*B \to 0$ is pure, there is a surjection

$$\mathrm{Pext}(A, B) \to \mathrm{Pext}(A, f_p^*B),$$

so, once again, we may assume $B = f_p^*B$, i.e., $t_p^*B = 0$.

By 57.4 of [1] there is a surjection

$$\mathrm{Pext}(A, B) \to \mathrm{Hom}(A^1, LB/B).$$

Since t_pB is countable and unbounded, it can be verified that $t_p(LB/B)$ is a divisible group of cardinality c. If $p^\omega t_pA \neq 0$, we could conclude that $\mathrm{Hom}(A^1, LB/B)$ has cardinality c, which implies that $\mathrm{Pext}(A, B)$ also has cardinality c. Therefore, we may assume $p^\omega t_pA = 0$; since A is countable, this implies that t_pA is Σ-cyclic.

Since $A_{(p)}$ is not Σ-cyclic (as a $\mathbf{Z}_{(p)}$-module), we can conclude that $fA_{(p)}$ is not Σ-cyclic (since otherwise $A_{(p)} \cong fA_{(p)} \oplus tA_{(p)}$ would be Σ-cyclic). Hence there is a finite-rank pure subgroup $A' \subseteq A$ containing tA such that $A'_{(p)}$ is not Σ-cyclic. There is a surjection $\mathrm{Pext}(A, B) \to \mathrm{Pext}(A', B)$, so we may assume $A = A'$ has finite torsion-free rank.

We pause for one further reduction. Let $N = t_pA \oplus F$ be a p-basic subgroup of A (i.e., N is p-pure in A and A/N is p-divisible), where F is a free subgroup of N. Note that F will also be p-pure in A. We next let $M = \{a \in A : \exists m \in \mathbf{Z}, ma \in F, (m, p) = 1\}$, and $A' = A/M$. We need to verify the following elementary facts:

(1) M/F is a torsion group with $t_p(M/F) = 0$. This is obvious from the definition.

(2) M is pure in A. If $a \in M$ and $ma \in F$ with $(m,p) = 1$, then computing p-heights we have $ht_M(a) = ht_M(ma) \geq ht_F(ma) = ht_A(ma) = ht_A(a) \geq ht_M(a)$. If $q \neq p$, $a \in M$ and $a = q^k y$ for some $y \in A$, then $y \in M$, so that q-heights in M and A also agree.

(3) $tA' = t_pA' \cong t_pA$, under the natural map $A \to A'$. If $a \in A$ and $na \in M$, then $n = mp^k$, where $(m,p) = 1$. It follows that $p^k a \in M$, so that $tA' = t_pA'$. To verify the last isomorphism, since M is pure in A, tA maps surjectively onto tA'. Note that $t_pA \cap M = 0$ and $t_p^*A \subseteq M$, so the isomorphism follows.

(4) fA' is p-divisible. This follows since fA' is an epimorphic image of A/N, which is p-divisible.

(5) $fA' \neq 0$. If $fA' = 0$, then $A \cong M \oplus t_pA$, so that $A_{(p)} \cong M_{(p)} \oplus t_pA \cong F_{(p)} \oplus t_pA$, which is a Σ-cyclic $\mathbf{Z}_{(p)}$-module.

The first fact implies $\mathrm{Hom}(M/F, B) = 0$ so that there is an injection $\mathrm{Hom}(M,B) \to \mathrm{Hom}(F,B)$. Since F has finite rank and B is countable, we can conclude $\mathrm{Hom}(F,B)$ is countable, which implies that $\mathrm{Hom}(M,B)$ is also countable. Considering the exact sequence

$$\mathrm{Hom}(M,B) \to \mathrm{Pext}(A',B) \to \mathrm{Pext}(A,B),$$

we can conclude that if $\mathrm{Pext}(A',B)$ has cardinality c, then so does $\mathrm{Pext}(A,B)$.

We therefore assume $A = A'$, and that $x \in A$ has infinite order. It follows that if $n_0, n_1, n_2, \ldots$ is the p-height sequence of x, then either 1) there is a j such that $n_j \geq \omega$, or 2) there are an infinite number of gaps (i.e., values of i such that $n_{i+1} > n_i + 1$).

Let $m_0, m_1, m_2, \ldots$ be a strictly increasing sequence of finite ordinals with an infinite number of gaps, such that $m_i < n_i$ for all but finitely many i.

Let $B = \{b_1, b_2, \ldots\}$ be an enumeration of B, and let $t_pB = \bigoplus_{u \in S} K_u$ where $S \subseteq \omega$ and for every $u \in S, K_u \neq 0$ is a direct sum of copies of $\mathbf{Z}_{p^u}$.

Since $B^1 = 0$, B embeds as a pure subgroup of $LB \cong L_pB \oplus \prod_{q \neq p} L_qB$ with LB/B divisible. We can think of elements of L_pB as vectors with coordinates in a p-basic subgroup of B, which contains the terms $K_u, u \in S$.

Choose inductively a strictly increasing sequence $u_i \in S$, and elements $v_i \in K_{u_i}$, such that

1) v_i generates a summand of K_{u_i},

2) $m_i < u_i$,

3) for every $j \leq i$, b_j has v_i-coordinate divisible by p^{m_i}.

Let $\{R_\alpha : \alpha < c\}$ be an almost disjoint family of subsets of S. For each $\alpha < c$, let

$$y_\alpha = \sum_{i \in R_\alpha} p^{m_i - i} v_i \in L_pB \subseteq LB$$

[the fact that $y_\alpha \in L_pB$ follows from the fact that $m_0, m_1, m_2, \ldots$ has an infinite number of gaps].

Now, LB/B is divisible and $x \in A$ has infinite order, so for every $\alpha < c$, there is a homomorphism

$$f_\alpha : A \to LB/B$$

such that $f_\alpha(x) = y_\alpha + B$. There is an exact sequence

$$\mathrm{Hom}(A, LB) \to \mathrm{Hom}(A, LB/B) \to \mathrm{Pext}(A, B).$$

We claim that the f_α map to distinct elements of $\mathrm{Pext}(A, B)$, proving that this has cardinality c, as required. If this failed, then for some distinct $\alpha, \beta < c$, $f_\alpha - f_\beta$ would factor through a homomorphism $g : A \to LB$. Since $g(x) + B = f_\alpha(x) - f_\beta(x) = y_\alpha - y_\beta + B$, we can find a $b_j \in B$ such that $g(x) = y_\alpha - y_\beta + b_j$. Choose $i \geq j$ such that $i \in R_\alpha - R_\beta$ and $m_i < n_i$. Then computing p-heights, we have

$$n_i = ht(p^i x) \leq ht(p^i g(x)) = ht(p^i y_\alpha - p^i y_\beta + p^i b_j).$$

Notice that the v_ith coordinate of $p^i b_j$ is divisible by $p^{m_i+i} > p^{m_i}$, $p^i y_\beta$ is 0 in this coordinate, and $p^i y_\alpha$ has p^{m_i} in this coordinate. It follows that the p-height of $p^i y_\alpha - p^i y_\beta + p^i b_j$ is m_i, but this contradicts $m_i < n_i$.

If p is a prime and $n \leq \omega$, then a subgroup L of A is *p^n-pure* if $p^m A \cap L = p^m L$ for every $m \leq n$. Note that if n is finite and L is torsion-free, then this reduces to requiring that $p^n A \cap L = p^n L$. We mention the following easy-to-check fact:

Lemma 3. *Suppose A is a group, J, K are subgroups of A, j and k are integers with $jJ \subseteq K$ and $kK \subseteq J$, p is a prime not dividing j, k and $n \leq \omega$. Then J is p^n-pure iff K is p^n-pure.*

Let $\sigma = (n_p : p \text{ is prime})$ be a sequence of finite ordinals. A subgroup L of A will be called *σ-pure* if for each prime p, L is p^{n_p}-pure in A. Similarly, L is *almost σ-pure* if it is p^{n_p}-pure for all but finitely many primes p.

Corollary 1. *Given the hypotheses of Lemma 3, J is almost σ-pure iff K is almost σ-pure.*

The main result

Theorem 1. *Suppose A and B are countable groups and B is reduced. Let*

$$\mathcal{P} = \{p : pfB \neq fB\},$$
$$\mathcal{Q} = \{p : t_pB \text{ is unbounded}\}.$$

In addition, let $\sigma = (n_p)$, where n_p is the exponent of t_pB if t_pB is bounded and is 0 *whenever t_pB is unbounded. Then* $\mathrm{Pext}(A, B) = 0$ *iff*

(1) *fA is $\mathcal{P}$-free,*
(2) *A is $\mathcal{Q}$-Σ-cyclic, and*
(3) *every finitely generated subgroup of A is almost σ-pure in A.*

Proof. We begin by supposing that (1)–(3) are valid and show that $\mathrm{Pext}(A, B) = 0$. By Lemma 1, (1) implies that $\mathrm{Pext}(A, fB) = 0$. We now show that (2) and (3) imply that $\mathrm{Pext}(A, tB) = 0$, which proves the result by considering

$$\mathrm{Pext}(A, tB) \to \mathrm{Pext}(A, B) \to \mathrm{Pext}(A, fB) \to 0.$$

To this end, we may assume that B is torsion. Since the kernel and cokernel of the natural map $A \to A_{(\mathcal{Q})}$ are torsion groups with no q-torsion for any $q \in \mathcal{Q}$, (2) implies that $\mathrm{Pext}(A, t_{\mathcal{Q}}B) \cong \mathrm{Pext}(A_{(\mathcal{Q})}, t_{\mathcal{Q}}B) = 0$, so we need to show $\mathrm{Pext}(A, t^*_{\mathcal{Q}}B) = 0$. It follows that we may assume $B = t^*_{\mathcal{Q}}B$, i.e., that B is torsion with bounded p-components.

Let

$$0 \longrightarrow B \longrightarrow X \xrightarrow{\pi} A \longrightarrow 0$$

be a pure sequence. Express A as an ascending union $0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ where each F_i is finitely generated. Our objective is to define a splitting $\phi : A \to X$, defining it inductively, extending it from each F_i to F_{i+1}. We will assume one particular property of such a partial splitting $\phi_i : F_i \to X$: If p is a prime, $m \leq n_p$ and $x \in F_i \cap p^m A$, then $\phi_i(x) \in p^m X$ (in other words, ϕ_i preserves p-heights up to n_p).

Since F_{i+1} is almost σ-pure, there is a finite set of primes $\mathcal{T}$ such that for every $p \notin \mathcal{T}$, F_{i+1} is p^{n_p}-pure in A. We can further expand $\mathcal{T}$ by adding a finite collection of primes in order to guarantee that $t^*_{\mathcal{T}}(F_{i+1}/F_i) = 0$. Let $B_0 = t_{\mathcal{T}} B$ and $B_1 = t^*_{\mathcal{T}} B$, so that $B = B_0 \oplus B_1$. Note that B_0 is bounded, so that $X = B_0 \oplus X_1$, and there is a pure sequence

$$0 \longrightarrow B_1 \longrightarrow X_1 \xrightarrow{\pi} A \longrightarrow 0.$$

Let $\rho : X \to B_0$ and $\mu : X \to X_1$ be the usual projections. Note that $\mu \circ \phi_i : F_i \to X_1$ is a partial splitting of π. Since F_{i+1} is finitely generated, by purity there is a partial splitting $\eta : F_{i+1} \to X_1$. It should be observed that we are not assuming η extends ϕ_i or $\mu \circ \phi_i$.

There is an exact sequence

$$\operatorname{Hom}(F_{i+1}, B_1) \to \operatorname{Hom}(F_i, B_1) \to \operatorname{Ext}(F_{i+1}/F_i, B_1)$$

and since $B_1 = t^*_{\mathcal{T}} B_1$ and $t^*_{\mathcal{T}}(F_{i+1}/F_i) = 0$, the right group vanishes. Next, observe that $\mu \circ \phi_i - \eta|_{F_i}$ is a homomorphism $F_i \to B_1$, so that $\mu \circ \phi_i - \eta|_{F_i} = \lambda|_{F_i}$ for some homomorphism $\lambda : F_{i+1} \to B_1$.

The last paragraph shows that $\mu \circ \phi_i$ extends to a partial splitting $\gamma = \eta + \lambda : F_{i+1} \to X_1$. Note that if $p \notin \mathcal{T}$, $m \leq n_p$, and $x \in p^m A \cap F_{i+1}$, then by p^{n_p}-purity, $x \in p^m F_{i+1}$, so that $\gamma(x) \in p^m X_1$, and hence γ preserves p-heights up to n_p for all $p \notin \mathcal{T}$.

Next, consider $\kappa = \rho \circ \phi_i : F_i \to B_0$. Let $n = \prod_{p \in \mathcal{T}} p^{n_p}$, so that $nB_0 = 0$. Note κ induces a homomorphism $\overline{\kappa} : F_i/(F_i \cap nA) \to B_0$ and that $\overline{\kappa}$ does not decrease p-heights (calculated in A/nA) for any $p \in \mathcal{T}$. Therefore, $\overline{\kappa}$ extends to a homomorphism $A/nA \to B_0$. Let κ' be the composition $F_{i+1} \to A \to A/nA \to B_0$, so that κ' preserves p-height up to n_p for each $p \in \mathcal{T}$.

We can now extend ϕ_i to $\phi_{i+1} : F_{i+1} \to X$ by setting it equal to $(\kappa', \gamma) : F_{i+1} \to B_0 \oplus X_1$. This is clearly a splitting; to show that it preserves heights, let p be a prime and $x \in F_{i+1}$. If $p \notin \mathcal{T}$, then $\kappa'(x)$ has infinite p-height, so that the p-height of $\phi_{i+1}(x)$ equals the p-height of $\gamma(x)$ which is at least the p-height of $x \in A$ (whenever this is at most n_p). Similarly, if $p \in \mathcal{T}$, then the p-height of $\phi_{i+1}(x)$ equals the p-height of $\kappa'(x)$ which is at least as large as the p-height of $x \in A$. Therefore, one direction of the proof is complete.

We now show that if (1), (2) or (3) fails, then $\operatorname{Pext}(A, B) \neq 0$. Suppose first that (1) fails. Lemma 1 implies that $\operatorname{Pext}(A, fB) \neq 0$, and since there is a surjection

$$\operatorname{Pext}(A, B) \to \operatorname{Pext}(A, fB) \to 0$$

we can conclude $\operatorname{Pext}(A, B) \neq 0$.

We next assume (3) fails. Suppose F is a finitely generated subgroup of A which is not almost σ-pure. Let k be an integer such that kF is free. By Corollary 1, kF also fails to be almost σ-pure, so there is no loss of generality in assuming that F

is free. Now, let $\mathcal{S}$ be an infinite set of primes such that $p^{n_p}A \cap F \neq p^{n_p}F$ for all $p \in \mathcal{S}$.

Let A' be the pure hull of F (i.e., $A'/F = t(A/F)$). Since A' is pure in A, for each $p \in \mathcal{S}$, F is not p^{n_p}-pure in A'. There is a surjection

$$\mathrm{Pext}(A, B) \to \mathrm{Pext}(A', B) \to 0,$$

so if we can show $\mathrm{Pext}(A', B) \neq 0$, we will be able to conclude $\mathrm{Pext}(A, B) \neq 0$, as required. We may therefore assume $A = A'$ has finite rank and A/F is torsion.

Claim: $\mathrm{Pext}(A, tB)$ has cardinality c.

Before continuing, we note how this implies $\mathrm{Pext}(A, B) \neq 0$. There is an exact sequence

$$\mathrm{Hom}(A, fB) \to \mathrm{Pext}(A, tB) \to \mathrm{Pext}(A, B).$$

Note that $\mathrm{Hom}(A, fB) \cong \mathrm{Hom}(fA, fB)$, and since fA has finite rank and fB has countable rank, this group is countable, and the claim follows.

Therefore, we may assume $B = tB$ is torsion. For each $p \in \mathcal{S}$, let $\langle r_p \rangle$ be a summand of t_pB of exponent n_p. If $B' = \bigoplus_{p \in \mathcal{S}} \langle r_p \rangle$ and $\mathrm{Pext}(A, B')$ has cardinality c, then it follows that $\mathrm{Pext}(A, B)$ also has cardinality c, as required. So we may assume $B = B'$.

For each $p \in \mathcal{S}$, let $m_p \leq n_p$ be minimal such that $p^{m_p}A \cap F \neq p^{m_p}F$, and let $x_p \in (p^{m_p}A \cap F) - p^{m_p}F$. The minimality of m_p implies that x_p has p-height $m_p - 1$ in F (since otherwise $x_p \in (p^{m_p-1}A \cap F) - p^{m_p-1}F$). Let $y_p \in F$ satisfy $p^{m_p-1}y_p = x_p$. Now choose $z_p \in A$ such that $p^{m_p}z_p = x_p$. The minimality of m_p also implies that $p^{m_p-1}z_p \notin F$ (since otherwise $p^{m_p-1}z_p \in (p^{m_p-1}A \cap F) - p^{m_p-1}F$). Let A' be the subgroup of A generated by F together with each z_p ($p \in \mathcal{S}$). There is a surjection

$$\mathrm{Pext}(A, B) \to \mathrm{Pext}(A', B) \to 0$$

[to verify this, if $0 \to Q \to P \xrightarrow{\rho} A \to 0$ is a pure sequence with P and Q Σ-cyclic and $P' = \pi^{-1}(A')$, then there are homomorphisms $\mathrm{Hom}(Q, B) \to \mathrm{Pext}(A, B) \to \mathrm{Pext}(A', B)$ whose composite is a surjection]. If we can show the latter group has cardinality c, it also follows that the first group has that cardinality. So, without loss of generality, we assume $A = A'$.

Let $\{\mathcal{S}_\alpha\}_{\alpha < c}$ be a pairwise almost disjoint family of infinite subsets of $\mathcal{S}$. For each $\alpha < c$, let X_α be generated by

(1.1) F,
(1.2) r_p, z_p, for $p \notin \mathcal{S}_\alpha$,
(1.3) u_p, v_p for $p \in \mathcal{S}_\alpha$,

subject to the relations,

(2.1) $p^{n_p}r_p = 0$, $p^{m_p}z_p = p^{m_p-1}y_p \in F$, for $p \notin \mathcal{S}_\alpha$,
(2.2) $p^{n_p}u_p = p^{n_p}y_p \in F$, $p^{m_p}v_p = p^{m_p-1}u_p$, for $p \in \mathcal{S}_\alpha$.

These generators and relators determine an extension

$$E_\alpha : 0 \longrightarrow B \xrightarrow{\lambda_\alpha} X_\alpha \xrightarrow{\pi_\alpha} A \longrightarrow 0$$

by assigning for each $p \in \mathcal{S}_\alpha$,

(3.1) $\lambda_\alpha(r_p) = u_p - y_p$,
(3.2) $\pi_\alpha(u_p) = y_p$,
(3.3) $\pi_\alpha(v_p) = z_p$.

The sequence E_α is pure; in fact, for every prime $p, 0 \to t_pB \to t_pX_\alpha \to t_pA \to 0$ actually splits. To see this, note that if $p \notin \mathcal{S}_\alpha$, then $t_pX_\alpha = \langle r_p\rangle \oplus \langle pz_p - y_p\rangle$, and if $p \in \mathcal{S}_\alpha$, then $t_pX_\alpha = \langle u_p - y_p\rangle \oplus \langle pv_p - u_p\rangle$. In either case, the first summand corresponds to t_pB and the second to t_pA.

Let $p \in \mathcal{S}$. We claim that F is p^{n_p}-pure in X_α iff $p \in \mathcal{S}_\alpha$. To verify this, note that if $p \notin \mathcal{S}_\alpha$, then $p^{m_p-1}y_p \in (p^{m_p}X_\alpha \cap F) - p^{m_p}F$, so that F is not p^{n_p}-pure in X_α. If $p \in \mathcal{S}_\alpha$, then $F/p^{n_p}F$ can be seen to embed naturally as a summand of $X_\alpha/p^{n_p}X_\alpha$, and the claim follows.

We now claim that for distinct $\alpha, \beta < c$, E_α is not equivalent to E_β, proving that $\mathrm{Pext}(A,B)$ has the required cardinality. In fact, we show that X_α and X_β are not isomorphic. To show this is the case, we assume that they are and derive a contradiction. Given such an isomorphism, we can identify X_α and X_β, giving a single group X. Note that F embeds in both X_α and X_β, but after this identification we may get two distinct subgroups which we denote by F_α and F_β, respectively. There are integers j and k such that $jF_\alpha \subseteq F_\beta$, $kF_\beta \subseteq F_\alpha$. Since $\mathcal{S}_\alpha - \mathcal{S}_\beta$ is infinite, there is a prime $p \in \mathcal{S}_\alpha - \mathcal{S}_\beta$ which does not divide j, k. This means that F_α is p^{n_p}-pure in A, while F_β fails to be p^{n_p}-pure in A, contradicting Lemma 3.

We now suppose that it is (2) that fails. If for some $p \in \mathcal{Q}, A_{(p)}$ fails to be a Σ-cyclic $\mathbf{Z}_{(p)}$-module, then by Lemma 2 we can conclude $\mathrm{Pext}(A,B) \neq 0$, so we may assume each such $A_{(p)}$ is Σ-cyclic. In particular, we may assume $t_{\mathcal{Q}}A = \bigoplus_{p\in\mathcal{Q}} t_pA$ is $\mathcal{Q}$-Σ-cyclic.

There is a sequence

$$0 \to t_{\mathcal{Q}}A \to A_{(\mathcal{Q})} \to (fA)_{(\mathcal{Q})} \to 0,$$

so we may assume fA is not $\mathcal{Q}$-free. This implies that there is a pure subgroup $A' \subseteq A$ of finite torsion-free rank such that fA' is not $\mathcal{Q}$-free. Since there is a surjection

$$\mathrm{Pext}(A,B) \to \mathrm{Pext}(A',B) \to 0,$$

if we can show $\mathrm{Pext}(A',B) \neq 0$, then $\mathrm{Pext}(A,B) \neq 0$, as required. It suffices, then, to assume that $A = A'$ has finite torsion-free rank.

Let F be a free subgroup of A of maximal rank, let $\mathcal{R}$ be the set of primes $p \in \mathcal{Q}$ such that F is p^ω-pure in A, and let $\mathcal{S} = \mathcal{Q} - \mathcal{R}$ be the remaining ones. Note that $F_{(\mathcal{R})}$ will be a pure $\mathbf{Z}_{(\mathcal{R})}$-submodule of $A_{(\mathcal{R})}$. Since $A_{(\mathcal{R})}/F_{(\mathcal{R})}$ is torsion, it follows that $A_{(\mathcal{R})}$ splits into $t_{\mathcal{R}}A \oplus F_{(\mathcal{R})}$, so that $(fA)_{(\mathcal{R})}$ is $\mathcal{R}$-free. Now for each $p \in \mathcal{S}$ we already know $(fA)_{(p)}$ is p-free, so that if $\mathcal{S}$ is finite, we can conclude $(fA)_{(\mathcal{Q})}$ is $\mathcal{Q}$-free, contrary to assumption.

It follows that $\mathcal{S}$ is finite and for each $p \in \mathcal{S}$ there exists a positive integer m_p for which F fails to be p^{m_p}-pure in A. For each $p \in \mathcal{S}$, let $t_pB = K_p \oplus L_p$ where K_p is a bounded subgroup of exponent $n_p \geq m_p$. If $B' = B/(\bigoplus_{p\in\mathcal{S}} L_p)$, then $t_{\mathcal{S}}B' \cong \bigoplus_{p\in\mathcal{S}} K_p$, so that for this A and B', condition (3) fails. By what we've proven already, this implies that $\mathrm{Pext}(A,B') \neq 0$, and since there is a surjection

$$\mathrm{Pext}(A,B) \to \mathrm{Pext}(A,B') \to 0,$$

we conclude that $\mathrm{Pext}(A,B) \neq 0$, as required.

We now point out two consequences of the above proof:

Corollary 2. *If A and B are countable groups, then* $\mathrm{Pext}(A,B) = 0$ *iff* $\mathrm{Pext}(A, fB) = \mathrm{Pext}(A, tB) = 0$.

Proof. We may clearly assume B is reduced. Essentially condition (1) is equivalent to $\mathrm{Pext}(A, fB) = 0$, and conditions (2) and (3) are equivalent to $\mathrm{Pext}(A, tB) = 0$.

Corollary 3. *If A and B are countable groups, then* $\mathrm{Pext}(A, B)$ *is either* 0 *or it has cardinality c.*

References

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Department of Mathematics, Auburn University, Auburn, Alabama 36849

Department of Mathematics, Whitman College, Walla Walla, Washington 99362