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THE STRUCTURE OF C^*-EXTREME POINTS IN SPACES OF COMPLETELY POSITIVE LINEAR MAPS ON C^*-ALGEBRAS

DOUGLAS R. FARENICK AND HONGDING ZHOU

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Abstract. If A is a unital C^*-algebra and if H is a complex Hilbert space, then the set $S_H(A)$ of all unital completely positive linear maps from A to the algebra $B(H)$ of continuous linear operators on H is an operator-valued, or generalised, state space of A. The usual state space of A occurs with the one-dimensional Hilbert space $\mathbb{C}$. The structure of the extreme points of generalised state spaces was determined several years ago by Arveson [*Acta Math.* 123(1969), 141-224]. Recently, Farenick and Morenz [*Trans. Amer. Math. Soc.* 349(1997), 1725-1748] studied generalised state spaces from the perspective of noncommutative convexity, and they obtained a number of results on the structure of C^*-extreme points. This work is continued in the present paper, and the main result is a precise description of the structure of the C^*-extreme points of the generalised state spaces of A for all finite-dimensional Hilbert spaces H.

1. Introduction

The subject of this paper concerns a certain quantization of the state space of a C^*-algebra and its accompanying convexity theory, whereby the values of the states and the values of the convex coefficients are operators acting on a Hilbert space. The operator-valued states are unital completely positive maps, and the form of convexity under consideration is C^*-convexity. Building on the work in [6], the objective of the present paper is to give a precise description of the structure of the C^*-extreme points in the space of all unital completely positive maps $A \to B(H)$, where A is an arbitrary unital C^*-algebra and H is any finite-dimensional Hilbert space.

Consider a unital C^*-algebra A and a complex Hilbert space H, and let $S_H(A)$ denote the set of all unital completely positive linear maps $\varphi : A \to B(H)$, where $B(H)$ denotes the C^*-algebra of continuous linear operators on H. In the bounded-weak-topology, $S_H(A)$ is a compact space whose elements function as generalised, or operator-valued, states on A. The set $S_H(A)$ is convex with respect to operator-valued convex coefficients, which means that for every finite subset $\{\varphi_1, \dots, \varphi_n\} \subseteq S_H(A)$ and for all operators $t_1, \dots, t_n \in B(H)$ such that $t_1^*t_1 + \cdots + t_n^*t_n = 1$, the

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completely positive map $x \mapsto \sum_{i=1}^n t_i^* \varphi_i(x) t_i$, for $x \in A$, is an element of $S_H(A)$. This form of convexity is called C^*-convexity, and it has been considered previously in [4], [5], [6], [8], [9], [10], [11]. There are, as well, other forms of operator-convexity that have been studied in connection with completely positive maps [3], [7], [15]; for example, the system $\{S_{\mathbb{C}^n}(A)\}_{n \in \mathbb{Z}^+}$ of convex sets under study in the present paper is "matrix convex" (in the dual space of A) in the sense of Wittstock [15] and Effros and Winkler [3]. For an overview of the developments in the field of convexity, and quantized functional analysis in general, see [2], [3], [15].

A unital completely positive map $\varphi \in S_H(A)$ is a *C^*-extreme point* of $S_H(A)$ if the only way to express φ as an operator-valued convex combination $\sum_{i=1}^n t_i^* \varphi_i t_i$ of finitely many unital completely positive linear maps φ_i, using *invertible* C^*-convex coefficients $t_i \in B(H)$, is with completely positive maps that are unitarily equivalent to φ. (Two completely positive maps $\varphi, \psi : A \to B(H)$ are unitarily equivalent if there exists a unitary operator $u \in B(H)$ such that $\psi(x) = u^* \varphi(x) u$ for all $x \in A$.) Observe that if φ is a C^*-extreme point, then so is every map unitarily equivalent to φ. It is known from [6] that for every C^*-algebra A and every Hilbert space H, there will always exist C^*-extreme points of $S_H(A)$, and any pure or multiplicative unital completely positive map is a C^*-extreme point; if H has finite dimension, then every C^*-extreme point is a linear extreme point, and the C^*-convex hull of the set of all C^*-extreme points of $S_H(A)$ is a dense subset, in the bounded-weak-topology, of $S_H(A)$. Of course when H is one-dimensional, the notions of generalised state, C^*-convexity, and C^*-extreme point all collapse to the ordinary ones of state, convexity, and linear extreme point.

Although it is natural to consider C^*-convex combinations of completely positive maps, in practice there are a number of technical difficulties that arise from the use of operators as convex coefficients, and the task of identifying C^*-extreme points among linear extreme points is compounded by the need to consider the entire unitary-equivalence class a completely positive map. For this reason, in Theorem 2.1 we describe, by way of the Stinespring decomposition, the structure of a particular representative in the unitary-equivalence class of a C^*-extreme point. This result has its antecedent in the characterisation of the extreme points of the state space via the Gelfand-Naimark-Segal decomposition of states, which, from our point of view, says that the extreme points of the state space are all the compressions to one-dimensional subspaces of all irreducible representations of the C^*-algebra. In Theorem 3.1 we obtain a characterisation of C^*-extreme points, akin to Arveson's theorem [1, Theorem 1.4.6] on extremal completely positive maps, that is independent of the choice of the representative from the unitary-equivalence class.

If φ is a completely positive linear map $A \to B(H)$, then we will write $\varphi = v^* \pi v$ to denote φ in its minimal Stinespring decomposition [12]. In this notation, π is a $*$-representation of A on a Hilbert space H_π and $v : H \to H_\pi$ is a bounded linear operator (and is an isometry if φ is unital), and the range of v is cyclic for the operator algebra $\pi(A)$. Recall further from [12] that all minimal decompositions of a completely positive map φ are unitarily equivalent. For a pair of completely positive maps ψ and φ, we write $\psi \le \varphi$ to indicate that $\varphi - \psi$ is completely positive. A completely positive map φ is said to be pure if the only completely positive maps ψ for which $\psi \le \varphi$ are those of the form $\psi = \lambda \varphi$ for some $\lambda \in [0,1] \subset \mathbb{R}$. The relation $\psi \le \varphi$ is described in full by an important theorem of Arveson [1, Theorem 1.4.2]: if $\varphi : A \to B(H)$ is a completely positive map with a minimal decomposition $\varphi = v^* \pi v$, then a completely positive map $\psi : A \to B(H)$ is such that $\psi \le \varphi$ if and only if there exists a (uniquely determined) positive contraction h in the commutant $\pi(A)'$ of the operator algebra $\pi(A) \subseteq B(H_\pi)$ such that $\psi(x) = v^* h \pi(x) v$ for every $x \in A$. In particular, φ is pure if and only if π is irreducible.

The linear extreme points of $S_H(A)$ are characterised by the following theorem of Arveson.

Theorem 1.1. ([1, Theorem 1.4.6]) *Suppose that $\varphi : A \to B(H)$ is a unital completely positive linear map with minimal decomposition $\varphi = v^*\pi v$. Let $\pi(A)'$ denote the commutant of $\pi(A)$ in $B(H_\pi)$ and let $\mathcal{C} : \pi(A)' \to B(v(H))$ be the linear map $\mathcal{C}(y) = v^* y v$, for all $y \in \pi(A)'$. Then φ is an extreme point of the convex set $S_H(A)$ if and only if $\mathcal{C}$ is an injection.*

2. Statememt of the main result

We first introduce some terminology. Suppose that $\varphi : A \to B(H)$ and $\psi : A \to B(K)$ are unital completely positive maps. We say that ψ is a *compression* of φ if there exists an isometry $w : K \to H$ such that $\psi(x) = w^*\varphi(x)w$ for all $x \in A$. An equivalent view is as follows. Suppose that $v^*\pi v$ is a minimal Stinespring decomposition of φ. Then ψ is a compression of φ if and only if ψ has a minimal decomposition of the form $v_1^*\pi v_1$, where the range of the isometry v_1 is contained in the range of v. A *nested sequence of compressions* of a $*$-representation π of A on a Hilbert space H_π is a sequence of unital completely positive maps $\varphi_j : A \to B(H_j)$ such that φ_{j+1} is a compression of φ_j, for each j, and φ_1 is a compression of π. Recall that if π_1 and π_2 are two representations of a C^*-algebra A, then π_1 and π_2 are said to be *disjoint* if no subrepresentation of one is unitarily equivalent to a subrepresentation of the other. In particular, two irreducible representations of A are disjoint if they are not unitarily equivalent. And, finally, some notation: given a completely positive map φ with minimal Stinespring decomposition $\varphi = v^*\pi v$, we will write φ^π in order to emphasize that φ is a compression of π (or, equivalently, that π is a dilation of φ).

Theorem 2.1. *Suppose that H is a finite-dimensional Hilbert space and that A is an arbitrary unital C^*-algebra, and consider an element $\varphi \in S_H(A)$. Then φ is a C^*-extreme point of $S_H(A)$ if and only if there exist finitely many pairwise nonequivalent irreducible representations $\pi_1, \pi_2, \ldots, \pi_k$ of A and nested sequences of compressions $\varphi_j^{\pi_i}$ $(1 \leq j \leq n_i)$ of each representation π_i such that φ is unitarily equivalent to the direct sum*

$$\sum_{i=1}^{k}{}^{\oplus} \left(\sum_{j=1}^{n_i}{}^{\oplus} \varphi_j^{\pi_i} \right)$$

of pure unital completely positive maps $\varphi_j^{\pi_i}$.

In [1], Arveson noted several instances where the results he had obtained for generalised states seemed to deviate significantly from the corresponding theorems for the standard state space. This is due, perhaps, to the fact that ordinary convexity was considered in [1] in conjunction with operator-valued states. One of the main points of [6] is that by basing the convexity theory for $S_H(A)$ on one in which operator-valued convex coefficients are used, the structure of the extremal elements in the quantum context better reflect the classical results concerning states on C^*-algebras. Below are two explicit examples of how C^*-extreme points behave more like classical extremal states than linear extreme points do.

Corollary 2.2. *Suppose that H has finite dimension and that $\varphi : A \to B(H)$ is a unital completely positive map. Let L_φ denote the left-kernel of φ, namely the set of all $x \in A$ for which $\varphi(x^*x) = 0$. If φ is a C^*-extreme point of $S_H(A)$, then the following two statements hold, although they may fail if φ is assumed to be only a linear extreme point.*

(1) $L_\varphi + L_\varphi^* = \ker \varphi$.

(2) *If A is commutative, then φ is multiplicative.*

Proof. For (1), Tsui has proved [13] that $L_\varphi + L_\varphi^* = \ker\varphi$ whenever φ is pure, and that there exist linear extreme points $\psi \in S_H(A)$ for which $L_\psi + L_\psi^* \neq \ker\psi$. Thus, for the completely positive map φ in its direct sum decomposition given by Theorem 2.1,

$$\ker\varphi = \bigcap_{i=1}^{k} \bigcap_{j=1}^{n_i} \ker \varphi_j^{\pi_i} = \bigcap_{i=1}^{k} \bigcap_{j=1}^{n_i} L_{\varphi_j^{\pi_i}} + L^*_{\varphi_j^{\pi_i}} = L_\varphi + L_\varphi^* .$$

(See, as well, Proposition 4.4 of [11].)

To prove (2), note that the irreducible representations of A take place on 1-dimensional Hilbert spaces, and so the completely positive maps that arise in the direct sum decomposition of a C^*-extreme point are all scalars, with nonequivalent irreducible representations being distinct characters of A. Nonmultiplicative linear extreme points are found in [1, Theorem 1.4.10] (see [6, Prop. 2.2] as well). $\square$

More interesting still is that Theorem 2.1 is very specific about the combinatorial structure of the direct sum decomposition. Let us consider briefly an example in a noncommutative C^*-algebra to better illustrate the essence of the theorem.

Consider the "mixing" state $\varphi \in S_{\mathbb{C}^4}(M_3)$ given by

$$\varphi\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{12} & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ 0 & 0 & x_{11} & x_{13} \\ 0 & 0 & x_{31} & x_{33} \end{pmatrix} .$$

Let $\xi_1, \xi_2, \xi_3 \in \mathbb{C}^3$ and $\eta_1, \eta_2, \dots, \eta_6 \in \mathbb{C}^6$ denote the standard orthonormal basis vectors for $\mathbb{C}^3$ and $\mathbb{C}^6$. A minimal decomposition of φ is given by $\varphi = v^*\pi v$, where $\pi : M_3 \to M_6$ is defined to be $\pi(x) = x \oplus x$, for all $x \in M_3$, and where $v = [\eta_1, \eta_2, \eta_4, \eta_6] : \mathbb{C}^4 \to \mathbb{C}^6$. Clearly φ is a direct sum of two pure generalised states $\varphi_i \in S_{\mathbb{C}^2}(M_3)$, namely

$$\varphi_1(x) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \qquad \text{and} \qquad \varphi_2(x) = \begin{pmatrix} x_{11} & x_{13} \\ x_{31} & x_{33} \end{pmatrix} .$$

It is very easy to compute the commutant of $\pi(M_3)$ and then to use Arveson's theorem (Theorem 1.1) to verify that φ is a linear extreme point of $S_{\mathbb{C}^4}(M_3)$; so, in this case, the direct sum $\varphi_1 \oplus \varphi_2$ of pure maps (and C^*-extreme points) is a linear extreme point. However, because the two generalised states φ_1 and φ_2 "mix" the 2-dimensional spaces on which their dilations, namely (in both cases) the identity representation I on M_3, are compressed, neither pure map is a compression of the other and therefore, by Theorem 2.1, this linear extreme point $\varphi_1 \oplus \varphi_2$ is not a C^*-extreme point. More precisely, φ_1 is a compression of I to the space L_1 spanned by ξ_1 and ξ_2, whereas φ_2 is the compression of I to the subspace L_2 spanned by ξ_1 and ξ_3; we have that $L_1 \cap L_2$ is one-dimensional rather than $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$, as would be the case if $\varphi_1 \oplus \varphi_2$ were a C^*-extreme point. To have a C^*-extreme point, one needs to compress φ_1, as is done in the map $\psi \in S_{\mathbb{C}^4}(M_3)$ given by

$$\psi(x) = \begin{pmatrix} x_{11} & x_{12} & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ 0 & 0 & x_{11} & 0 \\ 0 & 0 & 0 & x_{11} \end{pmatrix} .$$

We have, with this example, compressed every 3×3 matrix x to its 2×2 upper left-hand corner $\varphi_1(x)$ and added to it two copies of the compression of $\varphi_1(x)$ to its upper 1×1 left-hand corner.

The structure, therefore, of linear extreme points can be very subtle, allowing nontrivial interactions (or mixing) between pure operator-valued states; the structure of C^*-extreme points is much more rigid and predictable in that any two pure direct summands are either disjoint or one is a compression of the other.

The remainder of the paper is devoted to the proof of Theorem 2.1.

3. Characterisation of C^*-extreme points by compression of the commutant of the dilation

In this section we develop an analogue for C^*-convexity of Arveson's characterisation of linear extreme points (Theorem 1.1).

Definition. If $\varphi : A \to B(H)$ is a unital completely positive map with minimal Stinespring decomposition $\varphi = v^*\pi v$, then a positive operator $h \in \pi(A)'$ is said to be *φ-invertible* if v^*hv is invertible in $B(H)$.

Theorem 3.1. *Let A be a unital C^*-algebra and H be an arbitrary Hilbert space, and let $\varphi = v^*\pi v$ denote a minimal decomposition of a unital completely positive map $\varphi : A \to B(H)$. Then the following statements are equivalent.*

(1) *φ is a C^*-extreme point of $S_H(A)$.*

(2) *For every φ-invertible positive operator $h \in \pi(A)'$ there are a unitary operator $u \in \pi(A)'$ and an invertible operator $z \in B(H)$ such that $uh^{\frac{1}{2}}v = vz$.*

Proof. Assume that φ is a C^*-extreme point. Choose any φ-invertible positive element $h \in \pi(A)'$. Then there exists a real number $\alpha \in (0,1)$ such that $h_1 = \alpha h$ and $h_2 = 1 - \alpha h$ are φ-invertible positive operators. Observe that $h_1 + h_2 = 1$; thus,

$$\varphi = v^*\pi v = v^*\pi(h_1 + h_2)v = v^*h_1^{\frac{1}{2}}\pi h_1^{\frac{1}{2}}v + v^*h_2^{\frac{1}{2}}\pi h_2^{\frac{1}{2}}v .$$

For each i, let $s_i = v^*h_iv$ and $t_i = s_i^{\frac{1}{2}}$. Then t_1 and t_2 are invertible operators such that

$$t_1^*t_1 + t_2^*t_2 = s_1 + s_2 = v^*(h_1 + h_2)v = 1 .$$

Therefore

$$\varphi = \sum_{i=1}^{2} t_i^*(t_i^{*-1}v^*h_i^{\frac{1}{2}}\pi h_i^{\frac{1}{2}}vt_i^{-1})t_i$$

is a representation of φ as a proper operator-convex combination of generalised states $\varphi_i = (h_i^{\frac{1}{2}}vt_i^{-1})^*\pi(h_i^{\frac{1}{2}}vt_i^{-1}) \in S_H(A)$. Because φ is a C^*-extreme point of

$S_H(A)$, there exists a unitary $w \in B(H)$ such that

$$v^*\pi v = w^*\varphi_1 w = (h_1^{\frac{1}{2}} v t_1^{-1} w)^* \pi (h_1^{\frac{1}{2}} v t_1^{-1} w)\,.$$

This means that φ is now given by two different minimal Stinespring decompositions. As all minimal decompositions are unitarily equivalent, there is a unitary operator $u \in B(H_\pi)$ such that $u(h_1^{\frac{1}{2}} v t_1^{-1} w) = v$ and $u\pi(x) = \pi(x)u$ for all $x \in A$. Hence, $u \in \pi(A)'$ and $uh_1^{\frac{1}{2}} v = vz_1$, where $z_1 = w^* t_1$ is invertible in $B(H)$. Thus, $uh^{\frac{1}{2}} v = vz$, where $z = \frac{1}{\alpha} z_1$. This proves that statement (1) implies statement (2).

Suppose now that statement (2) holds. Let $\varphi = \sum_{i=1}^n t_i^* \varphi_i t_i$ be a representation of φ as a C^*-convex combination of generalised states φ_i using invertible operator-convex coefficients t_i. Fix i. From $t_i^*\varphi_i t_i \leq \varphi$, there is a unique positive operator $h_i \in \pi(A)'$ such that $t_i^*\varphi_i t_i = v^* h_i^{\frac{1}{2}} \pi h_i^{\frac{1}{2}} v$. Because $t_i^* t_i$ is invertible, $(v^* h_i v)^{-1}$ exists. Hence, by hypothesis, there exist a unitary $u_i \in \pi(A)'$ and an invertible operator $z_i \in B(H)$ such that $u_i h_i^{\frac{1}{2}} v = v z_i$. Now because u_i commutes with $\pi(A)'$,

$$\begin{aligned}\varphi_i &= (h_i^{\frac{1}{2}} v t_i^{-1})^* \pi (h_i^{\frac{1}{2}} v t_i^{-1}) = (h_i^{\frac{1}{2}} v t_i^{-1})^* \pi (h_i^{\frac{1}{2}} u_i^* u_i v t_i^{-1}) = (u_i h_i^{\frac{1}{2}} v t_i^{-1})^* \pi (u_i h_i^{\frac{1}{2}} v t_i^{-1}) \\ &= (v z t_i^{-1})^* \pi (v z t_i^{-1}) = (z t_i^{-1})^* \varphi (z t_i^{-1})\,.\end{aligned}$$

Therefore from $\varphi(1) = \varphi_i(1) = 1$, the operator $z t_i^{-1}$ is an invertible isometry and, hence, unitary. $\square$

Corollary 3.2 (cf. [6, Prop. 1.2(2)])**.** *If $p = vv^*$ commutes with $\pi(A)'$, then φ is a C^*-extreme point.*

Proof. Suppose that $h \in \pi(A)'$ is φ-invertible. Then

$$v^* h v = v^* v v^* h^{\frac{1}{2}} h^{\frac{1}{2}} v = v^* h^{\frac{1}{2}} v v^* h^{\frac{1}{2}} v = (v^* h^{\frac{1}{2}} v)(v^* h^{\frac{1}{2}} v)\,,$$

and so $z = v^* h^{\frac{1}{2}} v$ is invertible. Thus, $h^{\frac{1}{2}} v = h^{\frac{1}{2}} v v^* v = v v^* h^{\frac{1}{2}} v = vz$. By Theorem 3.1, φ must be an extreme point. $\square$

Corollary 3.3. *If H has finite dimension, then $\varphi = v^*\pi v$ is a C^*-extreme point of $S_H(A)$ if and only if for every positive operator $h \in \pi(A)'$ there are a unitary $u \in \pi(A)'$ and an operator $z \in B(H)$ such that $uh^{\frac{1}{2}} v = vz$.*

Proof. Assume that $\varphi = v^*\pi v$ is a C^*-extreme point of $S_H(A)$ and that $h \in \pi(A)'$ is positive. Then for every $\lambda > 0$, the positive operator $h_\lambda = \lambda 1 + h$ is φ-invertible. Therefore there are a unitary $u_\lambda \in \pi(A)'$ and an invertible operator $z_\lambda \in B(H)$ such that $u_\lambda h_\lambda^{\frac{1}{2}} v = v z_\lambda$. Consider a subsequence λ_n such that $\lim_n \lambda_n = 0$. Because the unitary group of $\pi(A)'$ is compact in the weak-operator-topology, the sequence $\{u_{\lambda_n}\}_n$ has a subsequence converging weakly to some unitary $u \in \pi(A)'$. (In fact, $\pi(A)'$ has finite dimension—because φ is a linear extreme point—and so the unitary group of $\pi(A)'$ is compact in the norm-topology.) Denote this subsequence once again by $\{\lambda_n\}_n$. Now because $h_{\lambda_n}^{\frac{1}{2}}$ converges to $h^{\frac{1}{2}}$ in norm, the sequence $\{z_{\lambda_n}\}_n$ is also convergent to, say, z. With the limit-operators u, h, and z we have $uh^{\frac{1}{2}} v = vz$.

To prove the converse, assume that the stated property holds for all positive $h \in \pi(A)'$; however Theorem 3.1 requires the use of φ-invertible operators. So, suppose that $h \in \pi(A)'$ is positive and φ-invertible. By hypothesis there are a unitary $u \in \pi(A)'$ and an operator $z \in B(H)$ such that $uh^{\frac{1}{2}} v = vz$. It remains only to show that z is invertible. Let $p = vv^*$; then $puh^{\frac{1}{2}}v = vuh^{\frac{1}{2}}v$ and therefore, with $z = v^*uh^{\frac{1}{2}}v$,

$$z^*z = (v^*h^{\frac{1}{2}}u^*v)(v^*uh^{\frac{1}{2}}v) = v^*h^{\frac{1}{2}}u^*puh^{\frac{1}{2}}v = v^*h^{\frac{1}{2}}u^*uh^{\frac{1}{2}}v = v^*hv\,.$$

Because v^*hv is invertible and H has finite dimension, z is invertible. $\square$

4. Proof of the necessity of the conditions in Theorem 2.1

Assume that H is a finite-dimensional Hilbert space and that $\varphi \in S_H(A)$ is a C^*-extreme point. By Theorem 2.1 of [6], φ is unitarily equivalent to a direct sum of pure completely positive maps; that is, there exist finitely many pairwise disjoint irreducible representations $\pi_1, \pi_2, \ldots, \pi_k$ of A, subspaces H_j^i of H $(1 \leq j \leq n_i)$, and compressions $\varphi_j^{\pi_i} : A \to B(H_j^i)$ $(1 \leq j \leq n_i)$ of each representation π_i such that

$$H = \sum_{i=1}^{k}{}^{\oplus} \left(\sum_{j=1}^{n_i}{}^{\oplus} H_j^i \right)$$

and, with respect to this decomposition of H,

$$\varphi = \sum_{i=1}^{k}{}^{\oplus} \left(\sum_{j=1}^{n_i}{}^{\oplus} \varphi_j^{\pi_i} \right) .$$

If for each $i = 1, \ldots, k$ we arrange the Hilbert spaces $H_1^i, \ldots, H_{n_i}^i$ so that the dimension of H_{j+1}^i is less than or equal to the dimension of H_j^i, then what remains to be proven is that for every $i = 1, \ldots, k$ and every $j = 1, \ldots, n_i - 1$ the map $\varphi_{j+1}^{\pi_i}$ is a compression of $\varphi_j^{\pi_i}$; to do this we use the following two lemmas.

Lemma 4.1. *Suppose that H is a finite-dimensional Hilbert space with a direct sum decomposition $H = H_1 \oplus \cdots \oplus H_k$ and that $\varphi = v^*\pi v \in S_H(A)$ has, with respect to this decomposition of H, a decomposition into a direct sum of unital completely positive maps $\varphi_j : A \to B(H_j)$, for $j = 1, \ldots, n$. If φ is a C^*-extreme point of $S_H(A)$, then $\varphi_1 \oplus \varphi_2$ is a C^*-extreme point of $S_{H_1 \oplus H_2}(A)$.*

Lemma 4.2. *If φ_1 and φ_2 are pure unital completely positive maps of A into $B(H_1)$ and $B(H_2)$, where H_1 and H_2 are finite-dimensional with* $\dim H_2 \leq \dim H_1$, *and if both φ_1 and φ_2 are compressions of the same irreducible representation π of A, then $\varphi_1 \oplus \varphi_2$ is a C^*-extreme point of $S_{H_1 \oplus H_2}(A)$ only if φ_2 is a compression of φ_1.*

With Lemma 4.1 we have that $\varphi_j^{\pi_i} \oplus \varphi_{j+1}^{\pi_i}$ is a C^*-extreme point in the space of all unital completely positive maps $A \to B(H_j^i \oplus H_{j+1}^i)$. Because $\varphi_j^{\pi_i}$ and $\varphi_{j+1}^{\pi_i}$ are compressions of the same irreducible representation π_i, Lemma 4.2 asserts that $\varphi_{j+1}^{\pi_i}$ is a compression of $\varphi_j^{\pi_i}$, which is the necessity condition of Theorem 2.1.

Proof of Lemma 4.1. To prove that $\varphi_1 \oplus \varphi_2$ is a C^*-extreme point of $S_{H_1 \oplus H_2}(A)$, suppose that $s_1 \ldots, s_r \in B(H_1 \oplus H_2)$ are invertible operator-valued convex coefficients and that $\theta_1, \ldots, \theta_r \in S_{H_1 \oplus H_2}(A)$ are such that

$$\varphi_1 \oplus \varphi_2 = \sum_{j=1}^{r} s_j^* \theta_j s_j \,.$$

Extend the equation above to $B(H)$: for $j = 1, \dots, r$, let

$$t_j = s_j \oplus \sqrt{\frac{1}{r}} 1_{H_3} \oplus \cdots \oplus \sqrt{\frac{1}{r}} 1_{H_r} \,,$$

which are invertible operators on $H = (H_1 \oplus H_2) \oplus H_3 \oplus \cdots \oplus H_r$ such that $\sum_j t_j^* t_j = 1$, and let

$$\psi_j = \theta_j \oplus \varphi_3 \oplus \cdots \oplus \varphi_r : A \to B((H_1 \oplus H_2) \oplus H_3 \oplus \cdots \oplus H_r)\,.$$

Then $\varphi = \sum_j t_j^* \psi_j t_j$. Because φ is a C^*-extreme point, there exist unitaries $w_1, \dots, w_r \in B(H)$ such that $w_j^* \varphi w_j = \psi_j$ for each j. Select any j; we will prove that there is a unitary operator $u_j \in B(H_1 \oplus H_2)$ such that $u_j^*(\varphi_1 \oplus \varphi_2) u_j = \theta_j$.

The linear manifold $\varphi(A)$ has finite dimension and therefore there exist $x_1, \dots, x_n \in A$ such that $\{\varphi(x_1), \dots, \varphi(x_n)\}$ is a basis of $\varphi(A)$. Let $\mathfrak{g}$ denote any word in $2n$ noncommuting variables. Then the operator

$$\mathfrak{g}(\varphi(x_1), \varphi(x_1)^*, \dots, \varphi(x_n), \varphi(x_n)^*)$$

is block-diagonal and unitarily equivalent (via the unitary w_j) to the block-diagonal operator

$$\mathfrak{g}(\psi_j(x_1), \psi_j(x_1)^*, \dots, \psi_j(x_n), \psi_j(x_n)^*)\,.$$

Thus these two operators have equal traces, which we now calculate. Note that

$$\varphi_{|H_3 \oplus \cdots \oplus H_k} = \psi_{j\,|H_3 \oplus \cdots \oplus H_k}\,,$$

and so

$$\begin{aligned}
&\operatorname{tr}(\mathfrak{g}(\varphi(x_1), \varphi(x_1)^*, \dots, \varphi(x_n), \varphi(x_n)^*)) \\
&\quad = \operatorname{tr}(\mathfrak{g}(\varphi_1 \oplus \varphi_2(x_1), \dots, \varphi_1 \oplus \varphi_2(x_n)^*)) \\
&\qquad + \sum_{i=3}^{k} \operatorname{tr}(\mathfrak{g}(\varphi_i(x_1), \varphi_i(x_1)^*, \dots, \varphi_i(x_n), \varphi_i(x_n)^*))
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{tr}(\mathfrak{g}(\psi_j(x_1), \psi_j(x_1)^*, \dots, \psi_j(x_n), \psi_j(x_n)^*)) \\
&= \operatorname{tr}(\mathfrak{g}(\theta_j(x_1), \dots, \theta_j(x_n)^*)) + \sum_{i=3}^{k} \operatorname{tr}(\mathfrak{g}(\varphi_i(x_1), \varphi_i(x_1)^*, \dots, \varphi_i(x_n), \varphi_i(x_n)^*))\,.
\end{aligned}$$

Therefore,

$$\operatorname{tr}(\mathfrak{g}(\varphi_1 \oplus \varphi_2(x_1), \dots, \varphi_1 \oplus \varphi_2(x_n)^*)) = \operatorname{tr}(\mathfrak{g}(\theta_j(x_1), \dots, \theta_j(x_n)^*))\,.$$

As the trace-equation above is true for every word $\mathfrak{g}$ in $2n$ noncommuting variables, there is, by the version [14] (for finite sets of matrices) of Specht's theorem, a unitary operator $u_j \in B(H_1 \oplus H_2)$ such that

$$u_j^* \begin{pmatrix} \varphi_1(x_i) & 0 \\ 0 & \varphi_2(x_i) \end{pmatrix} u_j = \theta_j(x_i) \qquad \text{for every} \quad i = 1 \dots, n\,.$$

We now extend the unitary-equivalence relations above from the elements $x_1, \dots, x_n \in A$ to all of A. Choose any $x \in A$. Then there are uniquely determined $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that $\varphi(x) = \sum_i \alpha_i \varphi(x_i)$. By restricting to $H_1 \oplus H_2$ we obtain

$$\varphi_1 \oplus \varphi_2(x) = \sum_{i=1}^{n} \alpha_i \varphi_1 \oplus \varphi_2(x_i) = \varphi_1 \oplus \varphi_2\Big(\sum_{i=1}^{n} \alpha_i x_i\Big)\,,$$

and thus

$$u_j^* \left(\varphi_1 \oplus \varphi_2(x)\right) u_j = \theta_j(\sum_{i=1}^{n} \alpha_i x_i) .$$

It remains to show that $\theta_j(\sum_i \alpha_i x_i) = \theta_j(x)$. Let $y = x - \sum_i \alpha_i x_i$. Because $\varphi(y)$ and $\psi_j(y)$ are unitarily equivalent, the operators $\varphi(y)$ and $\psi_j(y)$ have the same rank. However, if it were true that $\theta_j(\sum_i \alpha_i x_i) \neq \theta_j(x)$, then it would be true as well that the rank of $\varphi(y)$ is strictly less than the rank of $\psi_j(y)$, which is impossible. Thus $u_j^* \left(\varphi_1 \oplus \varphi_2(x)\right) u_j = \theta_j(x)$ for all $x \in A$, and so $\varphi_1 \oplus \varphi_2$ is a C^*-extreme point of $S_{H_1 \oplus H_2}(A)$. This completes the proof of Lemma 4.1. □

Proof of Lemma 4.2. Suppose that $\varphi_i = v_i^* \pi v_i \in S_{H_i}(A)$, for $i = 1, 2$, where π is an irreducible representation of A on a Hilbert space H_π. Let $\varrho = \pi \oplus \pi$, a representation of A on $H_\varrho = H_\pi \oplus H_\pi$. Let w_1, w_2 be the canonical injections of H_π into H_ϱ with ranges $H_\pi \oplus \{0\}$ and $\{0\} \oplus H_\pi$, and let $e_1, e_2 \in B(H_1 \oplus H_2)$ be projections onto H_1 and H_2. Then a minimal decomposition of $\varphi_1 \oplus \varphi_2$ is given by $v^* \varrho v$, where $v = \sum_j w_j v_j e_j$; that is $\varphi_1 \oplus \varphi_2 = \sum_j e_j v_j^* \pi v_j e_j$. The commutant of $\varrho(A)$ is isomorphic to M_2.

By Corollary 3.3, for every positive operator $h \in \varrho(A)'$ there are a unitary operator $u \in \varrho(A)'$ and an operator $z \in B(H_1 \oplus H_2)$ such that $uh^{\frac{1}{2}} v = vz$; writing this in matricial form, we get

$$\begin{pmatrix} \gamma_{11} v_1 & \gamma_{12} v_2 \\ \gamma_{21} v_1 & \gamma_{22} v_2 \end{pmatrix} = \begin{pmatrix} v_1 z_{11} & v_1 z_{12} \\ v_2 z_{21} & v_2 z_{22} \end{pmatrix} ,$$

where

$$\begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} = uh^{\frac{1}{2}} .$$

The unitary u and the matrix entries γ_{ij} depend on h; write this dependence as $\gamma_{ij} = \gamma_{ij}(h)$. If for some positive h, $\gamma_{12} \neq 0$, then we solve the matrix equation above to obtain $v_2 = \gamma_{12}^{-1}(v_1 z_{12})$, which shows that the range of v_2 is a subspace in the range of v_1 and so φ_2 is a compression of φ_1. Suppose, though, that $\gamma_{12} = 0$ for all positive $h \in \varrho(A)'$. If there is a positive operator h for which $\gamma_{21} \neq 0$, then the matrix equation reveals that $\gamma_{21} v_1 = v_2 z_{21}$; as $\dim H_2 \leq \dim H_1$, this means that the range of v_1 equals the range of v_2 and so, again, φ_2 is a compression of φ_1. It cannot happen that $\gamma_{12} = \gamma_{21} = 0$ for all positive $h \in \varrho(A)'$, for if this were true, then the square root of the rank-1 matrix $h = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ would be mapped to a diagonal matrix by some 2×2 unitary matrix u; however such a diagonal matrix would have identical columns and so its rank would be 0, not 1. This completes the proof of Lemma 4.2. □

5. Proof of the sufficiency of the conditions in Theorem 2.1

Assume now that $\pi_1, \ldots, \pi_k$ are pairwise disjoint irreducible representations of A on Hilbert spaces H_{π_i} and that

$$\varphi = \sum_{i=1}^{k}{}^{\oplus} (\sum_{j=1}^{n_i}{}^{\oplus} \varphi_j^{\pi_i}),$$

where for each i, $\varphi_{j+1}^{\pi_i}$ is a compression of $\varphi_j^{\pi_i}$. Here $H = H_1 \oplus \cdots \oplus H_k$, where each $H_i = H_1^i \oplus \cdots \oplus H_{n_i}^i$ and $\varphi_j^{\pi_i}$ is a unital completely positive map $A \to B(H_j^i)$. For each $i = 1, \ldots, k$, let $\varrho_i = \pi_i \otimes 1_{n_i}$, which is a representation of A on the Hilbert space $H_{\varrho_i} = H_{\pi_i} \otimes \mathbb{C}^{n_i}$. Let $\pi = \varrho_1 \oplus \cdots \oplus \varrho_k$, a representation of A on $H_\pi = \sum_i^\oplus H_{\varrho_i}$. Because $\varrho_1, \ldots, \varrho_k$ are pairwise disjoint, the commutant $\pi(A)'$ has a direct sum decomposition, with respect to the decomposition $H_\pi = \sum_i^\oplus H_{\varrho_i}$, such that each direct summand of $\pi(A)'$ is $\varrho_i(A)'$, which is isomorphic to the matrix algebra M_{n_i}. Let $p_1, \ldots, p_n \in B(H_\pi)$ be projections with the range of each p_i being H_{ϱ_i}. Now denote the minimal decomposition of $\varphi^{\varrho_i} : A \to B(H_i)$ by $\varphi^{\varrho_i} = w_i^* \varrho_i w_i$, where

$$\varphi^{\varrho_i} = \sum_{j=1}^{n_i}{}^{\oplus} \varphi_j^{\pi_i}$$

and $w_i : H_i \to H_{\varphi_i}$ is an isometry, and let $e_1, \ldots, e_n \in B(H)$ be projections such that $\operatorname{ran} e_j = H_j$ for every j. Then $\varphi^{\varrho_i} = e_i \varphi_{|_{H_i}}$ and $\varphi = \sum_j e_j \varphi e_j$. The operator $v : H \to H_\pi$ given by $v = \sum_i w_i e_i$ is an isometry, and

$$\varphi = v^* \pi v = \sum_{i=1}^n v^* p_i \pi p_i v = \sum_{i=1}^n e_i w_i^* \varrho_i w_i e_i$$

is a minimal decomposition of φ. Let $h \in \pi(A)'$ be any positive operator. Then there exist positive operators $h_i \in \varrho_i(A)'$ such that $h = h_1 \oplus \cdots \oplus h_k$. By Theorem 3.3 of [6], the completely positive maps φ^{ϱ_i} are C^*-extreme points of $S_{H_i}(A)$, and so by Corollary 3.3 there exist unitaries $u_i \in \varrho_i(A)'$ and operators $z_i \in B(H_i)$ such that $u_i h_i^{\frac{1}{2}} w_i = w_i z_i$ for every i. With $u = u_1 \oplus \cdots \oplus u_k$ and $z = z_1 \oplus \cdots \oplus z_k$, the operator u is a unitary element of $\pi(A)'$, z is an element of $B(H)$, and $u h^{\frac{1}{2}} v = vz$. Hence, again by Corollary 3.3, φ is a C^*-extreme point of $S_H(A)$, which completes the proof of Theorem 2.1.

Remark. We have concentrated on Hilbert spaces of finite dimension because the state space itself involves a one-dimensional Hilbert space. Suppose, though, that H has infinite dimension. Then the conditions of Theorem 2.1 are no longer necessary for C^*-extreme points of $S_H(A)$ (see Example 2 of [6]), although it seems reasonable to conjecture that these conditions remain sufficient. If the C^*-algebra A is already represented as an operator algebra acting on H, then many of the important completely positive maps $A \to B(H)$ are C^*-extreme points of $S_H(A)$. In particular, every endomorphism of $B(H)$ is, by virtue of being multiplicative, a C^*-extreme point among all unital completely positive maps $B(H) \to B(H)$. Indeed, if an endomorphism α of $B(H)$ has the form $\alpha(x) = \sum_i t_i^* x t_i$, for some co-isometries $t_1, \ldots, t_n \in B(H)$ such that $t_1^* t_1 + \cdots + t_n^* t_n = 1$, then α is obtained from the identity endomorphism $I \in S_H(B(H))$ through a C^*-convex combination.

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Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada S4S 0A2

E-mail address: farenick@math.uregina.ca

E-mail address: zhouho@math.uregina.ca