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ON THE TOPOLOGICAL BOUNDARY OF SEMI-FREDHOLM OPERATORS

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ABSTRACT. We prove several distance formulas from a fixed operator in $B(H)$ to some classes of operators connected with the semi-Fredholm ones. Here H is a separable Hilbert space. In particular, Fredholm and upper and lower semi-Fredholm operators have the same boundary in $B(H)$.

Let H be a separable Hilbert space and let $B(H)$ be the algebra of all bounded linear oprators on H. For an operator $T \in B(H)$, we will denote by T^*, $R(T)$, $N(T)$ and $\sigma(T)$ its adjoint, range, kernel and spectrum, respectively. Let $K(H)$ be the ideal of compact operators and $C(H) = B(H)/K(H)$ be the Calkin algebra. Denote by $\pi : B(H) \to C(H)$ the canonical projection. Endowed with the essential norm $||T||_e = ||\pi(T)||$, $C(H)$ is a C^*-algebra.

The index of an operator $T \in B(H)$ will be denoted by $\mathrm{ind}(T)$ and is defined by $\mathrm{ind}(T) = \dim N(T) - \dim N(T^*)$, with the convention $\infty - \infty = 0$.

We introduce the following notation for several classes of operators :

- $F_+ = \{T \in B(H) : R(T)$ is closed, $\dim N(T) < \infty\}$ is the set of all upper semi-Fredholm operators.
- $F_- = \{T \in B(H) : R(T)$ is closed, $\dim N(T^*) < \infty\}$ is the set of all lower semi-Fredholm operators.
- $F_\pm = F_+ \cup F_-$ is the set of semi-Fredholm operators.
- $F = F_+ \cap F_-$ is the set of Fredholm operators.
- $I_n = \{T \in B(H) : \mathrm{ind}(T) = n\}$ with $n \in \overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$.
- $F_\pm^n = F_\pm \cap I_n$, with $n \in \overline{\mathbb{Z}}$, the connected component of index n in $F_\pm$.

For a set X in $B(H)$, we will denote by $\mathrm{int}X$, $\overline{X}$ and ∂X the interior, closure and (topological) boundary, respectively.

For a linear operator $T \in B(H)$, we will denote by $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin F\}$ the essential spectrum of T. Let

$$m_e(T) = \inf\{\sigma_e(|T|)\}$$

(cf. [1]), where $|T| = (T^*T)^{1/2}$, and

$$M_e(T) = \max\{m_e(T); m_e(T^*)\}.$$

Using Theorem 1.1 of [4] and Theorem 3.1 of [7], we easily obtain the following result:

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Theorem 1. *Let $T \in B(H)$ and $n \in \overline{\mathbb{Z}}$. Then*

$$\operatorname{dist}(T, F_{\pm}^{n}) = \begin{cases} M_e(T) & \text{if } T \notin I_n, \\ \\ 0 & \text{if not.} \end{cases}$$

We recall the following theorem, which gives a characterization of the boundary of connected components of semi-Fredholm operators.

Theorem 2 ([5], Corollary 1.4). *The boundary of $\overline{F_{\pm}^{j}}$ does not depend on j, and, if $\Delta = \partial\overline{F_{\pm}^{j}}$, then $\Delta = \partial F_{\pm}^{j}$ and $B(H) = \Delta \cup \Big(\bigcup_{j\in\overline{\mathbb{Z}}} F_{\pm}^{j}\Big)$.*

Remarks. 1) We have $\Delta = \{T \in B(H) : M_e(T) = 0\}$. Indeed, using [4], Theorem 2.1 and Corollary 2.2, we obtain

$$\Delta = \partial F_{\pm}^{0} = \partial\overline{G} = \{T \in B(H) : M_e(T) = 0\},$$

where G is the group of invertible operators in $B(H)$.

2) The set Δ is stable with respect to compact perturbations: $\Delta + K(H) = \Delta$.

3) The set Δ is arcwise connected. Indeed, if $0 \in \Delta$ and $T \in \Delta$, then $tT \in \Delta$ for all $t \in [\,0, 1\,]$.

We prove now the following result.

Theorem 3. *Let $T \in B(H)$ and $J \subseteq \overline{\mathbb{Z}}$. Then*

$$\text{(a)} \qquad \operatorname{dist}\Big(T, \bigcup_{j\in J} F_{\pm}^{j}\Big) = \begin{cases} M_e(T) & \text{if } \operatorname{ind}(T) \notin J, \\ \\ 0 & \text{if not,} \end{cases}$$

and

$$\text{(b)} \qquad \operatorname{dist}\Big(T, \bigcup_{j\in J} I_j\Big) = \begin{cases} M_e(T) & \text{if } \operatorname{ind}(T) \notin J, \\ \\ 0 & \text{if not.} \end{cases}$$

Proof. (a) Let $n = \operatorname{ind}(T) \in \overline{\mathbb{Z}}$. If $n \in J$, then $F_{\pm}^{n} \subseteq \bigcup_{j\in J} F_{\pm}^{j}$. Therefore

$$0 \le \operatorname{dist}(T, \bigcup_{j\in J} F_{\pm}^{j}) \le \operatorname{dist}(T, F_{\pm}^{n}).$$

But, using Theorem 1, we obtain $\operatorname{dist}(T, F_{\pm}^{n}) = 0$. Hence, $\operatorname{dist}(T, \bigcup_{j\in J} F_{\pm}^{j}) = 0$.

Suppose now that $n \notin J$ and let $j_0 \in J$. Then, using once again Theorem 1 and the relation $n \neq j_0$, we get

$$\operatorname{dist}\Big(T, \bigcup_{j\in J} F_{\pm}^{j}\Big) \le \operatorname{dist}(T, F_{\pm}^{j_0}) = M_e(T).$$

We now show the converse inequality. For every $S \in \bigcup_{j\in J} F_\pm^j$, we have $\operatorname{ind}(S) \neq n = \operatorname{ind}(T)$. Then Theorem 1.1 of [6] implies $\|T - S\| \geq M_e(T)$. Therefore,

$$\operatorname{dist}\Big(T, \bigcup_{j\in J} F_\pm^j\Big) \geq M_e(T),$$

and (a) is proved.

In order to prove (b), it suffices to remark that $\bigcup_{j\in J} F_\pm^j \subseteq \bigcup_{j\in J} I_j$ and to use (a). Indeed, if $\operatorname{ind}(T) = n \in J$, then, using (a), we obtain

$$0 \leq \operatorname{dist}\Big(T, \bigcup_{j\in J} I_j\Big) \leq \operatorname{dist}\Big(T, \bigcup_{j\in J} F_\pm^j\Big) = 0.$$

On the other hand, if $n \notin J$, then for all $S \in \bigcup_{j\in J} I_j$, we have $\|T - S\| \geq M_e(T)$ (cf. [6], Theorem 1.1). This yields

$$\operatorname{dist}\Big(T, \bigcup_{j\in J} I_j\Big) \geq M_e(T).$$

Using (a) again, we also find that

$$\operatorname{dist}\Big(T, \bigcup_{j\in J} I_j\Big) \leq \operatorname{dist}\Big(T, \bigcup_{j\in J} F_\pm^j\Big) = M_e(T),$$

which completes the proof. ■

As consequences, the results below easily follow. Assertions (b) and (c) of the following corollary are also in [2, Theorems 12 and 13].

Corollary 4. *Let $T \in B(H)$. Then:*

$$\text{(a)} \qquad \operatorname{dist}(T, F) = \begin{cases} M_e(T) & \text{if } \operatorname{ind}(T) = \pm\infty, \\ 0 & \text{if not.} \end{cases}$$

$$\text{(b)} \qquad \operatorname{dist}(T, F_+) = \begin{cases} m_e(T^*) & \text{if } \operatorname{ind}(T) = +\infty, \\ 0 & \text{if } \operatorname{ind}(T) \neq +\infty. \end{cases}$$

$$\text{(c)} \qquad \operatorname{dist}(T, F_-) = \begin{cases} m_e(T) & \text{if } \operatorname{ind}(T) = -\infty, \\ 0 & \text{if } \operatorname{ind}(T) \neq -\infty. \end{cases}$$

Proof. (a) follows from Theorem 3 above with $J = \mathbb{Z}$, while (b) and (c) follow from [2, Lemma 7] and Theorem 3 above with $J = \mathbb{Z} \cup \{-\infty\}$ and $J = \mathbb{Z} \cup \{+\infty\}$, respectively. ■

Now we can prove the following equalities.

Theorem 5. *Let $J \subseteq \overline{\mathbb{Z}}$. Then*

$$\text{(a)} \qquad \overline{\bigcup_{j\in J} F_{\pm}^{j}} = \Delta \cup \Big(\bigcup_{j\in J} F_{\pm}^{j}\Big) = \bigcup_{j\in J} \overline{F_{\pm}^{j}}\ ;$$

$$\text{(b)} \qquad \partial\Big(\bigcup_{j\in J} F_{\pm}^{j}\Big) = \Delta.$$

Proof. (a) We prove the first equality. If $J = \overline{\mathbb{Z}}$, then the result follows from Theorem 2 since $\overline{\bigcup_{j\in J} F_{\pm}^{j}} = B(H)$.

Suppose that $J \neq \overline{\mathbb{Z}}$. Then, using Theorem 2 again, we have

$$B(H) = \Delta \cup \Big(\bigcup_{j\in J} F_{\pm}^{j}\Big) \cup \Big(\bigcup_{j\in \overline{\mathbb{Z}}\setminus J} F_{\pm}^{j}\Big).$$

It follows that $\Delta \cup \Big(\bigcup_{j\in J} F_{\pm}^{j}\Big)$ is closed, being the complement of the open set $\Big(\bigcup_{j\in \overline{\mathbb{Z}}\setminus J} F_{\pm}^{j}\Big)$ in $B(H)$. Therefore

$$\overline{\bigcup_{j\in J} F_{\pm}^{j}} \subseteq \Delta \cup \Big(\bigcup_{j\in J} F_{\pm}^{j}\Big).$$

In order to show the other inclusion, it suffices to see that $\Delta \subset \overline{\bigcup_{j\in J} F_{\pm}^{j}}$ (see Theorem 3).

The second equality follows from

$$\bigcup_{j\in J} \overline{F_{\pm}^{j}} = \bigcup_{j\in J} \Big(\Delta \cup F_{\pm}^{j}\Big).$$

(b) Since $\bigcup_{j\in J} F_{\pm}^{j}$ is open in $B(H)$, we have

$$\partial\Big(\bigcup_{j\in J} F_{\pm}^{j}\Big) = \Big(\overline{\bigcup_{j\in J} F_{\pm}^{j}}\Big) \setminus \Big(\bigcup_{j\in J} F_{\pm}^{j}\Big) = \Big(\Delta \cup \Big(\bigcup_{j\in J} F_{\pm}^{j}\Big)\Big) \setminus \Big(\bigcup_{j\in J} F_{\pm}^{j}\Big) = \Delta.$$

The proof is complete. ■

This easily implies the following consequence:

Corollary 6. *We have*

$$\text{(a)} \quad \overline{F} = F \cup \Delta\ ; \overline{F_+} = F_+ \cup \Delta \quad \textit{and} \quad \overline{F_-} = F_- \cup \Delta\ ;$$

$$\text{(b)} \quad \partial F = \partial F_+ = \partial F_- = \Delta.$$

We also have

Corollary 7. *Let $J \subseteq \overline{\mathbb{Z}}$. Then*

$$\overline{\bigcup_{j\in J} I_j} = \Delta \cup \Big(\bigcup_{j\in J} F_{\pm}^{j}\Big) = \Delta \cup \Big(\bigcup_{j\in J} I_j\Big).$$

Proof. Using Theorems 3 and 5, we obtain

$$\Delta \cup \Big(\bigcup_{j\in J} I_j\Big) \subseteq \overline{\bigcup_{j\in J} I_j} = \overline{\bigcup_{j\in J} F_\pm^j} = \Delta \cup \Big(\bigcup_{j\in J} F_\pm^j\Big) \subseteq \Delta \cup \Big(\bigcup_{j\in J} I_j\Big).$$

■

For the interior and for the boundary of the closure of sets considered in Theorem 5, we have

Theorem 8. *Let $J \subseteq \overline{\mathbb{Z}}$, $J \neq \overline{\mathbb{Z}}$. Then*

$$\text{(a)} \qquad \operatorname{int}\Big(\overline{\bigcup_{j\in J} F_\pm^j}\Big) = \bigcup_{j\in J} F_\pm^j = \operatorname{int}\Big(\bigcup_{j\in J} \overline{F_\pm^j}\Big)\ ;$$

$$\text{(b)} \qquad \partial\Big(\overline{\bigcup_{j\in J} F_\pm^j}\Big) = \partial\Big(\bigcup_{j\in J} \overline{F_\pm^j}\Big) = \Delta\ .$$

Proof. (a) We show the first equality. The inclusion

$$\bigcup_{j\in J} F_\pm^j \subseteq \operatorname{int}\Big(\overline{\bigcup_{j\in J} F_\pm^j}\Big)$$

is clear. In order to show the other one, let $T \in \operatorname{int}\Big(\overline{\bigcup_{j\in J} F_\pm^j}\Big)$. If T is semi-Fredholm, then $T \in \bigcup_{j\in J} F_\pm^j$. Indeed, using Theorem 5, we have

$$\begin{aligned} T \in \operatorname{int}\Big(\overline{\bigcup_{j\in J} F_\pm^j}\Big) &= \operatorname{int}\Big(\Delta \cup \Big[\bigcup_{j\in J} F_\pm^j\Big]\Big) \\ &\subset \Delta \cup \Big(\bigcup_{j\in J} F_\pm^j\Big) \end{aligned}$$

and, because $T \notin \Delta$, we get $T \in \bigcup_{j\in J} F_\pm^j$.

Suppose now that T is not semi-Fredholm. Then $T \in \Delta$. Consider $n \notin J$ (this is always possible since $J \neq \overline{\mathbb{Z}}$). Using Theorem 2, we obtain $T \in \Delta = \partial F_\pm^n$. Therefore

$$T \in \operatorname{int}\Big(\overline{\bigcup_{j\in J} F_\pm^j}\Big) \cap \partial F_\pm^n,$$

which implies the non-voidness of $\Big(\overline{\bigcup_{j\in J} F_\pm^j}\Big) \cap F_\pm^n$. The continuity of the index yields a contradiction.

The second equality follows from Theorem 5.

(b) Using (a) and Theorem 5, we obtain

$$\begin{aligned}\partial\Big(\bigcup_{j\in J}\overline{F_{\pm}^{j}}\Big) &= \partial\Big(\overline{\bigcup_{j\in J}F_{\pm}^{j}}\Big)\\ &= \Big(\overline{\bigcup_{j\in J}F_{\pm}^{j}}\Big)\setminus \operatorname{int}\Big(\overline{\bigcup_{j\in J}F_{\pm}^{j}}\Big)\\ &= \Big(\Delta\cup\Big[\bigcup_{j\in J}F_{\pm}^{j}\Big]\Big)\setminus\Big(\bigcup_{j\in J}F_{\pm}^{j}\Big) = \Delta\,.\end{aligned}$$

The proof is complete. ∎

The following consequence can be easily obtained.

Corollary 9. *We have*

$$\text{(a)}\quad \operatorname{int}(\overline{F}) = F\,;\operatorname{int}(\overline{F_+}) = F_+ \ \ \textit{and}\ \ \operatorname{int}(\overline{F_-}) = F_-\ ;$$

and

$$\text{(b)}\quad \partial\overline{F} = \partial F = \partial\overline{F_+} = \partial F_+ = \partial\overline{F_-} = \partial F_- = \Delta\,.$$

We also have

Corollary 10. *Let $J \subset \overline{\mathbb{Z}}$, $J \neq \overline{\mathbb{Z}}$. Then*

$$\text{(a)}\qquad \operatorname{int}\Big(\overline{\bigcup_{j\in J}I_j}\Big) = \bigcup_{j\in J}F_{\pm}^{j} = \operatorname{int}\Big(\bigcup_{j\in J}I_j\Big)\,;$$

$$\text{(b)}\qquad \partial\Big(\overline{\bigcup_{j\in J}I_j}\Big) = \partial\Big(\bigcup_{j\in J}I_j\Big) = \Delta\,.$$

Proof. (a) The first equality follows from Corollary 7 and Theorems 5 and 8. For the second equality, it is sufficient to note that

$$\bigcup_{j\in J}F_{\pm}^{j} \subseteq \operatorname{int}\Big(\bigcup_{j\in J}I_j\Big) \subseteq \operatorname{int}\Big(\overline{\bigcup_{j\in J}F_{\pm}^{j}}\Big) = \bigcup_{j\in J}F_{\pm}^{j}.$$

Indeed, the first two inclusions are obvious and the last equality is (a).
(b) is a direct consequence of (a) and of Corollary 7. ∎

We introduce the following notation:

- $G_+ = \{T \in B(H) : T$ is left invertible$\}$.
- $G_- = \{T \in B(H) : T$ is right invertible$\}$.
- $G_\pm = G_+ \cup G_-$: the set of one-sided invertible operators.
- $G = G_+ \cap G_-$: the set of invertible operators.

For $n \in \overline{\mathbb{Z}}$, let $G_{\pm}^{n} = G_\pm \cap I_n$. For $n \in \overline{\mathbb{Z}}_- = \mathbb{Z}_- \cup \{-\infty\}$, we denote $G_+^n = G_+ \cap I_n$, while for $n \in \overline{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{+\infty\}$, we set $G_-^n = G_- \cap I_n$.

Theorem 11. *Let $T \in B(H)$ and $J \subseteq \overline{\mathbb{Z}}$. Then*

$$\operatorname{dist}\Big(T, \bigcup_{j\in J}G_{\pm}^{j}\Big) = \begin{cases} M_e(T) & \textit{if } \operatorname{ind}(T) \notin J,\\[2ex] 0 & \textit{if not.}\end{cases}$$

Proof. Let $n = \operatorname{ind}(T)$. If $n \in J$, then $G_\pm^n \subseteq \bigcup_{j\in J} G_\pm^j$, so, using Theorem 3.1 of [7], we have

$$0 \leq \operatorname{dist}\Big(T, \bigcup_{j\in J} G_\pm^j\Big) \leq \operatorname{dist}(T, G_\pm^n) = 0.$$

Suppose now that $n \notin J$. Let $j_0 \in J$. Using Theorem 3.1 of [7], we get

$$0 \leq \operatorname{dist}\Big(T, \bigcup_{j\in J} G_\pm^j\Big) \leq \operatorname{dist}(T, G_\pm^{j_0}) = M_e(T). \tag{1}$$

On the other hand, for all $L \in \bigcup_{j\in J} G_\pm^j$, $\operatorname{ind}(L) \neq n = \operatorname{ind}(T)$. Therefore, using Theorem 1.1 of [6], we have $\|T - L\| \geq M_e(T)$. Thus

$$\operatorname{dist}\Big(T, \bigcup_{j\in J} G_\pm^j\Big) \geq M_e(T). \tag{2}$$

Now (1) and (2) imply the desired equality. ■

We obtain the following consequence.

Corollary 12. *Let $T \in B(H)$, and $J \subseteq \overline{\mathbb{Z}}_+$. Then*

$$\operatorname{dist}\Big(T, \bigcup_{j\in J} G_-^j\Big) = \begin{cases} M_e(T) & \text{if } \operatorname{ind}(T) \notin J, \\ 0 & \text{if } \operatorname{ind}(T) \in J. \end{cases}$$

A similar result can be stated for $\operatorname{dist}(T, \bigcup_{j\in J} G_+^j)$ if $J \subseteq \overline{\mathbb{Z}}_-$. For the distance of T to the set $G_- \setminus G$ we obtain the following formula:

Corollary 13. *Let $T \in B(H)$. Then*

$$\operatorname{dist}(T, G_- \setminus G) = \begin{cases} M_e(T) & \text{if } \operatorname{ind}(T) \leq 0, \\ 0 & \text{if not.} \end{cases}$$

The following result gives a description of the closure, the interior and the boundary of $\bigcup_{j\in J} G_\pm^j$ for $J \subseteq \overline{\mathbb{Z}}$.

Theorem 14. *Let $J \subseteq \overline{\mathbb{Z}}$. Then:*

(a) $\overline{\bigcup_{j\in J} G_\pm^j} = \Delta \cup \Big(\bigcup_{j\in J} F_\pm^j\Big) = \bigcup_{j\in J} \overline{G_\pm^j}$.

If in addition $J \neq \overline{\mathbb{Z}}$, then:

(b) $\operatorname{int}\Big(\overline{\bigcup_{j\in J} G_\pm^j}\Big) = \bigcup_{j\in J} F_\pm^j = \operatorname{int}\Big(\bigcup_{j\in J} \overline{G_\pm^j}\Big)$,

(c) $\partial\Big(\overline{\bigcup_{j\in J} G_\pm^j}\Big) = \partial\Big(\bigcup_{j\in J} \overline{G_\pm^j}\Big) = \Delta$.

Proof. (a) We prove the first equality. If $J = \overline{\mathbb{Z}}$, then, using [3, Problem 109], $G_\pm$ is dense in $B(H)$ and the result follows from Theorem 2.

Suppose now that $J \neq \overline{\mathbb{Z}}$. Using Theorem 5, we obtain that $\Delta \cup \Big(\bigcup_{j\in J} F_\pm^j\Big)$ is closed. But $\bigcup_{j\in J} G_\pm^j \subseteq \Delta \cup \Big(\bigcup_{j\in J} F_\pm^j\Big)$. Thus

$$\overline{\bigcup_{j\in J} G_\pm^j} \subseteq \Delta \cup \Big(\bigcup_{j\in J} F_\pm^j\Big). \tag{1}$$

On the other hand, using Theorem 11, we have:

$$\Delta \cup \Big(\bigcup_{j\in J} F_\pm^j\Big) \subseteq \overline{\bigcup_{j\in J} G_\pm^j}. \tag{2}$$

The first equality follows from (1) and (2).

The second equality follows from $\overline{G_\pm^j} = \overline{F_\pm^j}$, for all $j \in \overline{\mathbb{Z}}$. This follows easily from Theorem 1 and Theorem 11.

(b) is a direct consequence of (a), Theorem 5 and Theorem 8.

(c) Since the first equality follows from (a) and (b), it sufficient to show the second one. Using (a) and (b), we have

$$\partial\Big(\bigcup_{j\in J} \overline{G_\pm^j}\Big) = \Big(\Delta \cup \Big[\bigcup_{j\in J} F_\pm^j\Big]\Big) \setminus \Big(\bigcup_{j\in J} F_\pm^j\Big) = \Delta.$$

The proof is complete. ■

We obtain as consequences the following formulas :

Corollary 15. 1) a) $\overline{G_+} = \Delta \cup \Big(\bigcup_{j\le 0} F_\pm^j\Big)$,

b) $\overline{G_-} = \Delta \cup \Big(\bigcup_{j\ge 0} F_\pm^j\Big)$;

2) a) $\mathrm{int}(\overline{G_+}) = \bigcup_{j\le 0} F_\pm^j$,

b) $\mathrm{int}(\overline{G_-}) = \bigcup_{j\ge 0} F_\pm^j$;

3) $\partial\overline{G_+} = \partial\overline{G_-} = \Delta$.

Corollary 16. 1) a) $\overline{G_+ \setminus G} = \Delta \cup \Big(\bigcup_{j<0} F_\pm^j\Big)$,

b) $\overline{G_- \setminus G} = \Delta \cup \Big(\bigcup_{j>0} F_\pm^j\Big)$;

2) a) $\mathrm{int}\Big(\overline{G_+ \setminus G}\Big) = \bigcup_{j<0} F_\pm^j$,

b) $\mathrm{int}\Big(\overline{G_- \setminus G}\Big) = \bigcup_{j>0} F_\pm^j$;

3) $\partial\Big(\overline{G_+ \setminus G}\Big) = \partial\Big(\overline{G_- \setminus G}\Big) = \Delta$.

Similar formulas can be given for $\bigcup_{j\in J} G_+^j$, $J \subseteq \overline{\mathbb{Z}}_-$, and for $\bigcup_{j\in J} G_-^j$, $J \subseteq \overline{\mathbb{Z}}_+$.

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