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PERIODIC GROUPS OF OPERATORS IN BANACH SPACES

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Abstract. Spectral operators of scalar type in the sense of Dunford often occur in connection with unconditionally convergent series expansions, whereas conditionally convergent expansions under similar conditions may be described with the help of operators having a more general type of spectral decomposition. We show that under certain conditions even in the latter case we can restrict our considerations to a dense linear submanifold of the original Banach space with a stronger topology, where the convergence of the expansion under study will be unconditional. Though our conditions could be formulated in terms of a single operator, it seems to be more natural to state them in terms of (the generator of) a periodic group of operators.

1. Introduction

It is well-known that the generator of a periodic (C_0)-group of operators in a Banach space is, in general, not a spectral operator of scalar type in the sense of Dunford [DS]. The main result of this paper is that for each such group there is a reasonably large linear submanifold of the underlying Banach space and a new norm, majorizing the old one, on it such that the submanifold with the new norm is a Banach space invariant for all operators in the group, and the restrictions to this space form a (C_0)-group of operators with generator that is a spectral operator of scalar type. The main technique of the proofs is a suitably modified version of some ideas of Smart [Sma]. In Section 3 some classical examples concerning Fourier series and pertaining to these results will be presented. In Section 4 we shall show how the main result can be applied to the study of more general (non-periodic) semigroups of operators. Section 5 will consider the special case of a periodic group in a reflexive Banach space: our submanifold is then identical to the semisimplicity manifold defined in reflexive spaces by S. Kantorovitz (see [K1], [K2], [KH]), the two norms on it are equivalent, and the integral representations in the cited papers are valid in this topology, which is stronger than the original one of X.

Let X be a complex Banach space with norm $\|\cdot\|$, and let $T = \{T(t); t \in \mathbf{R}\}$ be a group of class (C_0) of bounded linear operators in X with period p. Without restricting the generality we shall assume that $p = 2\pi$. The generating operator of the group T will be denoted by A with domain $D(A)$. It is well-known (see,

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e.g., [Nag]) that the spectrum $\sigma(A)$ of A is a subset of $i\mathbf{Z}$, the resolvent of A is a meromorphic function having poles only of order one with the residues P_n at in defined by

$$P_n x = (2\pi)^{-1} \int_0^{2\pi} e^{-int} T(t) x \, dt \quad (x \in X, n \in \mathbf{Z}),$$

and the operators P_n (the spectral projections) satisfy $P_n P_k = \delta_{nk} P_n$. The set of the projections $\{P_n; n \in \mathbf{Z}\}$ is clearly bounded. However, if we define $Q(F) := \sum_{n \in F} P_n$, where F (as everywhere in what follows) denotes any finite subset of $\mathbf{C}$, then the family of all such projections can be unbounded (see, e.g., [Dav]). For further properties of these projections we refer also to [HP] or [Nag].

In the following we shall need some basic properties of summable families of vectors in Banach spaces. For these properties we refer e.g. to [BP, Chap. 19]. We shall denote the fact that the family $\{x_d; d \in D\}$ is summable in the norm $\|\cdot\|$ to the sum x by

$$x = \sum_{d \in D} x_d = \sum_D x_d \qquad (\|\cdot\|),$$

and we shall omit the reference to the norm if no misunderstanding is possible.

If the index family D is the set $\mathbf{N}$ or $\mathbf{Z}$, then [Sma, Lemma 1.2] shows that

$$|x| := \sup_J \|\sum_J x_d\| = \sup_F \|\sum_F x_d\| < \infty,$$

where J varies over *all* and F varies over *all finite* subsets of D. Some properties of $|\cdot|$ proved in [Sma, p.604] will be used in the sequel.

2. Periodic groups

Suitably modifying a definition in [Sma, p.605], we start with the following:

Definition. Given the group T of period 2π on the Banach space X, let $Y = Y(T)$ be the set of vectors $x \in X$ such that

$$x = \sum_{n \in \mathbf{Z}} P_n x \qquad (\|\cdot\|),$$

with the norm

$$|x| := \sup_F \|\sum_F P_n x\|.$$

Note that, by the above remarks, $|\cdot|$ clearly defines a norm on the vectors of Y.

Lemma 1. *Y with the norm $|\cdot|$ is a Banach space.*

Proof. The only nontrivial part is to show that Y is complete with respect to $|\cdot|$. By the definition above, we have $|y| \geq \|y\|$ for every $y \in Y$. Assume that $\{y_n\}$ is a Cauchy sequence in Y with respect to $|\cdot|$. Then it is also Cauchy with respect to $\|\cdot\|$. Hence there is $x \in X$ such that $\|y_n - x\| \to 0$. For each finite subset F of $\mathbf{Z}$ and for every $w \in X$ let

$$E(F)w := \sum_{n \in F} P_n w.$$

Then, for any fixed F, as $n \to \infty$ we have

$$\|E(F)y_n - E(F)x\| = \|\sum_{k\in F} P_k(y_n - x)\| \to 0.$$

By assumption, for any $\varepsilon > 0$ there is an $N(\varepsilon)$ such that $n, m > N(\varepsilon)$ implies for every finite F that $\|E(F)y_n - E(F)y_m\| < \varepsilon$. Letting $m \to \infty$, we obtain for every F that $\|E(F)y_n - E(F)x\| \le \varepsilon$. Hence the convergence

$$(*) \qquad \lim_{n\to\infty} \|E(F)y_n - E(F)x\| = 0$$

is uniform on all finite subsets F of $\mathbf{Z}$. On the other hand, since every y_n belongs to Y, we have

$$\lim_F \|E(F)y_n - y_n\| = 0,$$

where the limit is taken with respect to the naturally directed family of all finite subsets F of $\mathbf{Z}$. Applying the theorem of E.H.Moore on interchanging the limits (cf. [DS, I.7.6]), we obtain that

$$\lim_F \|E(F)x - x\| = 0.$$

In other words, the family $\{P_k x; k \in \mathbf{Z}\}$ is summable to x, i.e. x belongs to the manifold Y. From $(*)$ we see then that

$$\lim_{n\to\infty} |y_n - x| = 0,$$

i.e. Y with the norm $|\cdot|$ is complete. □

Lemma 2. *For any Borel set b of $\mathbf{C}$ and any $y \in Y$ define*

$$E(b)y := \sum_{in\in b} P_n y \qquad (\|\cdot\|).$$

Then this convergence is unconditional (i.e. the family is summable to its sum), and the operator $E(b)$ is a bounded linear operator in the Banach space $(Y, |\cdot|)$. Further, E is a strongly countably additive spectral measure in this Banach space in the sense of [DS].

Proof. Since $y \in Y$, the entire family $\{P_n y; n \in \mathbf{Z}\}$ is summable to y, and the bounded multiplier property (see, e.g., [Sma, Lemma 1.1]) shows that the convergence to $E(b)y$ is also unconditional. Further, $in \in b$ implies

$$P_n E(b)y = P_n \sum_{ik\in b} P_k y = P_n y;$$

hence $E(b)y \in Y$. By [Sma, Lemma 1.3], we obtain that

$$|E(b)y| = |\sum_{in\in b} P_n y| \le 2\sup\{0,1\}|y| \le 2|y|$$

for every Borel set b and $y \in Y$. The multiplicative property of a spectral measure obtains as follows for any Borel sets b and c:

$$E(b)E(c)y = \sum_{ik\in b} P_k E(c)y = \sum_{ik\in b} P_k \sum_{in\in c} P_n y.$$

Since the operators P_n are bounded linear in Y and have the "orthogonality" property, the right-hand side is

$$\sum_{ik\in b\cap c} P_k y = E(b\cap c)y.$$

The other algebraic properties of E obtain similarly. In order to prove the countable additivity of E in the $|\cdot|$-topology of Y, let $\{b_n\}$ be any increasing sequence of Borel sets with union b, and let $y \in Y$. Then

$$|E(b)y - E(b_n)y| = |E(b\setminus b_n)y| = |\sum_{ik\in b\setminus b_n} P_k y| = \sup_{F\subseteq b\setminus b_n} \|\sum_{ik\in F} P_k y\|.$$

By our assumptions and by [Sma, Lemma 1.4], the right-hand side will be arbitrarily small, if n is sufficiently large. □

Theorem 1. *With the previous notation the domain of the generator satisfies $D(A) \subseteq Y$, and the assertions (i) and (ii) are equivalent: (i) $y \in D(A)$ and $Ay \in Y$; (ii)*

$$y = \sum_{k\in \mathbf{Z}} P_k y \quad (\|\cdot\|) \quad \textit{and} \quad Ay = \sum_{k\in\mathbf{Z}} ikP_k y \quad (\|\cdot\|).$$

Let W denote the linear submanifold of $D(A)$ defined by (i) or (ii). Then the restriction $B := A|W$ is a closed linear operator in the Banach space $(Y, |\cdot|)$, and is a spectral operator of scalar type with resolution of the identity E. Finally, B is the generator of a (C_0)-group $G = \{G(t); t \in \mathbf{R}\}$ of period 2π of bounded linear operators in Y, for which $G(t) = T(t)|Y$.

Proof. The first statement is proved, e.g., in [Nag, p.81]. Now assume (i). Since $P_k Ay = ikP_k y$ for every k, we obtain both statements of (ii). For similar reasons, (ii) implies (i). Now consider the finite subsets F of $i\mathbf{Z}$, and the subspaces

$$E(F)Y = \{y \in Y; y = \sum_{ik\in F} P_k y\} \subseteq W \subseteq D(A) \subseteq Y.$$

It is clear that these subspaces are B-invariant, and the corresponding restrictions $B(F) := B|E(F)Y = A|E(F)Y$ are $|\cdot|$-bounded spectral operators of scalar type, for which

$$B(F) = \int_F zE(dz) \qquad (F \text{ is finite}),$$

the integrals being the respective finite sums. Consider the linear manifold

$$D := \{y \in Y; \lim_F B(F)y \text{ exists in } |\cdot|\text{-norm}\},$$

where the limit is taken with respect to the naturally directed family of all finite subsets F of $i\mathbf{Z}$. [DS, XVIII.2.6 and 27] show that D is exactly the domain of definition of the closed spectral operator of scalar type S defined in Y by the countably additive spectral measure E. We shall show that $S = B$.

Assume first that $d \in D$. Then there exists

$$Sd = \lim_F B(F)d = \lim_F \sum_{ik\in F} ikP_k d \qquad (|\cdot|).$$

Since this norm majorizes $\|\cdot\|$, the same equalities in the old norm follow. Since E is countably additive, we have

$$d = \lim_F E(F)d = \lim_F \sum_{ik\in F} P_k d \qquad (|\cdot|),$$

and again the same equalities in the old norm $\|\cdot\|$. Since $B(F)d = BE(F)d = AE(F)d$, and the generator A is a closed operator, we obtain that $d \in D(A)$ and $Ad = Sd \in Y$. Hence $d \in W$ and $Bd = (A|W)d = Sd$.

Assume now that $w \in W = D(B)$. Then

$$w = \sum_{k\in\mathbf{Z}} P_k w \quad \text{and} \quad Aw = \sum_{k\in\mathbf{Z}} ikP_k w \quad (\|\cdot\|).$$

Since both series are summable, [Sma, Lemma 1.5] shows that with the notation $b_n := \{z \in \mathbf{C}; |z| \le n\}$ we have

$$|Aw - SE(b_n)w| \to 0 \qquad (n \to \infty).$$

[DS, XVIII.2.27] shows that then there exists $Sw = Aw = Bw$. Hence $B = S$ is a closed spectral operator of scalar type in Y with resolution of the identity E.

Since E is concentrated on $i\mathbf{Z}$, [DS, XVIII.2.25] shows that the spectrum $\sigma(B)$ is a subset of $i\mathbf{Z}$. Define

$$G(t) := \int_{\sigma(B)} e^{tz} E(dz) \qquad (t \in \mathbf{R}).$$

The functional calculus for scalar-type operators, in particular [DS, XVIII.2.11, 17 and 18], shows that every $G(t)$ is a bounded scalar-type spectral operator with resolution of the identity E_t, the norms of which are bounded by the bound for the norms of $E(\cdot)$. Further, $G := \{G(t); t \in \mathbf{R}\}$ is a group of operators of class (C_0) with generating operator B. The structure of $\sigma(B)$ shows that the integral representation for $G(t)$ above can be written for every $y \in Y$ as

$$G(t)y = \lim_F \sum_{ik\in F} e^{tik} P_k y \qquad (|\cdot|).$$

Since the new norm majorizes the old one, we have the same equality for the norm $\|\cdot\|$. On the other hand, by the definition of Y we have

$$y = \sum_{k\in\mathbf{Z}} P_k y \quad (\|\cdot\|).$$

Since the operator $T(t)$ is bounded in the old norm, and $T(t)P_k y = e^{tik} P_k y$, we obtain

$$T(t)y = \sum_{k\in\mathbf{Z}} T(t)P_k y = \sum_{k\in\mathbf{Z}} e^{tik} P_k y,$$

and the convergence is again unconditional in the old norm $(\|\cdot\|)$. Hence

$$T(t)y = G(t)y \qquad (t \in \mathbf{R}, y \in Y).$$

Therefore B is the generator of the group $T|Y = G$, as asserted. $\square$

3. Examples

In the following examples we shall characterize the space Y from our definition in some classical cases, and shall point out some of the difficulties of an effective description even in these relatively simple cases.

Let X be either one of the spaces $L^p(0,2\pi) \quad (1 \le p < \infty)$ or the space $C[0,2\pi]$ of continuous functions with $x(0)=x(2\pi)$, with the usual norms. The group of operators will be the group of translations, i.e. will be defined by

$$[T(t)x](s) := x(s+t) \quad (\operatorname{mod} 2\pi; t \in \mathbf{R}, s \in [0,2\pi]).$$

It is well-known that the generators of these groups have the domains

$$D(A) = \{x \in X : x \text{ is absolutely continuous and } x' \in X\},$$

and $Ax = x'$. Further, the projections occurring in the general theory are

$$[P_n x](s) = \hat{x}(n)e^{ins} \quad (n \in \mathbf{Z}),$$

where $\hat{x}(n)$ denotes the nth Fourier coefficient of the function x. Applying the definition in our cases, the space Y consists exactly of those elements $x \in X$, the Fourier series of which converges unconditionally (is summable) to x in the norm topology of X.

Proposition 1. *If $X = C[0,2\pi]$, then Y consists precisely of those $x \in X$ for which $\hat{x} \in l^1(\mathbf{Z})$.*

Proof. If $x \in Y$, then for every bounded sequence $\{b_n; n \in \mathbf{Z}\} \subset \mathbf{C}$ the series $\sum_{n\in\mathbf{Z}} b_n P_n x$ converges unconditionally in the norm of X. Considering the Fourier series at a fixed point $s_0 \in [0,2\pi]$, and taking a suitable sequence $\{b_n\}$, we see that $\hat{x} \in l^1(\mathbf{Z})$. Conversely, if this holds, then the Fourier series of x converges absolutely, which implies unconditional convergence in X to x. □

Proposition 2. *If $X = L^p(0,2\pi)$ $(1 \le p < \infty)$, then, with the usual convention for $p = 1$,*

$$Y = Y(p) = \{x \in X : \hat{x}\hat{z} \in l^1(\mathbf{Z}) \textit{ for every } z \in L^q(0,2\pi),\ q := p/(p-1)\}.$$

Further, for $1 \le p \le 2$ we have $Y = Y(p) = L^2(0,2\pi)$. Finally, for $2 < p < \infty$ we have

$$\{x \in X : \hat{x} \in l^q(\mathbf{Z})\} \subset Y(p) = Y,$$

where q is again as above.

Proof. First assume that $x \in Y(p)$. For every $z \in L^q(0,2\pi)$ we have

$$\int_0^{2\pi} x(s)z(s)ds = \lim_F \sum_{k\in F} \hat{x}(k) \int_0^{2\pi} e^{iks} z(s)ds = \lim_F \sum_{k\in F} \hat{x}(k)\hat{z}(-k).$$

The convergence here is unconditional, i.e. the numerical series $\sum_{k\in\mathbf{Z}} \hat{x}(k)\hat{z}(-k)$ is summable. The properties of Fourier coefficients show that the above assertion holds with $\hat{z}(-k)$ replaced by $\hat{z}(k)$. Since the series are numerical, they are absolutely convergent.

Conversely, if all these series are absolutely convergent, then they are all summable. Hence the Fourier series of x is a weakly summable indexed family in the sense of, e.g., [BP, 19.Probl.I]. Since all L^p spaces for $1 \le p < \infty$ are weakly sequentially complete, a classical result of Orlicz [Orl] (cf. also [BP, 19.Probl.K]) gives that it is summable in the norm topology of X. It follows that the Fourier series converges in the classical (Cauchy) sense in the L^p norm, and its sum is x. Therefore $x \in Y(p)$.

Finally, as a consequence of our considerations above, the last two statements of the Proposition are contained in [E1, Ex.14.10(3), (4)]. $\square$

Remark. The author does not know any usable characterization of the space $Y(p)$ for the values $2 < p < \infty$ (cf. [E2, 7.2.3]).

4. An application

Theorem 2. *Let X be a complex Banach space, and let $U^+ = \{U(t); t \geq 0\}$ be a (C_0)-semigroup of operators in X such that $U(q)$ is a spectral operator of scalar type with spectrum in the unit circle for some $q > 0$. Assume that the periodic semigroup $T(t) := U(t)U(q)^{-t/q}$ of period q and class (C_0) has the property that its spectral projections $P_n = P_n(T)$ all have finite dimensional range spaces, and none of these dimensions is greater than a positive integer N. Then U^+ extends to a strongly continuous group $U = \{U(t); t \in \mathbf{R}\}$, and there is a linear manifold Y in X containing D(G) for the generating operator G=G(U) of U and a norm $|\cdot|$ on Y majorizing the old norm $\|\cdot\|$ with the following properties. Y is a Banach space with the new norm, is U(t)-invariant, and there is a strongly countably additive spectral measure E on the Borel sets of $\mathbf{R}$ with values in the projections of Y such that*

$$U(t)|Y = \int_{\mathbf{R}} e^{itr} E(dr) \qquad (t \in \mathbf{R}).$$

Proof. After rescaling the semigroup, if necessary, without loss of generality we may and will assume that $q = 2\pi$. Let E_q denote the resolution of the identity of $U(q)$, and define $H(b) := E_q(e^{2\pi ib})$ for every Borel set b in $(0, 1]$. Then

$$U(q)^t = \int_{(0,1]} e^{2\pi irt} H(dr) \qquad (t \in \mathbf{R}),$$

and $\{U(q)^{t/q}; t \in \mathbf{R}\}$ is a uniformly continuous group of scalar operators with $U(q)^1 = U(q)$ and with (bounded) generating operator

$$G_1 = i \int_{(0,1]} rH(dr) = q^{-1} \operatorname{Log} U(q),$$

where $\operatorname{Log} e^{iv} := iv$ for $v \in (0, 2\pi]$. Let $T(t) := U(t)U(q)^{-t/q} \quad (t \geq 0)$. Since all $U(t)$ and $U(q)^r$ commute, $\{T(t); t \geq 0\}$ is a strongly continuous semigroup of operators with period q. Therefore it extends to a periodic group $\{T(t); t \in \mathbf{R}\}$ of class (C_0). Now apply Theorem 1 to the periodic group $\{T(t)\}$ of period q, and consider the Banach space $Y = Y(T)$. The operators $H(\cdot)$ belong to the resolution of the identity of $U(q)$; hence they commute with every $U(t)$, and therefore with every $T(t)$. It follows that they also commute with the spectral projections $P_n = P_n(T)$ for the periodic group T. Hence $y = \lim_F \sum_{n \in F} P_n y$ implies $H(\cdot)y = \lim_F \sum_{n \in F} P_n H(\cdot)y$. Therefore every $U(q)^t$ leaves Y invariant. Since $U(t) = T(t)U(q)^{t/q}$, every $U(t)$ also leaves Y invariant. Further, denoting the generating operator of T by $G(T)$, and taking into account that G_1 is bounded, we obtain

$$G(U) = G(T) + G_1, \qquad D[G(U)] = D[G(T)] \subset Y(T) = Y.$$

These considerations show that each bounded linear operator V in X commuting with every P_n maps Y into Y, and, introducing the notation $P(b) := \sum_{in\in b} P_n$ (cf. Lemma 2), we see that

$$|Vy| = \sup_F \|P(F)Vy\| \leq \|V\| \sup_F \|P(F)y\| \leq \|V\||y|$$

implies that V is bounded in the norm of Y, and $|V| \leq \|V\|$. In particular, the group $\{U(q)^{t/q}\}$ consists of operators bounded in Y, and is continuous in the (uniform) $|\cdot|$ operator norm. Since the group $\{T(t)|Y\}$ is of class (C_0), the group $\{U(t)|Y\}$ is also of class (C_0) in $(Y, |\cdot|)$. Moreover, the spectral measure H is (uniformly) bounded in $(Y, |\cdot|)$. We shall show that it is also countably additive in the new norm.

It suffices to show that for every $y \in Y$ and every nonincreasing sequence $\{d_n\}$ of Borel sets with empty intersection in the plane we have

$$(**) \qquad \lim_n |H(d_n)y| = 0.$$

If y is such that only one, say the kth, of the elements $P_n y$ is nonzero, then

$$|H(d_n)y| = |P_k H(d_n)y| = \sup_F \|P(F)H(d_n)y\| = \|H(d_n)y\| \to 0 \quad (n \to \infty).$$

In view of the properties of the projections P_n, elements with the property $(**)$ form a fundamental set in Y. The uniform boundedness of H in Y implies therefore that the required relation holds for every $y \in Y$. The fact that the old operator norm majorizes the new operator norm in Y shows that the restricted operators $U(q)^t|Y$ are spectral operators of scalar type, for which the above representation also holds in the $|\cdot|$-norm topology.

Now we want to define the spectral measure E in the space $(Y, |\cdot|)$. First let b be a bounded Borel subset of $\mathbf{R}$. Then b has a unique representation $b = \bigcup_k b_k$, where each $b_k \subset (k, k+1]$, and only a finite number of them are nonempty. Therefore we can define

$$E(b) := \sum_{k=a}^{f} P_k H(b_k) \qquad \text{if} \quad b = \bigcup_{k=a}^{f} b_k.$$

Note that all these projections belong to the Boolean algebra of projections generated by the commuting spectral measures P and H. Applying a deep result of McCarthy ([McC, Theorem 2.2 and Section 4]), we shall show that the generated Boolean algebra of projections is bounded.

First note that the range $\mathbf{P}$ of the spectral measure P, i.e.

$$\mathbf{P} := \{P(b); b \subset i\mathbf{Z}\},$$

is a complete Boolean algebra of projections in the sense of Bade [Bad] (see also McCarthy [McC]). Further, $\mathbf{P}$ is clearly countably decomposable in the sense of Bade [Bad], and, by our assumption, every projection in $\mathbf{P}$ has multiplicity not greater than N. If $\mathbf{H}$ denotes the range of the spectral measure H, then the cited results of McCarthy show that the Boolean algebra of projections generated by $\mathbf{P}$ and $\mathbf{H}$ is bounded.

Now we extend the definition of the spectral measure E for any Borel subset $b \subset \mathbf{R}$. Any such b has the unique representation $b = \bigcup_{k=-\infty}^{\infty} b_k$, where b_k is a

Borel subset of $(k, k+1]$. Then for every $y \in Y$ we define

$$E(b)y := \sum_{-\infty}^{\infty} P_k H(b_k) y$$

in the $|\cdot|$-norm of Y. We have to and shall show that this sum exists. This is certainly the case if only a finite number of the elements $P_k y$ is nonzero. In this latter case we shall say in what follows that the element y has the "finite property" or, equivalently, is "finite". If M denotes an upper bound of the norms in the Boolean algebra of projections generated by $\mathbf{P}$ and $\mathbf{H}$, and y is an arbitrary element of Y, then for every $a, f \in \mathbf{Z}, a < f$, we have, with the notation $E(b)_a^f y := \sum_a^f P_k H(b_k) y$,

$$|E(b)_a^f y| \leq M|y|.$$

If $(a, f) \to (-\infty, \infty)$ and y has the "finite property" above, the above sum clearly tends to $E(b)y$. Since elements with this property form a dense set in Y, the uniformly bounded linear operators $E(b)_a^f$ converge to $E(b)$ in the strong operator topology, as asserted.

Now we shall show that the operator valued set function E is a spectral measure. In view of the commutativity of P and H and the convergence established above, the proof of the needed algebraic properties is more or less straightforward, and we omit it. We shall prove countable additivity in the strong operator topology of Y. Assume that $\{b^n := \bigcup_{k=-\infty}^{\infty} b_k^n; n \in \mathbf{N}\}$, where every $b_k^n \subset (k, k+1]$, is a nonincreasing sequence of Borel subsets of $\mathbf{R}$ with empty intersection and $y \in Y$. Then for every $k \in \mathbf{Z}$ the corresponding sequence $\{b_k^n; n \in \mathbf{N}\}$ has the same properties. In view of the countable additivity of H, for every pair $a, f \in \mathbf{Z}, a < f$, and for every $x \in Y$ we then have

$$\lim_{n\to\infty} \sum_{k=a}^{f} P_k H(b_k^n) x = 0.$$

Since we have $|E(\cdot)| \leq M$, and the elements with the "finite property" above form a dense set in Y, for every $\varepsilon > 0$ there is a "finite" $z \in Y$ such that for every $-\infty \leq a \leq f \leq \infty$ and for every $n \in \mathbf{N}$ we have

$$\Big|\sum_{k=a}^{f} P_k H(b_k^n)(z-y)\Big| \leq M|z-y| < \varepsilon.$$

Fixing such a "finite" z, pick integers $a \leq f$ such that $E(b)z$ vanishes if b is a subset of $(-\infty, a] \cup (f, \infty)$. Then

$$|E(b^n)y| \leq |E(b^n)(y-z)| + |E(b^n)_a^f z| < \varepsilon + |E(b^n)_a^f z|$$

for every $n \in \mathbf{N}$. Letting $n \to \infty$, we see that $\lim_{n\to\infty} E(b^n)y = 0$; hence E is countably additive in the strong operator topology.

Finally, the stated representation of $U(t)|Y$ with the help of the spectral measure E can be proved in the same way as e.g. in [RSz]. □

5. The case when X is reflexive

In the case of a reflexive Banach space X, S. Kantorovitz introduced and studied the semisimplicity manifold $Z = Z(V)$ of a (bounded or closed) operator V under the main assumption that the spectrum of V lies in $\mathbf{R}$ (see [K1], [K2], [KH]). The last two cited papers deal with the case of an unbounded operator V, and show that $Z(V)$ is a Banach space in a topology stronger than that of X, and that the restriction of V to Z has properties reminiscent of those of a spectral operator of scalar type. We note that the techniques of these papers make a strong use of reflexivity, and the countable additivity of the spectral measures and the existence of the scalar-like integral representations are proved there only in the original (weaker) topology of X. Nevertheless, it might be interesting in the case of (the generator $A = iV$ of) a periodic group in a reflexive space to compare the two manifolds Z and Y. In the next theorem we shall show that they are identical.

Recall that for a closed operator V with real spectrum, for which iV is a group generator, the semisimplicity manifold $Z = Z(V)$ is the set of all elements $x \in X$ for which

$$|||x||| := \sup_{\|x^*\| \le 1,\ q>0} (2\pi)^{-1} \int_{\mathbf{R}} |x^*[R(t-iq,V) - R(t+iq,V)]x| dt < \infty,$$

together with this new norm (cf. [KH, Definition 2.1 and Comment 2, p.543]). Note that the restriction $V|Z$ is defined as the operator V on the domain $D(V|Z) := \{x \in D(V) \cap Z :\ Vx \in Z\}$, and that we shall use these notions mutatis mutandis also for $A = iV$ or for the generated group T.

Theorem 3. *Let T be a group of operators of class (C_0) and of period 2π on the reflexive Banach space X, let $Y = Y(T)$ be as in Section 2, and let $Z = Z(T)$ denote the semisimplicity manifold of (the generator A of) the group T. Then $Y = Z$, and for every $z \in Z$ we have $|z| \le |||z|||$; hence the corresponding norms are equivalent.*

Proof. By Theorem 1, the restriction group $G(t) := T(t)|Y$ is a (C_0)-group of operators, and its generator $B := A|W$ is a spectral operator of scalar type with resolution of the identity E in the Banach space $(Y, |\cdot|)$. By the well-known operational calculus for spectral operators of scalar type, then

$$R(w,A)y = \int_{i\mathbf{R}} (w-s)^{-1} E(ds)y \qquad (y \in Y,\ w \in \mathbf{C} \setminus i\mathbf{R}),$$

where the integral is a Lebesgue-type vector-valued integral in the above norm topology of Y; hence it also exists in the (weaker) norm topology of X. By [KH, Proposition 2.4], if F denotes the spectral measure constructed there, we have $Y \subset Z$ and $E(b) = F(b)|Y$ for every Borel set $b \subset i\mathbf{R}$.

In the converse direction we shall start by scrutinizing a part of the proof of [KH, Theorem 2.3] under our conditions, so it will be more convenient to work with the operator $V := -iA$ instead of A, with the consequence that everything will happen on $\mathbf{R}$ rather than on $i\mathbf{R}$. For fixed $z \in Z$ and $x^* \in X^*$ the function

$$u(t,q) := (2\pi i)^{-1} x^*[R(t-iq,V) - R(t+iq,V)]z$$

is harmonic in the upper half-plane $(q > 0)$ and, by the definition of Z, satisfies for every $q > 0$ the estimate

$$\int_{\mathbf{R}} |u(t,q)| dt \le |||z|||\ \|x^*\| < \infty.$$

Hence u is the Poisson integral of a finite measure, i.e. there is a unique regular Borel measure $m = m(\cdot; z, x^*)$ on $\mathbf{R}$ such that

$$u(t,q) = \int_{\mathbf{R}} P(t-s;q) m(ds; z, x^*),$$

where $P(t,q) := q\{\pi(t^2+q^2)\}^{-1}$ is the Poisson kernel for (t,q) in the upper half-plane. Since the group T has the period 2π, the spectrum of V is contained in $\mathbf{Z}$. Hence, under our conditions, the function u is uniformly continuous on every compact subset of $\mathbf{C} \setminus \mathbf{Z}$, and $\lim_{q\to 0} u(t,q) = 0$ for $t \in \mathbf{R} \setminus \mathbf{Z}$. By [Gar, Ch.1, Theorem 3.1(c)], for any $g \in C(\mathbf{R})$ with support in any open interval $(k, k+1)$ for some $k \in \mathbf{Z}$ we obtain

$$\int g(s)m(ds; z, x^*) = \lim_{q\to 0} \int g(s)u(s,q)ds = 0.$$

Hence the measure $m(\cdot; z, x^*)$ lives only on $\mathbf{Z}$ and, since

$$x^* F(b)z = m(b; z, x^*) \qquad (z \in Z,\ x^* \in X^*)$$

for every Borel subset b of $\mathbf{R}$, the same holds for the spectral measure F.

For each $z \in Z$ the vector measure $F(\cdot)z$ is countably additive in the norm topology of X (see [KH, p.537]). Hence, introducing the notation $F(k) := F(\{k\})$ for every $k \in \mathbf{Z}$, we have

$$(*) \qquad z = \sum_{\mathbf{Z}} F(k)z \qquad (\|\cdot\|).$$

Further, for any $w \in \mathbf{C} \setminus \mathbf{R}$ the resolvent $R(w, V|Z)$ is a bounded linear operator in the Banach space $(Z, |||\cdot|||)$, and maps Z onto $D(V|Z)$. Hence the operator $V|Z - w$ is a closed linear operator in the above Banach space, and the same holds for $V|Z$. By [KH, Theorem 2.3(iii)], under our conditions we have

$$Vz = \lim_{N\to\infty} \sum_{k=-N}^{N} kF(k)z \qquad (z \in D(V|Z))$$

in the norm topology of X.

Assume now that $z = F(k)z$ for some $z \in Z$ and $k \in \mathbf{Z}$. By [KH, Theorem 2.3(ii)], then $z \in D(V|Z)$. By the preceding paragraph, $Vz = kz$. Hence, with the notation of Section 1, $z \in P_k X$, i.e. $z = P_k z$. We have proved that $F(k)Z \subset P_k Z$. Now multiplying $(*)$ by P_r from the left, we obtain

$$P_r z = \lim_{N\to\infty} \sum_{k=-N}^{N} P_r F(k)z = F(r)z \qquad (r \in \mathbf{Z},\ z \in Z),$$

in view of the continuity of P_r in the norm topology of X and of the "orthogonality" properties of the projections P_k $(k \in \mathbf{Z})$. Hence for every $z \in Z$ we have

$$z = \sum_{k\in\mathbf{Z}} P_k z \qquad (\|\cdot\|);$$

therefore $Z \subset Y$.

By the definition of the norm $|\cdot|$ and by [KH, p.537], we obtain

$$|z| := \sup_f ||E(f)z|| \leq |||z|||,$$

where the supremum is taken over all finite subsets $f \subset \mathbf{Z}$. The proof is complete. □

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