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A COUNTEREXAMPLE TO A QUESTION OF R. HAYDON, E. ODELL AND H. ROSENTHAL

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Abstract. We give an example of a compact metric space K, an open dense subset U of K, and a sequence (f_n) in $C(K)$ which is pointwise convergent to a non-continuous function on K, such that for every $u \in U$ there exists $n \in \mathbf{N}$ with $f_n(u) = f_m(u)$ for all $m \geq n$, yet (f_n) is equivalent to the unit vector basis of the James quasi-reflexive space of order 1. Thus c_0 does not embed isomorphically in the closed linear span $[f_n]$ of (f_n). This answers in the negative a question asked by H. Haydon, E. Odell and H. Rosenthal.

1. Introduction

A result of J. Elton [E], which was also proved later by R. Haydon, E. Odell and H. Rosenthal [HOR], states that if K is a compact metric space, and (f_n) is a uniformly bounded sequence in $C(K)$ such that

$$\sum_{n=1}^{\infty} |f_{n+1}(k) - f_n(k)| < \infty, \ \forall k \in K,$$

and the pointwise limit of (f_n) on K is a non-continuous function, then c_0 embeds isomorphically in the closed linear span $[f_n]$ of (f_n). Thus the following question was naturally raised by R. Haydon, E. Odell and H. Rosenthal:

Question 4.7 in [HOR]. Let K be a compact metric space, R be a residual subset of K (i.e. $K \backslash R$ is a first category set), and (f_n) be a sequence in $C(K)$ which converges pointwise on K to a non-continuous function, and

$$\sum_{n=1}^{\infty} |f_{n+1}(r) - f_n(r)| < \infty, \text{ for all } r \in R.$$

Does c_0 embed in the closed linear span $[f_n]$ of (f_n)?

We will construct a compact metric space K, an open dense subset U of K and a sequence $(g_n) \subset C(K)$ such that

(a) $(\sum_{i=1}^{n} g_i)_n$ is a uniformly bounded and pointwise convergent sequence on K to a non-continuous function;
(b) For every $u \in U$ there exists $n \in \mathbf{N}$ such that $g_m(u) = 0$ for every $m \geq n$;

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(c) $[g_n]$ is isomorphic to the James quasi-reflexive of order 1 space J.

Since, of course, c_0 does not embed isomorphically in J, this answers in the negative Question 4.7 of [HOR]. Our construction is very elementary and explicit, even though a shorter proof of the existence of a counterexample to Question 4.7 of [HOR] can be given along similar lines using more advanced machinery.

2. The construction

We recall the definition of the James space J and some simple facts. Let c_{00} denote the finitely supported sequences of real numbers. For $(x_n) \in c_{00}$ we define

$$\|(x_n)\|_J = \sup\{[x_{p_1}^2 + (x_{p_2} - x_{p_1})^2 + \cdots + (x_{p_k} - x_{p_{k-1}})^2]^{1/2} : k \in \mathbf{N}, 1 \le p_1 < p_2 < \cdots < p_{k-1} < p_k\}.$$

Then the James space J is the completion of $(c_{00}, \|\cdot\|_J)$. If (e_n) is the unit vector basis of c_{00}, then (e_n) becomes the unit vector basis of J, which is monotone and shrinking. Also, $(\sum_{i=1}^n e_n)_n$ is a weak-Cauchy sequence which is not weakly convergent in J. If $(a_n) \in c_0$ is a monotone sequence of real numbers (i.e. non-increasing, or non-decreasing) then $\|(a_n)\|_J = |a_1|$ (this is because if $a, b \in \mathbf{R}$ with $ab \ge 0$, then $a^2 + b^2 \le (a+b)^2$).

Notation. For $(a_n), (b_n) \in c_{00}$, we define $(a_n) \cdot (b_n) \in c_{00}$ by

$$(a_n) \cdot (b_n) = (a_n b_n).$$

Lemma 2.1. *For $(a_n), (b_n) \in c_{00}$ we have*

$$\|(a_n) \cdot (b_n)\|_J \le \|(a_n)\|_J \|(b_n)\|_\infty + \|(a_n)\|_\infty \|(b_n)\|_J.$$

Proof. For some $k \in \mathbf{N}$ and some finite sequence of positive integers $1 \le p_1 < p_2 < \cdots p_k$ we have

$$\begin{aligned} \|(a_n) \cdot (b_n)\|_J &= [(a_{p_1} b_{p_1})^2 + (a_{p_2} b_{p_2} - a_{p_1} b_{p_1})^2 + \cdots + (a_{p_k} b_{p_k} - a_{p_{k-1}} b_{p_{k-1}})^2]^{1/2} \\ &= [(a_{p_1} b_{p_1})^2 + (a_{p_2}(b_{p_2} - b_{p_1}) + (a_{p_2} - a_{p_1}) b_{p_1})^2 \\ &\quad + \cdots + (a_{p_k}(b_{p_k} - b_{p_{k-1}}) + (a_{p_k} - a_{p_{k-1}}) b_{p_{k-1}})^2]^{1/2}. \end{aligned}$$

Therefore by the triangle inequality in ℓ_2 we have that

$$\begin{aligned} \|(a_n) \cdot (b_n)\|_J &\le [a_{p_1}^2 b_{p_1}^2 + (a_{p_2} - a_{p_1})^2 b_{p_1}^2 + \cdots + (a_{p_k} - a_{p_{k-1}})^2 b_{p_{k-1}}^2]^{1/2} \\ &\quad + [a_{p_2}^2 (b_{p_2} - b_{p_1})^2 + \cdots + a_{p_k}^2 (b_{p_k} - b_{p_{k-1}})^2]^{1/2} \\ &\le [a_{p_1}^2 + (a_{p_2} - a_{p_1})^2 + \cdots + (a_{p_k} - a_{p_{k-1}})^2]^{1/2} \|(b_n)\|_\infty \\ &\quad + \|(a_n)\|_\infty [(b_{p_2} - b_{p_1})^2 + \cdots + (b_{p_k} - b_{p_{k-1}})^2]^{1/2} \\ &\le \|(a_n)\|_J \|(b_n)\|_\infty + \|(a_n)\|_\infty \|(b_n)\|_J, \end{aligned}$$

which finishes the proof of the lemma. ☐

Now we are ready to see the counterexample. Let $K := \{(a,b) \in \mathbf{R^2} : \mathbf{0 \le a \le 1, 0 \le b \le 1}\}$. Since $C[0,1]$ is universal for the class of separable spaces, there exist a sequence $(f_n) \subset C[0,1]$ and $M > 0$ such that (f_n) is M-equivalent to the unit vector basis of J. For $n \in \mathbf{N}$ set $K_n := \{(a,b) \in \mathbf{R^2} : \mathbf{0 \le a \le 1, 1/2^n \le b \le 1}\}$, $L_n := \{(a,b) \in \mathbf{R^2} : \mathbf{0 \le a \le 1, b = 1/2^n}\}$, $L := \{(a,0) : 0 \le a \le 1\}$ and $U = K \backslash L$. Now, for $n \in \mathbf{N}$ define $g_n : K \to \mathbf{R}$ by

- $g_n \mid K_n \equiv 0$,

- for every $0 \leq a \leq 1$, g_n restricted to the segment connecting the points $(a, 1/2^n)$ and $(a, 0)$, is linear,
- $g_n \mid L \equiv f_n$,
- g_n is continuous on K.

We will show that (g_n) is equivalent to the unit vector basis (e_i) of the James space. This will imply that $(\sum_{i=1}^n g_i)_n$ is a weak Cauchy sequence which is not weakly convergent, which will finish the proof. Let $n \in \mathbf{N}$ and $(\lambda_i)_{i=1}^n \subset \mathbf{R}$. We want to estimate $\|\lambda_1 g_1 + \cdots + \lambda_n g_n\|_\infty$. For $(a,b),(c,d) \in K$, let $[(a,b),(c,d)]$ denote the linear segment connecting the points (a,b) and (c,d). For every $0 \leq a \leq 1$ we have that

- $(\lambda_1 g_1 + \cdots + \lambda_n g_n) \mid [(a,1),(a,1/2)] \equiv 0$,
- $(\lambda_1 g_1 + \cdots + \lambda_n g_n) \mid [(a,1/2^i),(a,1/2^{i+1})]$ is linear, for every $i = 1, \ldots, n-1$,
- $(\lambda_1 g_1 + \cdots + \lambda_n g_n) \mid [(a,1/2^n),(a,0)]$ is linear,
- $\lambda_1 g_1 + \cdots + \lambda_n g_n$ is continuous on K.

Therefore we obtain

$$\begin{aligned}
&\|\lambda_1 g_1 + \cdots + \lambda_n g_n\|_\infty \\
&\qquad = \max_{2 \leq k \leq n} \|(\lambda_1 g_1 + \cdots + \lambda_n g_n) \mid L_k\|_\infty \vee \|(\lambda_1 g_1 + \cdots + \lambda_n g_n) \mid L\|_\infty \\
&\qquad = \max_{2 \leq k \leq n} \|(\lambda_1 g_1 + \cdots \lambda_{k-1} g_{k-1}) \mid L_k\|_\infty \vee \|\lambda_1 f_1 + \cdots + \lambda_n f_n\|_\infty.
\end{aligned}$$

Therefore we immediately obtain the lower estimate

$$\begin{aligned}
\|\lambda_1 g_1 + \cdots + \lambda_n g_n\|_\infty &\geq \|\lambda_1 f_1 + \cdots + \lambda_n f_n\|_\infty \\
&\geq \frac{1}{M}\|\lambda_1 e_1 + \cdots + \lambda_n e_n\|_J.
\end{aligned}$$

For the upper estimate we need to estimate $\|(\lambda_1 g_1 + \cdots + \lambda_n g_n \mid L_k)\|_\infty$ for $2 \leq k \leq n$. Note that for $0 \leq a \leq 1$ and $2 \leq k \leq n$ we have

$$\begin{aligned}
&(\lambda_1 g_1 + \cdots + \lambda_n g_n)(a, 1/2^k) \\
&\qquad = \lambda_1 \frac{\frac{1}{2} - \frac{1}{2^k}}{\frac{1}{2}} f_1(a) + \lambda_2 \frac{\frac{1}{2^2} - \frac{1}{2^k}}{\frac{1}{2^2}} f_2(a) + \cdots + \lambda_{k-1} \frac{\frac{1}{2^{k-1}} - \frac{1}{2^k}}{\frac{1}{2^{k-1}}} f_{k-1}(a) \\
&\qquad = \lambda_1 \frac{2^{k-1}-1}{2^{k-1}} f_1(a) + \lambda_2 \frac{2^{k-2}-1}{2^{k-2}} f_2(a) + \cdots + \lambda_{k-1} \frac{2-1}{2} f_{k-1}(a).
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
&\|\lambda_1 g_1 + \cdots + \lambda_{k-1} g_{k-1} \mid L_k\|_\infty \\
&\qquad = \|\lambda_1 \frac{2^{k-1}-1}{2^{k-1}} f_1 + \lambda_2 \frac{2^{k-2}-1}{2^{k-2}} f_2 + \cdots + \lambda_{k-1} \frac{2-1}{2} f_{k-1}\|_\infty \\
&\qquad \leq M\|\lambda_1 \frac{2^{k-1}-1}{2^{k-1}} e_1 + \lambda_2 \frac{2^{k-2}-1}{2^{k-2}} e_2 + \cdots + \lambda_{k-1} \frac{2-1}{2} e_{k-1}\|_J \\
&\qquad = M\|(\lambda_1, \lambda_2, \cdots, \lambda_{k-1}, 0, \ldots) \\
&\qquad\qquad \cdot (\frac{2^{k-1}-1}{2^{k-1}}, \frac{2^{k-2}-1}{2^{k-2}}, \ldots, \frac{2-1}{2}, 0, \ldots)\|_J \\
&\qquad \leq M\|\lambda_1 e_1 + \cdots + \lambda_{k-1} e_{k-1}\|_J \cdot 1 \\
&\qquad\quad + M\|(\lambda_i)_{i=1}^{k-1}\|_\infty \|(\frac{2^{k-1}-1}{2^{k-1}}, \ldots, \frac{2-1}{2}, 0, \ldots)\|_J \quad \text{(by Lemma 2.1)}
\end{aligned}$$

$$\leq M\|\lambda_1 e_1 + \cdots + \lambda_{k-1}e_{k-1}\|_J + M\|(\lambda_i)\|_\infty \frac{2^{k-1}-1}{2^{k-1}} \quad \text{(since the sequence } \left(\frac{2^{k-1}-1}{2^{k-1}}, \frac{2^{k-2}-1}{2^{k-2}}, \dots, \frac{2-1}{2}, 0, \dots,\right) \text{ is decreasing)}$$

$$\leq 2M\|\lambda_1 e_1 + \cdots + \lambda_{k-1}e_{k-1}\|_J \quad \text{(since } \|(\lambda_i)_{i=1}^{k-1}\|_\infty \leq \|(\lambda_i)_{i=1}^{k-1}\|_J).$$

Also, since $\|\lambda_1 f_1 + \cdots + \lambda_n f_n\|_J \leq M\|\lambda_1 e_1 + \cdots + \lambda_n e_n\|_J$, we obtain

$$\|\lambda_1 g_1 + \cdots + \lambda_n g_n\|_\infty \leq 2M\|\lambda_1 e_1 + \cdots + \lambda_n e_n\|_J.$$

This finishes the proof. □

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