COHOMOLOGY OF CERTAIN CONGRUENCE SUBGROUPS OF THE MODULAR GROUP

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Abstract. In this note we compute the integral cohomology groups of the subgroups \( \Gamma_0(n) \) of \( SL(2, \mathbb{Z}) \) and the corresponding subgroups \( P\Gamma_0(n) \) of its quotient, the classical modular group, \( PSL(2, \mathbb{Z}) \).

1. Introduction

Let \( \Gamma_0(n) \) denote the subgroup of \( SL(2, \mathbb{Z}) \) consisting of matrices whose lower left entry is divisible by the integer \( n \geq 2 \) and let \( P\Gamma_0(n) = \Gamma_0(n)/\langle \pm I \rangle \). It is well-known (see [7], p. 11) that the group \( PSL(2, \mathbb{Z}) \) is the free product of a group of order 2 and a group of order 3. The Kurosh subgroup theorem then tells us that \( P\Gamma_0(n) \) is the free product of finitely many copies of \( \mathbb{Z} \), \( \mathbb{Z}/2 \), and \( \mathbb{Z}/3 \). Hence the integral cohomology of \( P\Gamma_0(n) \) is free abelian in dimension one, trivial for higher odd dimensions, and sums of copies of \( \mathbb{Z}/2 \) and \( \mathbb{Z}/3 \) in positive even dimensions. Determining the number of summands of each type in the cohomology is equivalent to determining the number of summands of each type in the free product (see [7], p. 127). These were computed for arbitrary \( n \) in [2], but that computation is inaccurate. (In fact, the numbers given in [2] fail in general to be integers; the difficulties appear to begin with problems in counting the number of torsion summands.) The rational Euler characteristic of these groups was computed in [4], while in [1] their integral cohomology was computed for \( n \) prime. In [3], the cohomology was computed for \( n \) prime or a product of two distinct primes. In this paper we shall give a quick, easy computation of the integral cohomology of these groups for all \( n \). To state our result, we define functions \( a(n) \), \( b(n) \), and \( c(n) \) as follows. Let \( a(n) \) be the index of \( \Gamma_0(n) \) in \( SL(2, \mathbb{Z}) \). Let \( b(n) \) be the number of roots of \( x^2 + x + 1 \ (mod \ n) \), and let \( c(n) \) be the number of roots of \( x^2 + 1 \ (mod \ n) \). Finally, define

\[
r(n) = 1 + \frac{a(n) - 4b(n)}{6} - \frac{c(n)}{2}.
\]

We have the following two theorems.

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Theorem 1.1. The cohomology of $P\Gamma_0(n)$ is given by the formula

$$H^i(P\Gamma_0(n), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}^r(n), & \text{if } i = 1; \\ (\mathbb{Z}/3)^{b(n)} \oplus (\mathbb{Z}/2)^{c(n)}, & \text{if } i \geq 2 \text{ is even}; \\ 0, & \text{for other values of } i. \end{cases}$$

Theorem 1.2. The cohomology of $\Gamma_0(n)$ is given by the formula

$$H^i(\Gamma_0(n), \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ \mathbb{Z}^r(n), & \text{if } i = 1; \\ (\mathbb{Z}/2)^{r(n)}, & \text{if } i \geq 3 \text{ is odd}; \\ (\mathbb{Z}/3)^{b(n)} \oplus (\mathbb{Z}/2)^{c(n)-1} \oplus \mathbb{Z}/4, & \text{if } i \geq 2 \text{ is even and } c(n) > 0; \\ \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^{b(n)}, & \text{if } i \geq 2 \text{ is even and } c(n) = 0. \end{cases}$$

The functions $a(n), b(n),$ and $c(n)$ are well-known. We have $a(n) = n \prod_{p|n}(\frac{p-1}{p})$; $b(n) = 2^k$ if $n = 3^a p_1^{l_1} \ldots p_k^{l_k}$ is the prime factorization of $n$, where $l_0 \leq 1$ and all $p_i \equiv 1 \pmod{6}$, and is zero otherwise; and $c(n) = 2^k$ if $n = 2^a p_1^{l_1} \ldots p_k^{l_k}$, where $l_0 \leq 1$ and all $p_i \equiv 1 \pmod{4}$, and is zero otherwise.

We shall conclude this paper with some easy applications of our results to the cohomology of the groups $PSL(\mathbb{Z}[1/n], 2)$.

2. Preliminaries

Henceforth, let $G$ denote the group $SL(2, \mathbb{Z})$, and let $K$ denote the subgroup $\Gamma_0(n)$ for some fixed $n$. Let $PK$ denote $P\Gamma_0(n)$. We shall begin by describing an action of $G$ on the cubic tree. (Recall that the cubic tree is the tree whose vertices all have order three.) It is well-known that $G$ acts on the cubic tree; cf. [7]. Our goal here is to give a description that facilitates the calculation of the action of $K$.

We define $X'$ to be the subset of the integer lattice in the plane given by

$$X' = \left\{ \left( \begin{array}{c} w \\ y \end{array} \right) \big| (w, y) = 1 \right\},$$

and define $X$ to be the quotient of $X'$ by the action of $\pm I$. Then the action of $G$ on the integer lattice preserves $X$, and $\pm I$ are the only elements of $G$ that fix two distinct points of $X$. We identify, by abuse of notation, $X$ with the subset of $X'$ consisting of points $\left( \begin{array}{c} w \\ y \end{array} \right)$ with $w \geq 1$, together with the point $\pm \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$. Let $\Delta$ be the set of triangles whose vertices are elements

$$\left( \begin{array}{c} w \\ y \end{array} \right), \left( \begin{array}{c} x \\ z \end{array} \right), \text{ and } \left( \begin{array}{c} t \\ u \end{array} \right),$$

for which $t = w + x, u = y + z,$ and $wz - xy = \pm 1$. We shall represent such a triangle in matrix form,

$$T = \left( \begin{array}{ccc} w & x & t \\ y & z & u \end{array} \right).$$

Without loss of generality, we may require that

$$\det \left( \begin{array}{cc} w & x \\ y & z \end{array} \right) = 1.$$

Note that the triangles of this form provide a triangulation of a subset of the right half-plane.
The action of $G$ on $X$ induces an action of $G$ on $\Delta$. For $T$ as above, we define three associated elements of $G$ by

$$T_1 = \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \quad T_2 = \begin{pmatrix} t & -w \\ u & -y \end{pmatrix}, \quad T_3 = \begin{pmatrix} x & -t \\ z & -u \end{pmatrix}.$$ 

Now consider the action of $K$ on $\Delta$. We observe that if at least one of $y, z, u$ is relatively prime to $n$, then $T$ is equivalent under the action of $K$ to a triangle of the form $\begin{pmatrix} 1 & 0 & 1 \\ k & 1 & k+1 \end{pmatrix}$ for some $0 \leq k \leq n-1$. We further note that if a triangle $S \in \Delta$ is $K$-equivalent to $T$, then for some $1 \leq i \leq 3$, $T_iS^{-1} \in K$.

We conclude that under the projection $G/K \to \Delta/K$, the inverse image of the class of an element $T$ of $\Delta/K$ consists of three distinct elements of $G/K$, except in the case for which $T_2T_1^{-1}$ (and hence $T_3T_1^{-1}$) is in $K$. We call triangles in the first case triangles of Type 1 and in the second case, triangles of Type 2. The triangle $T$ is of Type 2 if and only if $y^2 + yz + z^2 \equiv 0 \pmod{n}$. Also in this case, $T$ is $K$-equivalent to a unique triangle of the form $\begin{pmatrix} 1 & 0 & 1 \\ k & 1 & k+1 \end{pmatrix}$ for some $0 \leq k \leq n-1$.

As an example, take $n = 3$, and in Figure 1 consider the triangle with vertices $(3)_1, (1)_1, (5)_2$, which we represent in our notation as

$$T = \begin{pmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \end{pmatrix}.$$ 

We have

$$\begin{pmatrix} 2 & -5 \\ 3 & -7 \end{pmatrix} \begin{pmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 1 \end{pmatrix},$$

which is a representative of the triangle

$$S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$
We observe that the matrix
\[
S_2 S_1^{-1} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}
\]
is an element of $K$ of order 6 that preserves the triangle $S$.

From the above discussion, we conclude

**Lemma 2.1.** The cardinality of the set $X/K$ is given by
\[
\text{card}(X/K) = \frac{a(n) + 2b(n)}{3}.
\]

We may now consider the dual graph to this complex: The set of vertices of this graph is $\Delta$, and two vertices have an edge in common provided the corresponding triangles are adjacent. This graph is clearly the cubic tree. To compute the cohomology of $K$, we shall appeal to the following theorem of Serre [7].

**Theorem 2.2.** Let a group $\Gamma$ act without inversion on a tree $Y$. Let $\Sigma_0$ (resp. $\Sigma_1$) denote a system of representatives of the vertices (resp. the edges) of $Y$, and for each vertex $x$ (resp. edge $y$) let $\Gamma_x$ (resp. $\Gamma_y$) be its stabilizer in $\Gamma$. For each $\Gamma$-module $M$, one has an exact cohomology sequence
\[
\ldots H^i(\Gamma, M) \to \prod_{x \in \Sigma_0} H^i(\Gamma_x, M) \to \prod_{y \in \Sigma_1} H^i(\Gamma_y, M) \to H^{i+1}(\Gamma, M) \to \ldots .
\]
The connecting maps are induced by the respective inclusions of groups.

Here, to act without inversion means that there is no element of $\Gamma$ that exchanges two adjacent vertices of $Y$. So, to apply this theorem, we shall alter our tree by adding a new vertex at the midpoint of each edge that is inverted by an element of $K$. If an edge of the tree is inverted by $K$, the two triangles corresponding to the ends of this edge have two of their vertices in common which are exchanged by an element of $K$. Suppose these two vertices are $(w_1 y)$ and $(w_2 z)$. Then the only matrices in $G$ that interchange them are $\pm \begin{pmatrix} wy + xz & -(w^2 + x^2) \\ y^2 + z^2 & -(w y + x z) \end{pmatrix}$. If these matrices are elements of $K$, then $y^2 + z^2$ must be divisible by $n$; hence both $y$ and $z$ must be relatively prime to $n$. By the remarks above, there is a unique pair of triangles of the form $\begin{pmatrix} 1 & 0 & 1 \\ k & 1 & k + 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ k - 1 & 1 & k \end{pmatrix}$ for $1 \leq k \leq n$ that are exchanged by $K$. Hence the number of $K$-equivalence classes of edges that are inverted is $c(n)$.

We see that by adding vertices as described above to bisect each edge inverted by $K$, we have added $c(n)$ $K$-equivalence classes of vertices. We call vertices of this type vertices of Type 3, and the vertices corresponding to triangles of Types 1 and 2, vertices of Types 1 and 2, respectively. Thus the total number of equivalence classes of vertices is given by
\[
v(n) = \frac{a(n) + 2b(n)}{3} + c(n).
\]
To obtain the number $e(n)$ of $K$-equivalence classes of edges, we do some elementary counting to get
\[
e(n) = \frac{a(n) + c(n)}{2}.
\]
3. Cohomology calculations

Under the action of $K$ (resp. $PK$), the stabilizer of vertices of Type 1 is $\mathbb{Z}/2$ (resp. $\{0\}$), the stabilizer of vertices of Type 2 is $\mathbb{Z}/6$ (resp. $\mathbb{Z}/3$), and that of vertices of Type 3 is $\mathbb{Z}/4$ (resp. $\mathbb{Z}/2$). The cohomology of cyclic groups is given by

$$H^i(\mathbb{Z}/n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } i = 0; \\ 0, & \text{if } i \text{ is odd}; \\ \mathbb{Z}/n, & \text{otherwise}. \end{cases}$$

The initial portion of the cohomology sequence

$$0 \to H^0(\Gamma, M) \to \prod_{x \in \Sigma_0} H^0(\Gamma_x, M) \to \prod_{y \in \Sigma_1} H^0(\Gamma_y, M) \to H^1(\Gamma, M) \to 0.$$  

is

$$0 \to \mathbb{Z} \to (\mathbb{Z})^c(n) \to (\mathbb{Z})^{c(n)} \to H^1(\Gamma) \to 0,$$

for both $\Gamma = K$ and $\Gamma = PK$. In higher degrees, we have

$$0 \to H^{2i}(\Gamma, M) \to \prod_{x \in \Sigma_0} H^{2i}(\Gamma_x, M) \to \prod_{y \in \Sigma_1} H^{2i}(\Gamma_y, M) \to H^{2i+2}(\Gamma, M) \to 0.$$  

For $\Gamma = PK$, the fourth term in this sequence is trivial, so the second and third are isomorphic and the fifth is also trivial. For $\Gamma = K$, the sequence is

$$0 \to H^{2i}(K) \to (\mathbb{Z}/2)^{c(n)-c(n)} \oplus (\mathbb{Z}/3)^{b(n)} \oplus (\mathbb{Z}/4)^{c(n)}$$

$$\to (\mathbb{Z}/2)^{c(n)} \to H^{2i+2}(K) \to 0.$$  

From this we easily obtain the values stated in Theorems 1.1 and 1.2.

4. Applications

Let $n$ be an integer and let $p_1, \ldots, p_k$ be the distinct prime factors of $n$. There is a spectral sequence (see [6], p. 95, and [4], pp. 813–816) converging to $H^*(PSL(2, \mathbb{Z}[1/n]))$ whose $E_1$-term is given by

$$E_1^{s,t} = \bigoplus H^t(PTG_0(p_1, \ldots, p_k))^{2k-s},$$

where the summation is over all $s$-element subsets of $\{p_1, \ldots, p_k\}$. In particular, for $s = 0$, $E_1^{0,*}$ is the sum of $2^k$ copies of the cohomology of $PSL(2, \mathbb{Z})$.

From the existence of this spectral sequence, we may use the results of this paper to obtain immediately the following results, which were computed inductively in [3].

**Proposition 4.1.** In dimension one we have $H^1(PSL(2, \mathbb{Z}[1/n])) = \{0\}$.

**Proposition 4.2.** For $m \geq k + 2$, $H^m(PSL(2, \mathbb{Z}[1/n]))$ is a finite abelian group possessing only 2- and 3-torsion.

We remark that for $t \neq 1$ it is fairly easy to compute $d_1$ and thus obtain $E_2^{s,t}$.

To state the results of this computation, for $n$ as above and $j = 0, 1, 2$ we define the functions

$$\beta_j(n) = \text{card}\{p_i \mid 1 \leq i \leq k \text{ and } b(p_i) = j\}$$

and

$$\gamma_j(n) = \text{card}\{p_i \mid 1 \leq i \leq k \text{ and } c(p_i) = j\}.$$
Proposition 4.3. For \( t \geq 2 \), \( t \) even,
\[
E_{2}^{s,t} = \left( \mathbb{Z}/3 \right)^{2^{\beta_0(n)}} \oplus \left( \mathbb{Z}/2 \right)^{2^{\gamma_0(n)}}.
\]
Furthermore, \( E_{2}^{0,0} = \mathbb{Z} \), and \( E_{2}^{s,0} = 0 \) for \( s > 0 \).

This permits us to bound the abelianization of \( \text{PSL}(2, \mathbb{Z}[1/n]) \).

Corollary 4.4. The abelianization \( H_1(\text{PSL}(2, \mathbb{Z}[1/n])) \) of \( \text{PSL}(2, \mathbb{Z}[1/n]) \) is a subgroup of
\[
\left( \mathbb{Z}/3 \right)^{2^{\beta_0(n)}} \oplus \left( \mathbb{Z}/2 \right)^{2^{\gamma_0(n)}}.
\]

We have no idea how to compute higher differentials or how to solve the extension problem at \( E_\infty \), so we shall not pursue this further at present. We note, however, that in dimensions greater than \( k + 1 \), the integral cohomology coincides with the Farrell cohomology, which has been computed by N. Naffah in [5]. Naffah’s results show that for \( m \geq k + 2 \),
\[
H^m(\text{PSL}(2, \mathbb{Z}[1/n])) \simeq \bigoplus_{s=0}^{k} E_{s,m-s}^{s,m-s},
\]
so in this range, \( E_{s,m-s}^{s,m-s} = E_{2}^{s,m-s} \) and the extensions are trivial.

Since in rational cohomology the spectral sequence is concentrated on the line \( t = 1 \), the only differential is \( d_1 \) and \( H^{s+1}(\text{SL}(2, \mathbb{Z}[1/n]), \mathbb{Q}) \simeq E_{2}^{s,1} \otimes \mathbb{Q} \). Although the computation of this differential is beyond the scope of this note, we plan to return to it in future work. From \( E_1 \), we get the following crude upper bounds.

Proposition 4.5. The rational cohomology \( H^s(\text{SL}(2, \mathbb{Z}[1/n]), \mathbb{Q}) \) is trivial in dimensions greater than \( k + 1 \), and for \( 2 \leq s \leq k + 1 \) we have
\[
\text{rank}(H^s(\text{SL}(2, \mathbb{Z}[1/n]), \mathbb{Q})) \leq 2^{k+1-s} \sum r(p_{i_1} \ldots p_{i_s}),
\]
where the summation is over all \( s \)-element subsets of \( \{p_1, \ldots, p_k\} \).

References


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