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THE QUADRATIC FORM IN THE LÉVY-KHINCHIN FORMULA ON SEMIGROUPS

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Abstract. In this paper we obtain the quadratic form in the Lévy-Khinchin formula on a commutative involutive semigroup, with a neutral element, as a sum of two simpler quadratic forms.

The Lévy-Khinchin representation for a negative definite function, with the real part bounded below, defined on a commutative involutive semigroup with a neutral element, was proved in [1, p. 108, Theorem 3.19] .

In Section 2 of this paper we give a proof for this Lévy-Khinchin representation in which we obtain the quadratic form q from [1] as a sum of an additive function and a quadratic form associated with a bi-additive function.

Section 3 is concerned with negative definite functions which have real part bounded below, defined on the semigroup $(\mathbb{Z}^2, +)$ with the involution $(m,n)^* = (n,m)$. We obtain the Lévy-Khinchin representation of these functions using the result of Section 2.

1. Notation

Let $(S,+,*)$ be a commutative involutive semigroup with neutral element ([1, p. 86]). We say that a function $\varphi : S \to \mathbb{C}$ is positive definite if for each natural number $n \geq 1$, each family $c_1, \dots, c_n$ of complex numbers and each family $x_1, \dots, x_n$ of elements of S, we have

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi(x_j + x_k^*) \geq 0.$$

A function $\varphi : S \to \mathbb{C}$ is hermitian if $\varphi(x^*) = \overline{\varphi(x)}$ for each $x \in S$.

We say that an hermitian function $\varphi : S \to \mathbb{C}$ is negative definite if for each natural number $n \geq 2$, each family $c_1, \dots, c_n$ of complex numbers such that $c_1 + \dots + c_n = 0$, and each family $x_1, \dots, x_n$ of elements of S we have

$$\sum_{j,k=1}^{n} c_j \bar{c}_k \varphi(x_j + x_k^*) \leq 0.$$

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We denote by Γ the set

$$\{\rho : S \to \mathbb{C} | \rho(x+y) = \rho(x)\rho(y); \rho(x^*) = \overline{\rho(x)}; |\rho(x)| \leq 1; \rho(0) = 1\}$$

and by Ω the set $\{\rho \in \Gamma | \rho \not\equiv 1\}$.

With the product topology Γ is a compact space and Ω a locally compact space.

We denote by $\mathcal{M}(S)$ the set of positive Radon measure on Ω such that the functions $(\rho \mapsto 1 - \text{ Re } \rho(x))_{x \in S}$ are μ-integrable for each μ in $\mathcal{M}(S)$.

An element of $\mathcal{M}(S)$ is called a Lévy measure. We denote by $\mathcal{Q}(S)$ the set

$$\{q : S \to [0, \infty[| 2(q(x) + q(y)) = q(x+y) + q(x+y^*), x, y \in S\}.$$

An element of $\mathcal{Q}(S)$ is called a quadratic form on S. This notion of quadratic form was introduced in [5, p. 211].

Let $\mathcal{A}(S)$ be the set

$$\{a : S \to [0, \infty[| a(x+y) = a(x) + a(y), a(x^*) = a(x), x, y \in S\}$$

and $\mathcal{B}(S)$ the set

$$\{b : S \times S \to \mathbb{R} | b(x,y) = b(y,x), b(x^*, y) = -b(x,y),$$
$$b(x+y, z) = b(x,z) + b(y,z), b(x,x) \in [0, \infty[, x, y, z \in S\}.$$

We denote by $\mathcal{L}(S)$ the set

$$\{L : S \times \Omega \to \mathbb{R} | L(x+y, \rho) = L(x, \rho) + L(y, \rho), L(x^*, \rho) = -L(x, \rho),\ x, y \in S;$$
$$\rho \mapsto L(x, \rho) \text{ is continuous on } \Omega,\ x \in S;$$
$$\rho \mapsto L(x, \rho) - \text{ Im } \rho(x) \text{ is } \mu\text{-integrable},\ x \in S,\ \mu \in \mathcal{M}(S)\}.$$

An element of $\mathcal{L}(S)$ is called a Lévy function. In [4] is proved that the set $\mathcal{L}(S)$ is nonvoid.

Let $\mathcal{T}(S)$ be the set

$$\{\ell : S \to \mathbb{R} | \ell(x+y) = \ell(x) + \ell(y), \ell(x^*) = -\ell(x), x, y \in S\}.$$

2. The Lévy-Khinchin representation

Theorem. *For a function $\varphi : S \to \mathbb{C}$ the following conditions are equivalent:*

(i) *the function φ is negative definite and has real part bounded below;*

(ii) *for every $L \in \mathcal{L}(S)$ there are $C \in \mathbb{R}, a \in \mathcal{A}(S), b \in \mathcal{B}(S), \ell \in \mathcal{T}(S)$ and $\mu \in \mathcal{M}(S)$ which satisfy*

$$\varphi(x) = C + a(x) + b(x,x) + i\ell(x) + \int_\Omega (1 - \rho(x) + iL(x, \rho)) d\mu(\rho),\ x \in S.$$

C, a, b and μ are uniquely determined by φ; ℓ is uniquely determined by φ and L. We have the relations

$$a(x) = \lim_{n \to \infty} \frac{\varphi(n(x+x^*))}{2n} \quad \text{and} \quad b(x,x) = \lim_{n \to \infty} \frac{\text{Re } \varphi(nx)}{n^2},\ x \in S.$$

Proof. Let U be the vector space

$$\{f : \Gamma \to \mathbb{R} | f(\rho) = \sum_{k=1}^n a_k \rho(x_k), \sum_{k=1}^n a_k = 0, n \in \mathbb{N}, n \geq 2, a_k \in \mathbb{C}, x_k \in S\}$$

and $U_+ = \{f \in U | f \geq 0\}$.

For every $t \in]0, \infty[$ the function $\psi_t : S \to \mathbb{C}$ defined by $\psi_t(x) = e^{-t\varphi(x)}$ is positive definite (cf. [1, p. 74, Theorem 2.2]) and bounded.

It follows from [1, p. 93, Theorem 2.5] that for each $t \in]0, \infty[$ there is a positive Radon measure μ_t on Γ such that

$$e^{-t\varphi(x)} = \int_\Gamma \rho(x) d\mu_t(\rho).$$

Let $n \geq 2$ be a natural number, $a_1, \ldots, a_n$ complex numbers such that $a_1 + \ldots + a_n = 0$, and $x_1, \ldots, x_n$ elements of S.

We have

$$\sum_{k=1}^{n} a_k \left(\frac{e^{-t\varphi(x_k)} - 1}{t} \right) = \frac{1}{t} \int_\Gamma \sum_{k=1}^{n} a_k \rho(x_k) d\mu_t(\rho). \tag{1}$$

Letting t tend to 0, we obtain that the function $F : U \to \mathbb{R}$ defined by

$$F(\rho \mapsto \sum_{k=1}^{n} a_k \rho(x_k)) = -\sum_{k=1}^{n} a_k \varphi(x_k)$$

is well defined. If in (1) we take $x_k + y$ instead of x_k and assume that the function $\rho \mapsto \sum_{k=1}^{n} a_k \rho(x_k)$ is in U_+, we obtain, also letting t tend to 0, that the function

$$y \mapsto -\sum_{k=1}^{n} a_n \varphi(x_n + y)$$

is positive definite and bounded.

Consequently Theorem 2.5 from [1] , p. 93, implies that for every $g \in U_+$ there is a positive Radon measure μ_g on Γ such that if $g(\rho) = \sum_{k=1}^{n} a_k \rho(x_k), \sum_{k=1}^{n} a_k = 0$, we have

$$-\sum_{k=1}^{n} a_k \varphi(x_k + y) = \int_\Gamma \rho(y) d\mu_g(\rho),\ y \in S.$$

Let $f \in U, g \in U_+$. We suppose that

$$f(\rho) = \sum_{k=1}^{n} a_k \rho(x_k), \sum_{k=1}^{n} a_k = 0 \quad \text{and} \quad g(\rho) = \sum_{\ell=1}^{m} b_\ell \rho(y_\ell), \sum_{\ell=1}^{m} b_\ell = 0.$$

We have

$$F(fg) = -\sum_{k=1}^{n} \sum_{\ell=1}^{m} a_k b_\ell \varphi(x_k + y_\ell) = \int_\Gamma f(\rho) d\mu_g(\rho). \tag{2}$$

This implies that if $h, g \in U_+$ we have

$$F(\rho \mapsto \rho(x) h(\rho) g(\rho)) = \int_\Gamma \rho(x) h(\rho) d\mu_g(\rho) = \int_\Gamma \rho(x) g(\rho) d\mu_h(\rho),\ x \in S,$$

and consequently that

$$h\mu_g = g\mu_h. \tag{3}$$

It follows from (3) that we can define a positive Radon measure μ on Ω such that

$$\mu|_{O_g} = \frac{1}{g|_{O_g}} \mu_g|_{O_g}$$

where $O_g = \{\rho \in \Gamma | g(\rho) > 0\}$.

It is easy to verify that

$$\mu_g|_\Omega = g|_\Omega \mu. \tag{4}$$

The measure μ is the Lévy measure for φ (cf. [1, p. 103, 3.12]).

Using (4), for g in U_+ we obtain

$$\int_\Omega g(\rho)d\mu(\rho) = \int_\Omega d\mu_g \leq \int_\Gamma d\mu_g = F(g). \tag{5}$$

We denote by E the set

$$\{f \in U | F(f) = \int_\Omega f(\rho)d\mu(\rho)\}.$$

The relations (2) and (4) yield

$$fg \in E \quad \text{for} \quad f \in U \quad \text{and} \quad g \in U_+ \tag{6}$$

because $f(\theta) = 0$, where $\theta : S \to \mathbb{R}$ is identically 1.

We denote by $B : S \times S \to \mathbb{R}$ the function defined by

$$B(x,y) = F(\rho \mapsto \text{Im } \rho(x)\text{Im } \rho(y)) - \int_\Omega \text{Im } \rho(x)\text{Im } \rho(y)d\mu(\rho).$$

It is immediate that

$$B(x^*, y) = -B(x,y) \quad \text{and} \quad B(x,x) \geq 0.$$

It follows from (6) that the functions

$$\rho \mapsto (1 - \text{Re } \rho(x))\text{Im } \rho(y)\text{Im } \rho(z)$$

and

$$\rho \mapsto (1 - \text{Re } \rho(y))\text{Im } \rho(x)\text{Im } \rho(z)$$

are in E.

The equality

$$\begin{aligned} &F(\rho \mapsto (1 - \text{Re } \rho(x))\text{Im } \rho(y)\text{Im } \rho(z) + (1 - \text{Re } \rho(y))\text{Im } \rho(x)\text{Im } \rho(z)) \\ &= \int_\Omega ((1 - \text{Re } \rho(x))\text{Im } \rho(y)\text{Im } \rho(z) + (1 - \text{Re } \rho(y))\text{Im } \rho(x)\text{Im } \rho(z))d\mu(\rho) \end{aligned}$$

is equivalent to

$$B(x+y, z) = B(x,z) + B(y,z) \quad x, y, z \in S.$$

We have proved that $B \in \mathcal{B}(S)$.

Define $a : S \to \mathbb{R}$ by

$$\begin{aligned} a(x) &= -\varphi(0) + \text{Re } \varphi(x) - \frac{1}{2}B(x,x) - \int_\Omega (1 - \text{Re } \rho(x))d\mu(\rho) \\ &= F(\rho \mapsto 1 - \text{Re } \rho(x) - \frac{1}{2}(\text{Im } \rho(x))^2) - \int_\Omega (1 - \text{Re } \rho(x) - \frac{1}{2}(\text{Im } \rho(x))^2)d\mu(\rho). \end{aligned}$$

First we have $a(x^*) = a(x)$.

Using the relations

$$\begin{aligned} &1 - \text{Re } \rho(x) - \frac{1}{2}(\text{Im } \rho(x))^2 \\ &= \frac{1}{2}(1 - \text{Re } \rho(x))^2 + \frac{1}{2}(1 - (\text{Re } \rho(x))^2 - (\text{Im } \rho(x))^2) \geq 0, \end{aligned}$$

we see from (5) that $a(x) \geq 0$ for every $x \in S$.

The fact that the function

$$\rho \mapsto (1 - \text{Re } \rho(x))(1 - \text{Re } \rho(y))$$

is in E yields

$$
\begin{aligned}
B(x,y) &= F(\rho \mapsto \text{Im } \rho(x)\text{Im } \rho(y) - (1 - \text{Re } \rho(x))(1 - \text{Re } \rho(y))) \\
&\quad - \int_\Omega (\text{Im } \rho(x)\text{Im } \rho(y) - (1 - \text{Re } \rho(x))(1 - \text{Re } \rho(y)))d\mu(\rho) \\
&= F(\text{Re } \rho(x) + \text{Re } \rho(y) - 1 - \text{Re } \rho(x+y)) \\
&\quad - \int_\Omega (\text{Re } \rho(y) + \text{Re } \rho(x) - 1 - \text{Re } \rho(x+y))\, d\mu(\rho).
\end{aligned} \tag{7}
$$

Using (7) and the equality

$$B(x,y) = \frac{1}{2}(B(x+y, x+y) - B(x,x) - B(y,y)),$$

we obtain

$$a(x+y) = a(x) + a(y), \quad x, y \in S.$$

We have proved that $a \in \mathcal{A}(S)$.

The function $\ell : S \to \mathbb{R}$ defined by

$$\ell(x) = \text{Im } \varphi(x) - \int_\Omega ((-\text{Im } \rho(x)) + L(x,\rho))d\mu(\rho)$$

satisfies

$$\ell(x^*) = -\ell(x).$$

We have

$$
\begin{aligned}
&\text{Im } \varphi(x) + \text{Im } \varphi(y) - \text{Im } \varphi(x+y) \\
&\quad = \int_\Omega (\text{Im } \rho(x+y) - \text{Im } \rho(x) - \text{Im } \rho(y))d\mu(\rho)
\end{aligned}
$$

because the function

$$
\begin{aligned}
&\rho \mapsto \text{Im } \rho(x) + \text{Im } \rho(y) - \text{Im } \rho(x+y) \\
&\quad = (1 - \text{Re } \rho(x))\text{Im } \rho(y) + (1 - \text{Re } \rho(y))\text{Im } \rho(x)
\end{aligned}
$$

is in E. Consequently the function ℓ also satisfies

$$\ell(x+y) = \ell(x) + \ell(y).$$

Taking $b(x,y) = \frac{1}{2}B(x,y)$, we finish the proof of the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Let $b \in \mathcal{B}(S)$. Using the relation

$$b(x+y^*, x+y^*) = b(x,x) + b(y,y) - 2b(x,y)$$

we see, as in [1, p. 103], that the function $x \mapsto b(x,x)$ is negative definite. Now the implication (ii) $\Rightarrow$ (i) is clear.

If we have a representation as in (ii), we obtain

$$
\begin{aligned}
\varphi(n(x+x^*)) &= \varphi(0) + 2na(x) + n^2 b(x+x^*, x+x^*) \\
&\quad + \int_\Omega (1 - \rho(n(x+x^*))d\mu(\rho), \quad x \in S, n \in \mathbb{N}^*.
\end{aligned} \tag{8}
$$

The dominated convergence theorem implies

$$\lim_{n\to\infty} \frac{1}{2n} \int_\Omega (1 - |\rho(x)|^{2n})d\mu(\rho) = 0, \tag{9}$$

because $\frac{1-|\rho(x)|^{2n}}{2n} \le 1 - |\rho(x)|^2$.

The relations $b(x+x^*, x+x^*) = 0$, (8) and (9) yield

$$a(x) = \lim_{n\to\infty} \frac{1}{2n}\varphi(n(x+x^*)). \tag{10}$$

From the equality (7) we obtain

$$\begin{aligned} 2b(nx, nx^*) &= \varphi(0) - \varphi(nx) - \varphi(nx^*) + \varphi(n(x+x^*)) \\ &+ \int_\Omega 1 - ((\rho(x))^n)(1 - (\rho(x^*))^n) d\mu(\rho) \end{aligned} \tag{11}$$

We have, using again the dominated convergence theorem,

$$\lim_{n\to\infty} \frac{1}{n^2} \int_\Omega (1 - (\rho(x))^n)(1 - (\rho(x^*))^n)\, d\mu(\rho) = 0 \tag{12}$$

because $\frac{1}{n^2}|(1 - (\rho(x))^n)(1 - (\rho(x^*))^n)| \le |1 - \rho(x)|^2$.

The relations $b(x, x^*) = -b(x, x)$, (10), (11) and (12) now give

$$b(x, x) = \lim_{n\to\infty} \frac{1}{n^2} \text{Re } \varphi(nx).$$

The representation from (ii) implies that

$$\begin{aligned} &\varphi(x+y) + \varphi(x+y^*) - 2\varphi(x) \\ &= 2(a(y) + b(y, y) + \int_\Omega \rho(x)(1 - \text{ Re } \rho(y)) d\mu(\rho)), \end{aligned}$$

which proves the unicity of the measure μ. This completes the proof of the theorem. □

Corollary. *A function $q : S \to [0, \infty[$ is an element of $\mathcal{Q}(S)$ if and only if there are $a \in \mathcal{A}(S)$ and $b \in \mathcal{B}(S)$ such that*

$$q(x) = a(x) + b(x, x).$$

The functions a and b are uniquely determined by q according to the relations

$$a(x) = \frac{1}{2} q(x + x^*) \quad \textit{and} \quad b(x, x) = \lim_{n\to\infty} \frac{q(nx)}{n^2},\ x \in S.$$

The corollary is an immediate consequence of the unicity results in the theorem of this section and [1, p. 108, Theorem 3.19] . It also results from [1] , p. 102, because if we denote by $a : S \to \mathbb{R}$ the function defined by

$$a(x) = q(x) - b(x, x),$$

where $b(x, y) = \frac{1}{2}(-q(x) - q(y) + q(x+y))$, we have

$$a(x+y) = a(x) + a(y),\ a(x^*) = a(x),$$

and the definition of $\mathcal{Q}(S)$ gives

$$a(x) = q(x) - \frac{1}{2}(-2q(x) + q(2x)) = \frac{1}{2} q(x + x^*).$$

Remark 1. Using the method of Bloom and Ressel from [2] , we deduce from the representation given in this paper that the quadratic form on some commutative hypergroups may be written as a sum of two quadratic forms (see [2, p. 248, Theorem 2.4, and p. 250, Theorem 2.6]).

Remark 2. The relation $q(x) = a(x) + b(x, x)$ of the corollary is also a consequence of [3, p. 636, Theorem 8].

3. An application

Proposition. *Consider on the semigroup* $(\mathbb{Z}^2, +)$ *the involution* $(m,n)^* = (n,m)$. *For a function* $\varphi : \mathbb{Z}^2 \to \mathbb{C}$ *the following conditoins are equivalent:*

(i) *the function* φ *is negative definite and has real part bounded below;*

(ii) *there are real numbers* C, α, β, γ, *such that* $\alpha, \beta \geq 0$, *and a positive Radon measure* μ *on* $\{z \in \mathbb{C} | |z| = 1, z \neq 1\}$ *such that the function* $z \mapsto 1 - \mathrm{Re}\ z$ *is* μ*-integrable, which satisfy*

$$\varphi(m,n) = C + (m+n)\alpha + (m-n)^2\beta + i(m-n)\gamma + \int_{T\setminus\{1\}} (1 - z^m \bar{z}^n + (m-n)\mathrm{Im}\ z) d\mu(z),$$

where $T = \{z \in \mathbb{C} | |z| = 1\}$.

C, α, β, γ *and* μ *are uniquely determined by* φ.

Proof. We note that the function

$$z \mapsto ((m,n) \mapsto z^m \bar{z}^n)$$

is a homeomorphism of T onto the space of bounded characters of $\mathbb{Z}^2$.

It is easy to see that $a \in \mathcal{A}(\mathbb{Z}^2)$ if and only if there is $\alpha \in [0, \infty[$ such that $a(m,n) = (m+n)\alpha$, that $b \in \mathcal{B}(\mathbb{Z}^2)$ if and only if there is $\beta \in [0, \infty[$ such that $b((m,n),(p,q)) = (mp + nq - mq - np)\beta$, and that $\ell \in \mathcal{T}(\mathbb{Z}^2)$ if and only if there is a real number γ such that $\ell(m,n) = (m-n)\gamma$.

Let μ be a positive Radon measure on $T \setminus \{1\}$ such that the function $z \mapsto 1 - \mathrm{Re}\ z$ is μ-integrable. We note that the function $z \mapsto (1-z)^2$ is also μ-integrable.

We show that the functions $z \to 1 - z^m \bar{z}^n + m(z-1) + n(\bar{z}-1)$ and $z \mapsto 1 - \mathrm{Re}\ z^m \bar{z}^n$ are μ-integrable for every $(m,n) \in \mathbb{Z}^2$. Take, for example, $m < 0$ and $n \geq 0$. Using the binomial theorem, we obtain that the function

$$\big(\frac{1}{z}\big)^{-m} \bar{z}^n + m\big(\frac{1}{z} - 1\big) - n\big(\bar{z} - 1\big) - 1$$

is μ-integrable.

We have

$$\begin{aligned} &z^m \bar{z}^n - m(z-1) - n(\bar{z}-1) - 1 \\ &= \left(\frac{1}{z}\right)^{-m} \bar{z}^n + m\left(\frac{1}{z} - 1\right) - n(\bar{z}-1) - 1 - m\frac{(1-z)^2}{z}, \end{aligned}$$

which means that the functions

$$z \mapsto 1 - z^m \bar{z}^n + m(z-1) + n(\bar{z}-1)$$

and $z \mapsto 1 - \mathrm{Re}\ z^m \bar{z}^n$ are μ-integrable.

The other cases are proved in a similar way. It follows that we can choose the function $L : \mathbb{Z}^2 \times (T \setminus \{1\}) \to \mathbb{C}$ defined by $L((m,n),z) = (m-n)\mathrm{Im}\ z$ as a Lévy function for $\mathbb{Z}^2$, and that $\mathcal{M}(\mathbb{Z}^2) = \{\mu$ positive Radon measure on $T \setminus \{1\} |$ the function $z \mapsto 1 - \mathrm{Re}\ z$ is μ-integrable$\}$.

Now the proposition is a particular case of the theorem given in Section 2. □

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