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CONTINUITY OF K-THEORY: AN EXAMPLE IN EQUAL CHARACTERISTICS

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ABSTRACT. If k is a perfect field of characteristic $p > 0$, we show that the Quillen K-groups $K_i(k[[t]])$ are uniquely p-divisible for $i = 2, 3$. In fact, the Milnor K-groups $K_n^M(k((t)))$ are uniquely p-divisible for all $n > 1$. This implies that $K(A) \to \operatorname{holim}_{\overleftarrow{n}} K(A/\mathfrak{m}^n)$ is 4-connected after profinite completion for A a complete discrete valuation ring with perfect residue field.

Let A be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$. Let

$$K^{top}(A) = \operatorname*{holim}_{\overleftarrow{n}} K(A/\mathfrak{m}^n)$$

We say that K-theory is *continuous* (at A) if it commutes with the (inverse) limit, in the sense that

$$K(A)^{\widehat{}} \to K^{top}(A)^{\widehat{}}$$

is an equivalence, where $X \to X^{\widehat{}}$ denotes profinite completion.

This question of continuity has acquired new relevance since the fibers of

$$K(A/\mathfrak{m}^n)^{\widehat{}} \to K(A/\mathfrak{m})^{\widehat{}}$$

are now better understood, and have been shown by McCarthy [Mc] to agree with the corresponding fibers in topological cyclic homology. Hence we are in a position where we sometimes can calculate $K^{top}(A)$.

One situation where we have an affirmative answer is the theorem of Suslin and Panin [Su], [P], which says that if A is a Henselian discrete valuation ring with maximal ideal $\mathfrak{m}$, then

$$K(A)^{\widehat{}}_{\ell} \to \operatorname*{holim}_{\overleftarrow{n}} K(A/\mathfrak{m}^n)^{\widehat{}}_{\ell}$$

is an equivalence for all primes ℓ different from the characteristic of (the field of fractions of) A. So, if A is of characteristic zero, then K-theory is continuous at A.

This theorem was used critically in Bökstedt and Madsen's calculation [BM] of the K-theory of the p-adic integers in order to get the correspondence with topological cyclic homology (here the situation was a bit special, as a similar statement holds for TC).

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The limiting condition in Suslin/Panin's result is that ℓ has to be different from the characteristic of A. If $\ell \neq char(A/\mathfrak{m})$ then a result of Gabber [Ga] tells us furthermore that $K(A)\hat{}_\ell \to K(A/\mathfrak{m}^n)\hat{}_\ell \to K(A/\mathfrak{m})\hat{}_\ell$ are equivalences, and so the situation is really rather degenerate. However, at the characteristic of A, K-theory is largely unknown, except for some rather old results in dimensions ≤ 2.

In dimension 2, much insight can be deduced from a generators–and–relations presentation of $K_2(A)$ (e.g. [DS]). It is the hope that a presentation of $K(A)$ as in [D] can shed light on the general situation. In this paper we choose a different approach, in that we try to study the lower K-groups directly by means of Milnor K-theory, and then use Merkur′ev and Suslin's results [MS] to get information about the K–groups themselves.

We take results showing that $K(A)\hat{} \to K^{top}(A)\hat{}$ is somewhat connected for evidence that $K(A)$ may be continuous.

In this note we shall prove

Theorem 1. *If A is a complete discrete valuation ring with perfect residue field, then*

$$K(A)\hat{} \to K^{top}(A)\hat{}$$

is 4-connected.

Note added in proof (Aug. 26, 1997). It appears that Thomas Geisser has shown that $K_i^{ind}(F)$ is uniquely p-divisible for any field F of characteristic $p > 0$. Using a slight modification of lemma 6, the observations in this note then show that $K(A)\hat{} \to K^{top}(A)\hat{}$ is an equivalence for the rings in theorem 1.

Remark 2. Note that our definition of $K_i^{top}(A) = \pi_i K^{top}(A)$ differs slightly from, e.g., the notion in [W], as we are using the homotopy limit. This is connected to the inverse limit of the homotopy groups by the short exact sequence

$$0 \to \lim_{\leftarrow}{}^{(1)} K_{i+1}(A/\mathfrak{m}^n) \to K_i^{top}(A) \to \lim_{\leftarrow} K_i(A/\mathfrak{m}^n) \to 0.$$

We will be using completion of spectra as in [B]. Note that completions commute with homotopy inverse limits. Recall that if X is a spectrum, then there is an exact sequence

$$0 \to Ext(\mathbf{Z}[1/p]/\mathbf{Z}, \pi_n X) \to \pi_n(X\hat{}_p) \to Hom(\mathbf{Z}[1/p]/\mathbf{Z}, \pi_{n-1}X) \to 0.$$

Remark 3. The reason we have to profinitely complete everything before we ask for an equivalence, is that $K(A) \to K^{top}(A)$ is certainly not an equivalence integrally. For instance, if A is the p-adic integers, $\mathbf{Z}\hat{}_p$, then $F = \mathbf{Q}\hat{}_p$ is uncountable, and hence by [M, 11.10] $K_2(F)$ is uncountable. By the localization sequence

$$0 = K_2(\mathbf{F}_p) \to K_2(A) \to K_2(F) \to K_1(\mathbf{F}_p) \to 0$$

we get that $K_2(A)$ is uncountable too. But, as each $K_3(\mathbf{Z}/p^k\mathbf{Z})$ is a finite group, we get that the $\lim_{\leftarrow}{}^{(1)}$ term vanishes, and $K_2^{top}(A) \cong \lim_{\leftarrow} K_2(\mathbf{Z}/p^k\mathbf{Z})$, which by [M, p. 180] and [W] is trivial.

A similar consideration rules out integral continuity for the case of equal characteristics too.

Proof of theorem 1. Let F be the field of fraction of A, let k be the residue field, and let p be the characteristic of k. F has characteristic either zero or p. The characteristic zero part is taken care of by the Suslin/Panin result mentioned above.

The only remaining piece is the case where $char(F) = char(k) = p > 0$. Then A must be isomorphic to the ring of formal power series $k[[t]]$ [Se, II, 4.2]. This is stated separately as proposition 7 below, and the proof will occupy the rest of the paper. □

First we prove some useful lemmas regarding the p-divisibility of the groups $K_i(A)$. This is important in this context, since we have by [He] that $\pi_i K^{top}(A)\hat{}_p = 0$ for $i > 1$.

Lemma 4. *Let $A = k[[t]]$ be the ring of formal power series in a perfect field k of characteristic $p > 0$, and let $F = k((t)) = k[[t]][t^{-1}]$ be its field of fractions. Let $n > 1$. Then $K_n(A)$ is uniquely p-divisible if and only if $K_n(F)$ is.*

Proof. As k is perfect, we get by by [K] or [Hi] that $K_n(k)$ is uniquely p-divisible for $n > 0$. By [Ge, 1.3] the localization sequence breaks up into short exact sequences

$$0 \to K_n(A) \to K_n(F) \to K_{n-1}(k) \to 0.$$

By hypothesis, multiplication by p is an isomorphism on two of the three groups, and hence also on the third. □

For a field F, note that $K_1(F)$ is the multiplicative group $F^* = F - \{0\}$. Milnor K-theory, $K^*(F)$, is defined as the graded ring represented as the quotient of the tensor algebra $T_*(F^*) = \bigoplus_{n=0}^{\infty}(F^*)^{\otimes n}$ by the homogeneous ideal generated by the elements $x \otimes (1 - x) \in F^* \otimes F^*$. The product in K-theory then defines a map of graded rings $K_*^M(F) \to K_*(F)$ which is an isomorphism for $* < 3$.

Lemma 5. *Let $F = k((t))$, where k is a perfect field of characteristic p. Then $K_n^M(F)$ is uniquely p-divisible for $n > 1$.*

Proof. Since F is of characteristic p, the main result of Izboldin [I] gives that $K_n^M(F)$ has no p-torsion, so we just have to show that it is p-divisible. We do this by showing that $K_2^M(F)$ is p-divisible. This is enough, for the surjection $(F^*)^{\otimes n} \to K_n^M(F)$ factors through $K_2^M(F) \otimes (F^*)^{\otimes n-2}$, which is p-divisible, and so $K_n^M(F)$ must be p-divisible for all $n > 1$.

Consider the generator $\{x, y\} \in K_2^M(F)$, where $x, y \in F^*$. We will show that it has a pth root (cf. [M, A.14]). This is clear if there is a $z \in F$ such that $y = z^p$, for then $\{x, y\} = \{x, z^p\} = \{x, z\}^p$.

On the other hand, suppose y has no pth root in F, and consider the inseparable extension $F \subseteq F(t^{1/p})$ of degree p. Note that, as k is perfect, the norm map $F(t^{1/p})^* \to F^*$ given by $z \mapsto z^p$ is surjective ($at^{-N}\prod_{i=0}^{\infty}(1 - a_i t^i)$ is hit by $a^{1/p}t^{-\frac{N}{p}}\prod_{i=0}^{\infty}(1 - a_i^{1/p}t^{\frac{i}{p}})$). In particular, y has a pth root in $F(t^{1/p})$. This means that $F \subset F[z]/(z^p - y) \subseteq F(t^{1/p})$, and since the first inclusion is not an isomorphism, the latter must be, since p is prime.

So, $F \subseteq E = F[z]/(z^p - y)$ is a field extension of degree p, and x is in the image of the norm map, and hence [M, 14.3] applies to show that $\{x, y\}$ has a pth root in $K_2(F)$. □

Lemma 6. *If $A = k[[t]]$, where k is perfect of characteristic $p > 0$, then $K_2(A)$ and $K_3(A)$ are uniquely p-divisible.*

Proof. We have to show that $K_2(F)$ and $K_3(F)$ are uniquely p-divisible for $F = k((t))$. For $i = 2$, note that $K_2^M(F) \cong K_2(F)$ and use the two foregoing lemmas.

Consider the map $K_3^M(F) \to K_3(F)$. Call the kernel K and the cokernel $K_3^{\mathrm{ind}}(F)$, and we are done if both groups turn out to be uniquely p-divisible. By Merkur′ev and Suslin [MS], $K_3^{\mathrm{ind}}(F)$ is uniquely p-divisible. Also, K is annihilated by $(3-1)! = 2$. If $p = 2$, the unique 2-divisibility of $K_3^M(F)$ implies that $K = 0$, and if $p \neq 2$ then $p = 2i + 1$ acts as the identity on K. Anyhow, K is uniquely p-divisible too. □

Proposition 7. *Let k be a perfect field. Then*

$$K(k[[t]])\hat{} \to K^{top}(k[[t]])\hat{}$$

is 4*-connected.*

Proof. Let p be the characteristic of k. First note that $K_i(k[[t]]) \to K_i^{top}(k[[t]])$ is an isomorphism if $i < 2$, and the groups are without p-torsion. If ℓ is a prime different from p, then $\pi_i K(k[[t]])\hat{}_\ell \cong \pi_i K(k[[t]]/t^n)\hat{}_\ell \cong \pi_i K(k)\hat{}_\ell$ by [Ga], and so

$$K(k[[t]])\hat{}_\ell \xrightarrow{\simeq} K^{top}(k[[t]])\hat{}_\ell \xrightarrow{\simeq} K(k)\hat{}_\ell.$$

By Hesselholt [He], $K^{top}(k[[t]])\hat{}_p$ has vanishing homotopy groups in dimension greater than 1. This is actually only stated for finite fields, but the proof works equally well for perfect fields.

If G is an Abelian group, we have by [BK, VI] that

$$Ext(\mathbf{Z}[1/p]/\mathbf{Z}, G) = 0$$

if and only if G is p-divisible, and

$$Hom(\mathbf{Z}[1/p]/\mathbf{Z}, G) = 0$$

if the p-torsion elements in G are of bounded order.

So the proposition follows from the exact sequence

$$0 \to Ext(\mathbf{Z}[1/p]/\mathbf{Z}, K_i(A)) \to \pi_i(K(A)\hat{}_p) \to Hom(\mathbf{Z}[1/p]/\mathbf{Z}, K_{i-1}(A)) \to 0,$$

the unique p-divisibility of $K_i(k[[t]])$ for $i = 2, 3$, and the lack of p-torsion in $K_1(k[[t]])$. □

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