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A SOLUTION TO A PROBLEM ON INVERTIBLE DISJOINTNESS PRESERVING OPERATORS

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Abstract. We construct an invertible disjointness preserving operator T on a normed lattice such that T^{-1} is not disjointness preserving.

Two elements x_1, x_2 in an (Archimedean) vector lattice X are said to be disjoint (in symbols: $x_1 \perp x_2$) if $|x_1| \wedge |x_2| = 0$. Recall that a (linear) operator $T : X \to Y$ between vector lattices is said to be **disjointness preserving** if T sends any two disjoint elements in X to disjoint elements in Y, that is, $Tx_1 \perp Tx_2$ in Y whenever $x_1 \perp x_2$ in X.

Assume now that a disjointness preserving operator $T : X \to Y$ is bijective, i.e., one-to-one and onto, so that $T^{-1} : Y \to X$ exists. It was conjectured by the first named author (see, for example, the problem section in [HL, page 143]) that T^{-1} is also disjointness preserving. The purpose of this note is to give a counterexample to this conjecture. Moreover, we will construct a bijective disjointness preserving automorphism T on a normed lattice G such that T^{-1} is not disjointness preserving. We precede our construction by several comments on some recent developments regarding the above conjecture.

The two most important classes of vector lattices are the Banach lattices and the Dedekind complete vector lattices. The vector lattice G that we will construct is in neither of these classes. It cannot be a Banach lattice in view of the fact that, as has been recently shown by Huijsmans and de Pagter [HP] and independently by Koldunov [K], the above conjecture has an affirmative solution for Banach lattices. More precisely, they proved the following theorem.

Theorem 1 ([HP, Theorem 2.1], [K, Theorem 3.6])**.** *Let T be a one-to-one disjointness preserving operator from a relatively uniformly complete vector lattice X into a normed lattice Y. Then T is a d-isomorphism, that is, $x_1 \perp x_2$ in X if and only if $Tx_1 \perp Tx_2$ in Y. In particular, if $TX = Y$ then $T^{-1} : Y \to X$ is a disjointness preserving operator.*

Since each Banach lattice is relatively uniformly complete, the previous theorem includes the case of Banach lattices. Here we should also cite the paper by K. Jarosz [J], which preceded [HP] and [K] and in which a special case of Theorem 1 was

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handled for $X = C(K_1)$ and $Y = C(K_2)$, the spaces of continuous functions on the compact Hausdorff spaces K_1 and K_2.

For completeness of the picture let us mention as well the remaining known cases when the above conjecture has an affirmative solution. Huijsmans and de Pagter proved in the same paper [HP, Theorem 2.6] that *if the domain space X is a discrete vector lattice, then $T^{-1} : Y \to X$ is also disjointness preserving* (without any additional assumptions on Y).

Finally, we mention a paper by Font and Hernandez [FH] dealing with disjointness preserving operators between spaces of continuous functions vanishing at infinity, and also a very recent one by Araujo, Beckenstein and Narici [ABN] who consider the disjointness preserving operators between spaces $C(\Omega_1)$ and $C(\Omega_2)$ for Tikhonov spaces Ω_1 and Ω_2. It is proved in [ABN] that *if Ω_1 is zerodimensional or Ω_2 is connected, then T^{-1} preserves disjointness*; [ABN] contains also an alternative proof of a special case of Theorem 1.

The case of a bijective disjointness preserving operator between two arbitrary Dedekind complete vector lattices is still open.

Recall that an operator $T : X \to X$ is said to be band preserving if $x_1 \perp x_2$ implies $Tx_1 \perp x_2$. Band preserving operators form an important subclass of the disjointness preserving operators, and for them the above conjecture was resolved positively in [HW] under the very mild assumptions that X either is relatively uniformly complete or has the principal projection property. We refer to [A], [AVK] and [HW] for some related information regarding band preserving operators. Let us mention in passing that in [AVK] a bijective band preserving operator T on $L^0[0,1]$ (the space of equivalence classes of all measurable functions on [0,1]) was constructed which was not order bounded and thus did not admit a multiplicative representation. Nevertheless, the inverse T^{-1} of that operator remains band preserving. This fact can be easily verified directly, or it follows from the above mentioned theorem in [HW]. We refer to [AB] for general references on vector lattices and operators on them.

A construction of the counterexample

We will show here that in general, i.e., without some additional assumptions, the answer to our conjecture is negative. Moreover, we will show that there exists a disjointness preserving bijection $T : G \to G$ on a normed lattice G satisfying the projection property[1] for which T^{-1} is not disjointness preserving. A crucial step in this example depends on a simple modification of a well known construction exploiting piecewise linear functions and used by many authors (see for example, [M], [HP], [K]). As far as we know, it was M. Meyer [M] who was the first to use piecewise linear functions in connection with disjointness preserving operators.

Theorem 2. *There exist a normed lattice G and a disjointness preserving bijection $T : G \to G$ such that T^{-1} is not disjointness preserving.*

Proof. We start by introducing the appropriate vector lattices. We say that a finite collection $\{A_1, \dots, A_n\}$ of measurable subsets of [0,1) is a partition of [0,1) if

i) the sets are pairwise disjoint,

[1] A vector lattice X satisfies the projection property if for each band B in X there exists a band-projection from X onto this band; in other words, $X = B + B^d$, where by definition, $B^d = \{x \in X : x \perp y \ \ \forall y \in B\}$.

ii) each set has positive Lebesgue measure, and

iii) $\bigcup_j A_j = [0,1)$.

A function f on [0,1) is said to be **piecewise linear** if there exist a partition $\{A_1, \ldots, A_n\}$, $n \in \mathbb{N}$, and real numbers a_j, b_j $(j = 1, \ldots, n)$ such that $f(t) = a_j t + b_j$ for each $t \in A_j$.

We denote by X the collection of all piecewise linear functions on [0,1). It is easy to see that under the natural linear operations and pointwise order X is a vector lattice. A simple verification shows that this vector lattice satisfies the projection property. Equipped with the uniform norm, X becomes a normed lattice.

Also we will consider a vector lattice Y of all simple functions on [0,2); that is, a function g belongs to Y if and only if there exist a partition $\{A_1, \ldots, A_m\}$ of [0,2) and real numbers c_j $(j = 1, \ldots, m)$ such that $g(t) = c_j$ for all $t \in A_j$. We equip Y with the uniform norm.

For these normed lattices X and Y we will construct a disjointness preserving bijection S from X onto Y such that S^{-1} fails to be disjointness preserving. Take any f in X. Let $\{A_1, \ldots, A_n\}$ and a_j, b_j $(j = 1, \ldots, n)$ be the corresponding partition and the "values" of f respectively. For $j = n+1, n+2, \ldots, 2n$ let

$$A_j := A_{j-n} + 1 = \{a + 1 : a \in A_{j-n}\}.$$

Clearly the collection $\{A_{n+1}, A_{n+2}, \ldots, A_{2n}\}$ forms a partition of $[1,2)$, and hence, the collection $\{A_1, \ldots, A_n, A_{n+1}, \ldots, A_{2n}\}$ forms a partition of $[0,2)$.

Now we can define the function Sf by letting

$$Sf = \sum_{j=1}^{n} b_j \chi_{A_j} + \sum_{j=n+1}^{2n} a_j \chi_{A_j}.$$

It is easy to see that this mapping $f \mapsto Sf$ defines a linear operator from X to Y. Obviously $Sf \neq 0$ provided $f \neq 0$, and so S is one-to-one.

Let us check that S is onto. Clearly, it is enough to verify that each characteristic function $\chi_A \in Y$ is in the range space of the operator S. Assume first that $A \subseteq [0,1)$. Then clearly $S\chi_A = \chi_A$. Now assume that $A \subseteq [1,2)$ and consider $A' = \{a - 1 : a \in A\}$. Again it is clear that $Sf = \chi_A$, where $f = t\chi_{A'}$, i.e., $f(t) = t$ for $t \in A'$ and $f(t) = 0$ for $t \notin A'$. The case of a general $A \subseteq [0,2)$ can be trivially reduced to the previous two cases.

We omit a straightforward verification that S preserves disjointness. The fact that S^{-1} does not preserve disjointness is obvious since the functions $g_1 = \chi_{[0,1)}$ and $g_2 = \chi_{[1,2)}$ are disjoint in Y, while their images $f_1 = \chi_{[0,1)}$ and $f_2(t) = t$ under S^{-1} are not disjoint in X. (Notice in passing that S is a discontinuous operator.)

Consider now the following space $G = \ldots \oplus X \oplus X \oplus X \oplus Y \oplus Y \oplus Y \oplus \ldots$ of vector-valued sequences, that is, a two-sided sequence $\bar{f} = (\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots)$ belongs to G if and only if

i) $f_k \in X$ for each negative index $k < 0$,

ii) $f_k \in Y$ for each non-negative index $k \geq 0$, and

iii) $\|\bar{f}\| = \sup_{k \in \mathbf{Z}} \|f_k\| < \infty$.

Now we are ready to define our operator $T : G \to G$. Namely, for the elements $\bar{f} = (\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots)$ and $\bar{g} = (\ldots, g_{-2}, g_{-1}, g_0, g_1, g_2, \ldots)$ in G we let $\bar{g} = T\bar{f}$ if $g_0 = Sf_{-1}$ and $g_k = f_{k-1}$ for all $k \neq 0$. That is, T shifts all the coordinates to the right and, additionally, on the (-1)-coordinate, T acts as S.

It is obvious that T is one-to-one and preserves disjointness. Since $S : X \to Y$ is onto it follows that $T : G \to G$ is also onto. Finally, T^{-1} clearly does not preserve disjointness since S^{-1} does not. $\square$

The "usual" space PL of piecewise linear functions on an interval $[a, b]$ consists of the functions which are linear on intervals, that is, their corresponding partitions of $[a, b]$ are the Riemann partitions rather than the Lebesgue partitions exploited by us. Basically preserving the proof of Theorem 2, we can use the space PL as well, but then the resulting normed lattice won't have the projection property. Another advantage of our more general construction lies in the fact that it can be adapted to more general underlying spaces rather than just to the intervals.

Notice also that if instead of finite partitions we allow countable partitions, and replace Y by the space of all step functions with countably many steps, then the same construction holds, and we additionally obtain the conditional lateral completeness[2] in X, and therefore in G. Being conditionally laterally complete, the vector lattice G is "very close" to being Dedekind complete. However, as can easily be seen, G is not Dedekind complete, and presently it is not clear whether we can modify our construction to include the Dedekind complete vector lattices as well. We conclude by formulating explicitly the two most interesting remaining open problems regarding the topic at hand.

Problem 1. Let X and Y be Dedekind complete vector lattices, and let $T : X \to Y$ be a bijective disjointness preserving operator. Is $T^{-1} : Y \to X$ also disjointness preserving? In particular, is it so if X and Y are universally complete?

Problem 2. Let $T : X \to X$ be a bijective band preserving operator on an Archimedean vector lattice. Is $T^{-1} : X \to X$ also band preserving?

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