ON THE TOPOLOGICAL BOUNDARY OF SEMI-FREDHOLM OPERATORS

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ABSTRACT. We prove several distance formulas from a fixed operator in \( B(H) \) to some classes of operators connected with the semi-Fredholm ones. Here \( H \) is a separable Hilbert space. In particular, Fredholm and upper and lower semi-Fredholm operators have the same boundary in \( B(H) \).

Let \( H \) be a separable Hilbert space and let \( B(H) \) be the algebra of all bounded linear operators on \( H \). For an operator \( T \in B(H) \), we will denote by \( T^* \), \( R(T) \), \( N(T) \) and \( \sigma(T) \) its adjoint, range, kernel and spectrum, respectively. Let \( K(H) \) be the ideal of compact operators and \( C(H) = B(H)/K(H) \) be the Calkin algebra. Denote by \( \pi : B(H) \to C(H) \) the canonical projection. Endowed with the essential norm \( ||T||_e = ||\pi(T)|| \), \( C(H) \) is a \( C^* \)-algebra.

The index of an operator \( T \in B(H) \) will be denoted by \( \text{ind}(T) \) and is defined by \( \text{ind}(T) = \text{dim } N(T) - \text{dim } N(T^*) \), with the convention \( \infty - \infty = 0 \).

We introduce the following notation for several classes of operators:

- \( F_+ = \{ T \in B(H) : R(T) \text{ is closed}, \text{dim } N(T) < \infty \} \) is the set of all upper semi-Fredholm operators.
- \( F_- = \{ T \in B(H) : R(T) \text{ is closed}, \text{dim } N(T^*) < \infty \} \) is the set of all lower semi-Fredholm operators.
- \( F_{\pm} = F_+ \cup F_- \) is the set of semi-Fredholm operators.
- \( F = F_+ \cap F_- \) is the set of Fredholm operators.
- \( I_n = \{ T \in B(H) : \text{ind}(T) = n \} \) with \( n \in \mathbb{Z} = \mathbb{Z} \cup \{-\infty, +\infty\} \).
- \( F_{\pm}^n = F_{\pm} \cap I_n \), with \( n \in \mathbb{Z} \), the connected component of index \( n \) in \( F_{\pm} \).

For a set \( X \) in \( B(H) \), we will denote by \( \text{int } X, \overline{X} \) and \( \partial X \) the interior, closure and (topological) boundary, respectively.

For a linear operator \( T \in B(H) \), we will denote by \( \sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin F \} \) the essential spectrum of \( T \). Let

\[
\me(T) = \inf \{ \sigma_e(|T|) \}
\]

(cf. [1]), where \( |T| = (T^*T)^{1/2} \), and

\[
M_e(T) = \max \{ \me(T); \me(T^*) \}.
\]

Using Theorem 1.1 of [4] and Theorem 3.1 of [7], we easily obtain the following result:
Theorem 1. Let $T \in B(H)$ and $n \in \mathbb{Z}$. Then
\[
\text{dist}(T, F^n_{\pm}) = \begin{cases} 
M_e(T) & \text{if } T \notin I_n, \\
0 & \text{if not.} 
\end{cases}
\]

We recall the following theorem, which gives a characterization of the boundary of connected components of semi-Fredholm operators.

Theorem 2 ([5], Corollary 1.4). The boundary of $F^j_{\pm}$ does not depend on $j$, and, if $\Delta = \partial F^j_{\pm}$, then $\Delta = \partial F^0_{\pm}$ and $B(H) = \Delta \cup \left( \bigcup_{j \in \mathbb{Z}} F^j_{\pm} \right)$.

Remarks. 1) We have $\Delta = \{ T \in B(H) : M_e(T) = 0 \}$. Indeed, using [4], Theorem 2.1 and Corollary 2.2, we obtain $\Delta = \partial F^0_{\pm} = \partial G = \{ T \in B(H) : M_e(T) = 0 \}$, where $G$ is the group of invertible operators in $B(H)$.

2) The set $\Delta$ is stable with respect to compact perturbations: $\Delta + K(H) = \Delta$.

3) The set $\Delta$ is arcwise connected. Indeed, if $0 \in \Delta$ and $T \in \Delta$, then $tT \in \Delta$ for all $t \in [0, 1]$.

We prove now the following result.

Theorem 3. Let $T \in B(H)$ and $J \subseteq \mathbb{Z}$. Then
\[
\text{dist}\left(T, \bigcup_{j \in J} F^j_{\pm}\right) = \begin{cases} 
M_e(T) & \text{if ind}(T) \notin J, \\
0 & \text{if not}, 
\end{cases}
\]

and
\[
\text{dist}\left(T, \bigcup_{j \in J} I_j\right) = \begin{cases} 
M_e(T) & \text{if ind}(T) \notin J, \\
0 & \text{if not}, 
\end{cases}
\]

Proof. (a) Let $n = \text{ind}(T) \in \mathbb{Z}$. If $n \in J$, then $F^n_{\pm} \subseteq \bigcup_{j \in J} F^j_{\pm}$. Therefore
\[
0 \leq \text{dist}(T, \bigcup_{j \in J} F^j_{\pm}) \leq \text{dist}(T, F^n_{\pm}).
\]

But, using Theorem 1, we obtain $\text{dist}(T, F^n_{\pm}) = 0$. Hence, $\text{dist}(T, \bigcup_{j \in J} F^j_{\pm}) = 0$.

Suppose now that $n \notin J$ and let $j_0 \in J$. Then, using once again Theorem 1 and the relation $n \neq j_0$, we get
\[
\text{dist}\left(T, \bigcup_{j \in J} F^j_{\pm}\right) \leq \text{dist}(T, F^{j_0}_{\pm}) = M_e(T).
\]
We now show the converse inequality. For every \( S \in \bigcup_{j \in J} F_j \), we have \( \text{ind}(S) \neq n = \text{ind}(T) \). Then Theorem 1.1 of [6] implies \( \|T - S\| \geq M_e(T) \). Therefore,

\[
\text{dist}
\left(
T, \bigcup_{j \in J} F_j \right) \geq M_e(T),
\]

and (a) is proved.

In order to prove (b), it suffices to remark that \( \bigcup_{j \in J} F_j \subseteq \bigcup_{j \in J} I_j \) and to use (a).

Indeed, if \( \text{ind}(T) = n \in J \), then, using (a), we obtain

\[
0 \leq \text{dist}
\left(
T, \bigcup_{j \in J} I_j
\right) \leq \text{dist}
\left(
T, \bigcup_{j \in J} F_j
\right) = 0.
\]

On the other hand, if \( n \notin J \), then for all \( S \in \bigcup_{j \in J} I_j \), we have \( \|T - S\| \geq M_e(T) \) (cf. [6], Theorem 1.1). This yields

\[
\text{dist}
\left(
T, \bigcup_{j \in J} I_j
\right) \geq M_e(T).
\]

Using (a) again, we also find that

\[
\text{dist}
\left(
T, \bigcup_{j \in J} I_j
\right) \leq \text{dist}
\left(
T, \bigcup_{j \in J} F_j
\right) = M_e(T),
\]

which completes the proof.

As consequences, the results below easily follow. Assertions (b) and (c) of the following corollary are also in [2, Theorems 12 and 13].

**Corollary 4.** Let \( T \in B(H) \). Then:

(a) \( \text{dist}(T, F) = \begin{cases} 
M_e(T) & \text{if } \text{ind}(T) = \pm\infty, \\
0 & \text{if not.}
\end{cases} \)

(b) \( \text{dist}(T, F_+) = \begin{cases} 
m_e(T^*) & \text{if } \text{ind}(T) = +\infty, \\
0 & \text{if } \text{ind}(T) \neq +\infty.
\end{cases} \)

(c) \( \text{dist}(T, F_-) = \begin{cases} 
m_e(T) & \text{if } \text{ind}(T) = -\infty, \\
0 & \text{if } \text{ind}(T) \neq -\infty.
\end{cases} \)

**Proof.** (a) follows from Theorem 3 above with \( J = \mathbb{Z} \), while (b) and (c) follow from [2, Lemma 7] and Theorem 3 above with \( J = \mathbb{Z} \cup \{-\infty\} \) and \( J = \mathbb{Z} \cup \{+\infty\} \), respectively. \( \blacksquare \)
Now we can prove the following equalities.

**Theorem 5.** Let $J \subseteq \mathbb{Z}$. Then

(a) \[ \bigcup_{j \in J} F_j^\pm = \Delta \cup \left( \bigcup_{j \in J} F_j^\pm \right) = \bigcup_{j \in J} \overline{F_j^\pm} ; \]

(b) \[ \partial \left( \bigcup_{j \in J} F_j^\pm \right) = \Delta. \]

**Proof.** (a) We prove the first equality. If $J = \mathbb{Z}$, then the result follows from Theorem 2 since $\bigcup_{j \in J} F_j^\pm = B(H)$.

Suppose that $J \neq \mathbb{Z}$. Then, using Theorem 2 again, we have

\[ B(H) = \Delta \cup \left( \bigcup_{j \in J} F_j^\pm \right) \cup \left( \bigcup_{j \in \mathbb{Z} \setminus J} F_j^\pm \right). \]

It follows that $\Delta \cup \left( \bigcup_{j \in J} F_j^\pm \right)$ is closed, being the complement of the open set $\left( \bigcup_{j \in \mathbb{Z} \setminus J} F_j^\pm \right)$ in $B(H)$. Therefore

\[ \bigcup_{j \in J} \overline{F_j^\pm} \subseteq \Delta \cup \left( \bigcup_{j \in J} F_j^\pm \right). \]

In order to show the other inclusion, it suffices to see that $\Delta \subset \bigcup_{j \in J} \overline{F_j^\pm}$ (see Theorem 3).

The second equality follows from

\[ \bigcup_{j \in J} \overline{F_j^\pm} = \bigcup_{j \in J} \left( \Delta \cup F_j^\pm \right). \]

(b) Since $\bigcup_{j \in J} F_j^\pm$ is open in $B(H)$, we have

\[ \partial \left( \bigcup_{j \in J} F_j^\pm \right) = \left( \bigcup_{j \in J} \overline{F_j^\pm} \right) \setminus \left( \bigcup_{j \in J} F_j^\pm \right) = \left( \Delta \cup \left( \bigcup_{j \in J} F_j^\pm \right) \right) \setminus \left( \bigcup_{j \in J} F_j^\pm \right) = \Delta. \]

The proof is complete. \[ \blacksquare \]

This easily implies the following consequence:

**Corollary 6.** We have

(a) \[ \overline{F} = F \cup \Delta ; \quad \overline{F_+} = F_+ \cup \Delta \quad \text{and} \quad \overline{F_-} = F_- \cup \Delta ; \]

(b) \[ \partial \overline{F} = \partial \overline{F_+} = \partial \overline{F_-} = \Delta. \]

We also have

**Corollary 7.** Let $J \subseteq \mathbb{Z}$. Then

\[ \bigcup_{j \in J} I_j = \Delta \cup \left( \bigcup_{j \in J} \overline{F_j^\pm} \right) = \Delta \cup \left( \bigcup_{j \in J} I_j \right). \]
Proof. Using Theorems 3 and 5, we obtain
\[
\Delta \cup \left( \bigcup_{j \in J} I_j \right) \subseteq \bigcup_{j \in J} F^j = \Delta \cup \left( \bigcup_{j \in J} F^j_\pm \right) \subseteq \Delta \cup \left( \bigcup_{j \in J} I_j \right).
\]

For the interior and for the boundary of the closure of sets considered in Theorem 5, we have

**Theorem 8.** Let \( J \subseteq \mathbb{Z}, J \neq \mathbb{Z} \). Then

(a) \( \text{int} \left( \bigcup_{j \in J} F^j_\pm \right) = \bigcup_{j \in J} F^j = \text{int} \left( \bigcup_{j \in J} F^j_\pm \right) \);

(b) \( \partial \left( \bigcup_{j \in J} F^j_\pm \right) = \partial \left( \bigcup_{j \in J} F^j_\pm \right) = \Delta \).

Proof. (a) We show the first equality. The inclusion
\[
\bigcup_{j \in J} F^j_\pm \subseteq \text{int} \left( \bigcup_{j \in J} F^j_\pm \right)
\]
is clear. In order to show the other one, let \( T \in \text{int} \left( \bigcup_{j \in J} F^j_\pm \right) \). If \( T \) is semi-Fredholm, then \( T \in \bigcup_{j \in J} F^j_\pm \). Indeed, using Theorem 5, we have
\[
T \in \text{int} \left( \bigcup_{j \in J} F^j_\pm \right) = \text{int} \left( \Delta \cup \left[ \bigcup_{j \in J} F^j_\pm \right] \right)
\subset \Delta \cup \left( \bigcup_{j \in J} F^j_\pm \right)
\]
and, because \( T \notin \Delta \), we get \( T \in \bigcup_{j \in J} F^j_\pm \).

Suppose now that \( T \) is not semi-Fredholm. Then \( T \in \Delta \). Consider \( n \notin J \) (this is always possible since \( J \neq \mathbb{Z} \)). Using Theorem 2, we obtain \( T \in \Delta = \partial F^*_\pm \). Therefore
\[
T \in \text{int} \left( \bigcup_{j \in J} F^j_\pm \right) \cap \partial F^*_\pm,
\]
which implies the non-voidness of \( \left( \bigcup_{j \in J} F^j_\pm \right) \cap F^*_\pm \). The continuity of the index yields a contradiction.

The second equality follows from Theorem 5.
(b) Using (a) and Theorem 5, we obtain
\[
\partial \left( \bigcup_{j \in J} F_j^{\pm} \right) = \partial \left( \bigcup_{j \in J} F_j^{\pm} \right) = \left( \bigcup_{j \in J} F_j^{\pm} \right) \setminus \text{int} \left( \bigcup_{j \in J} F_j^{\pm} \right) = \left( \Delta \cup \left[ \bigcup_{j \in J} F_j^{\pm} \right] \right) \setminus \left( \bigcup_{j \in J} F_j^{\pm} \right) = \Delta.
\]

The proof is complete. ■

The following consequence can be easily obtained.

**Corollary 9.** We have

(a) \( \text{int}(F) = F \); \( \text{int}(F) = F_+ \) and \( \text{int}(F) = F_- \);

and

(b) \( \partial F = \partial F = \partial F = \partial F = \partial F = \partial F = \Delta \).

We also have

**Corollary 10.** Let \( J \subset \mathbb{Z}, J \neq \mathbb{Z} \). Then

(a) \( \text{int} \left( \bigcup_{j \in J} I_j \right) = \bigcup_{j \in J} F_j^{\pm} = \text{int} \left( \bigcup_{j \in J} I_j \right) \);

(b) \( \partial \left( \bigcup_{j \in J} I_j \right) = \partial \left( \bigcup_{j \in J} I_j \right) = \Delta \).

**Proof.** (a) The first equality follows from Corollary 7 and Theorems 5 and 8. For the second equality, it is sufficient to note that

\[
\bigcup_{j \in J} F_j^{\pm} \subseteq \text{int} \left( \bigcup_{j \in J} I_j \right) \subseteq \text{int} \left( \bigcup_{j \in J} F_j^{\pm} \right) = \bigcup_{j \in J} F_j^{\pm}.
\]

Indeed, the first two inclusions are obvious and the last equality is (a). (b) is a direct consequence of (a) and of Corollary 7. ■

We introduce the following notation:

- \( G_+ = \{ T \in B(H) : T \text{ is left invertible} \} \).
- \( G_- = \{ T \in B(H) : T \text{ is right invertible} \} \).
- \( G_\pm = G_+ \cup G_- \): the set of one-sided invertible operators.
- \( G = G_+ \cap G_- \): the set of invertible operators.

For \( n \in \mathbb{Z}, \) let \( G^n_\pm = G_\pm \cap I_n \). For \( n \in \overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}, \) we denote \( G^n_+ = G_+ \cap I_n \), while for \( n \in \overline{\mathbb{Z}} = \mathbb{Z} \cup \{ +\infty \}, \) we set \( G^n_- = G_- \cap I_n \).

**Theorem 11.** Let \( T \in B(H) \) and \( J \subset \overline{\mathbb{Z}} \). Then

\[
\text{dist} \left( T, \bigcup_{j \in J} G_j^{\pm} \right) = \begin{cases} M_\varepsilon(T) & \text{if ind}(T) \notin J, \\
0 & \text{if not.} \end{cases}
\]
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Proof. Let \( n = \text{ind}(T) \). If \( n \in J \), then \( G^n_\pm \subseteq \bigcup_{j \in J} G^j_\pm \), so, using Theorem 3.1 of [7], we have

\[
0 \leq \text{dist}(T, \bigcup_{j \in J} G^j_\pm) \leq \text{dist}(T, G^n_\pm) = 0.
\]

Suppose now that \( n \notin J \). Let \( j_0 \in J \). Using Theorem 3.1 of [7], we get

\[
0 \leq \text{dist}(T, \bigcup_{j \in J} G^j_\pm) \leq \text{dist}(T, G^j_\pm) = M_e(T).
\]

On the other hand, for all \( L \in \bigcup_{j \in J} G^j_\pm \), \( \text{ind}(L) \neq n = \text{ind}(T) \). Therefore, using Theorem 1.1 of [6], we have

\[
\|T - L\| \geq M_e(T).
\]

Thus

\[
\text{dist}(T, \bigcup_{j \in J} G^j_\pm) \geq M_e(T).
\]

Now (1) and (2) imply the desired equality. \( \blacksquare \)

We obtain the following consequence.

**Corollary 12.** Let \( T \in B(H) \), and \( J \subseteq \mathbb{Z}_+ \). Then

\[
\text{dist}(T, \bigcup_{j \in J} G_j^-) = \begin{cases} M_e(T) & \text{if } \text{ind}(T) \notin J, \\ 0 & \text{if } \text{ind}(T) \in J. \end{cases}
\]

A similar result can be stated for \( \text{dist}(T, \bigcup_{j \in J} G_j^+) \) if \( J \subseteq \mathbb{Z}_- \). For the distance of \( T \) to the set \( G_- \setminus G \) we obtain the following formula:

**Corollary 13.** Let \( T \in B(H) \). Then

\[
\text{dist}(T, G_- \setminus G) = \begin{cases} M_e(T) & \text{if } \text{ind}(T) \leq 0, \\ 0 & \text{if not}. \end{cases}
\]

The following result gives a description of the closure, the interior and the boundary of \( \bigcup_{j \in J} G^j_\pm \) for \( J \subseteq \mathbb{Z} \).

**Theorem 14.** Let \( J \subseteq \mathbb{Z} \). Then:

(a) \( \bigcup_{j \in J} G^j_\pm = \Delta \cup \left( \bigcup_{j \in J} F^j_\pm \right) = \bigcup_{j \in J} \overline{G^j_\pm} \).

If in addition \( J \neq \mathbb{Z} \), then:

(b) \( \text{int} \left( \bigcup_{j \in J} G^j_\pm \right) = \bigcup_{j \in J} F^j_\pm = \text{int} \left( \bigcup_{j \in J} \overline{G^j_\pm} \right) \),

(c) \( \partial \left( \bigcup_{j \in J} G^j_\pm \right) = \partial \left( \bigcup_{j \in J} \overline{G^j_\pm} \right) = \Delta \).
Proof. (a) We prove the first equality. If \( J = \mathbb{Z} \), then, using [3, Problem 109], \( G_{\pm} \) is dense in \( B(H) \) and the result follows from Theorem 2.

Suppose now that \( J \neq \mathbb{Z} \). Using Theorem 5, we obtain that \( \Delta \cup \left( \bigcup_{j \in J} F^j_{\pm} \right) \) is closed. But \( \bigcup_{j \in J} G^j_{\pm} \subseteq \Delta \cup \left( \bigcup_{j \in J} F^j_{\pm} \right) \). Thus

\[
\bigcup_{j \in J} G^j_{\pm} \subseteq \Delta \cup \left( \bigcup_{j \in J} F^j_{\pm} \right).
\]

On the other hand, using Theorem 11, we have:

\[
\Delta \cup \left( \bigcup_{j \in J} F^j_{\pm} \right) \subseteq \bigcup_{j \in J} G^j_{\pm}.
\]

The first equality follows from (1) and (2).

The second equality follows from \( G^j_{\pm} = F^j_{\pm} \), for all \( j \in \mathbb{Z} \). This follows easily from Theorem 1 and Theorem 11.

(b) is a direct consequence of (a), Theorem 5 and Theorem 8.

(c) Since the first equality follows from (a) and (b), it sufficient to show the second one. Using (a) and (b), we have

\[
\partial \left( \bigcup_{j \in J} G^j_{\pm} \right) = \left( \Delta \cup \left( \bigcup_{j \in J} F^j_{\pm} \right) \right) \setminus \left( \bigcup_{j \in J} F^j_{\pm} \right) = \Delta.
\]

The proof is complete.

We obtain as consequences the following formulas:

**Corollary 15.** 1) a) \( \overline{G_+} = \Delta \cup \left( \bigcup_{j \leq 0} F^j_{\pm} \right) \),

b) \( \overline{G_-} = \Delta \cup \left( \bigcup_{j \geq 0} F^j_{\pm} \right) \);

2) a) \( \text{int}(G_+) = \bigcup_{j \leq 0} F^j_{\pm} \),

b) \( \text{int}(G_-) = \bigcup_{j \geq 0} F^j_{\pm} \);

3) \( \partial G_+ = \partial G_- = \Delta \).

**Corollary 16.** 1) a) \( \overline{G_+} \setminus G = \Delta \cup \left( \bigcup_{j < 0} F^j_{\pm} \right) \),

b) \( \overline{G_-} \setminus G = \Delta \cup \left( \bigcup_{j > 0} F^j_{\pm} \right) \);

2) a) \( \text{int}(G_+ \setminus G) = \bigcup_{j < 0} F^j_{\pm} \),

b) \( \text{int}(G_- \setminus G) = \bigcup_{j > 0} F^j_{\pm} \);

3) \( \partial \left( \overline{G_+} \setminus G \right) = \partial \left( \overline{G_-} \setminus G \right) = \Delta \).

Similar formulas can be given for \( \bigcup_{j \in J} G^j_{\pm}, J \subseteq \mathbb{Z}_- \), and for \( \bigcup_{j \in J} G^j_{\pm}, J \subseteq \mathbb{Z}_+ \).
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References


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