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ON UNIQUENESS OF RIEMANN'S EXAMPLES

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Abstract. We prove that a properly embedded minimal annulus with one flat end, bounded in a slab by lines or circles, is a part of a Riemann's example.

In 1956 Shiffman [16] proved that a minimal annulus bounded by a pair of circles in two parallel planes intersects every plane between the two planes by a circle. Therefore, by a result of Riemann and Enneper ([12], pages 85-90), it must be a part of the catenoid or a part of a Riemann's example. In [7], Hoffman, Karcher and Rosenberg proved that a properly embedded minimal annulus bounded by two parallel lines in a slab must be a piece of a Riemann's example. Toubiana [17] proved that no properly embedded minimal annulus in a slab can be bounded by a pair of nonparallel lines. In [5] the first author generalized these results to a properly embedded minimal annulus in a slab bounded by a combination of circles or lines in parallel planes. He showed that the annulus must be a part of a catenoid or a part of a Riemann's example; thus if the boundary consists of two lines, the lines must be parallel.

In this paper, we generalize the first author's result to allow the minimal annulus to have a flat end in the slab. Denote $S = \{(x_1, x_2, x_3) : -1 \leq x_3 \leq 1\}$ and let S_{-1} and S_1 be the two boundary planes of S at $x_3 = -1, 1$, respectively. We prove the following theorem.

Theorem 1. *Suppose $A \subset S$ is a properly embedded minimal annulus with a flat end. If ∂A consists of a line or circle in S_{-1} and a line or circle in S_1, then A intersects every plane between S_{-1} and S_1 by a circle, except at the height of the end, where the intersection is a line. Consequently, A is a piece of a Riemann's example. In particular, if the boundary consists of two lines, the lines must be parallel.*

Some remarks are in order before we proceed to the proof. It is known that Riemann's examples are properly embedded minimal annuli with infinitely many flat ends. Any slab parallel to the ends cut such a surface by an annulus with a finite number of ends. It is natural to ask whether or not Riemann's examples are the only properly embedded minimal annuli with infinitely many flat ends, or whether they are the only properly embedded minimal annuli in a slab with a finite number of ends. The above theorem only deals with annuli in a slab with one end. Prior to it, Romon [15] proved the same result under the assumption that the

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boundary consists of two parallel lines. A more general result of Pérez and Ros [14] states that no properly embedded minimal annulus with a finite number of flat ends can be bounded in a slab by two nonparallel lines. More recently, López, Ritoré and the second author [10] proved that Riemann's examples are the only properly embedded minimal tori with two planar ends in $\mathbb{R}^3/T$, where T is the subgroup generated by a non-trivial translation in $\mathbb{R}^3$.

The proof of the theorem follows three lemmas given below. Lemma 1 determines the conformal type of the annulus A; Lemma 2 gives a lower bound for the total curvature of A; Lemma 3 establishes a Jacobi field on A which is derived from the derivative of the curvature of the level curves. These lemmas are proved in similar ways as in [5]. As in that paper, the proof of the theorem involves finding an upper bound for the total curvature by studying the nodal set of the Jacobi field. The comparison of the upper bound and the lower bound obtained in Lemma 2 shows that the Jacobi field vanishes on the annulus. However, due to the equality part of the estimate in Lemma 2, which is not a case in [5], a subtler investigation is needed for the current case. We resolve the problem by studying the image of the nodal domains under the Gauss map, and a non-trivial application of the Riemann-Hurwitz formula.

We begin with the determination of the conformal type of A. The conformal type of the interior of A is

$$\{z \in \mathbf{C} \mid \rho < |z| < \sigma\} - \{p_0\}, \quad 0 \leq \rho < \sigma \leq \infty \quad \text{and} \quad \rho < |p_0| < \sigma.$$

Since A has one-dimensional boundaries which are separated by the interior of A, we have $\rho > 0$ and $\sigma < \infty$. Thus there is an $R \in (1, \infty)$ such that the interior of A is conformally a plane ring as follows:

$$\{z \in \mathbf{C} \mid 1/R < |z| < R\} - \{p_0\} := A_R - \{p_0\}, \quad 1/R < |p_0| < R.$$

Let $X : A_R - \{p_0\} \to S$ be the proper minimal embedding of the interior of A. It is well known that X is smooth up to the boundary. Thus there is a $C \subset \partial A_R$ such that X can be smoothly extended to $\partial A_R - C$. Since X is a proper embedding, $\partial A_R - C$ is homeomorphic to $A(1) \cup A(-1)$. Therefore $A = X(\overline{A_R} - \{p_0\} - C)$. Actually, C can be described as the set such that $|X(z_n)| \to \infty$ whenever $z_n \to z \in C$.

Assume that $\partial A \subset S_1 \cup S_{-1}$ and $A(1) = \partial A \cap S_1$, $A(-1) = \partial A \cap S_{-1}$ are circles or straight lines. Let $g : A_R - \{p_0\} \to \mathbf{C}$ be the Gauss map of X. We have

Lemma 1. *If $A(1)$ (resp. $A(-1)$) is a circle, then $C \cap \{|z| = R\} = \emptyset$ (resp. $C \cap \{|z| = 1/R\} = \emptyset$). If $A(1)$ (resp. $A(-1)$) is a straight line, then $C \cap \{|z| = R\} = \{p\}$ (resp. $C \cap \{|z| = 1/R\} = \{q\}$).*

The third coordinate function X^3 is given by $X^3(z) = \log |z| / \log R$.

The Gauss map g can be extended to a neighborhood of A_R such that the extended g at p_0, p and q has either zero or pole of order 2. Moreover, the Gauss map g has neither zero nor pole in a neighborhood of A_R except at p_0, p and q.

Proof. Clearly if $A(1)$ or $A(-1)$ is a circle, then $C \cap \{|z| = R\} = \emptyset$ or $C \cap \{|z| = 1/R\} = \emptyset$. Therefore, we need only consider the case that $A(1)$ or $A(-1)$ is a straight line.

Let $J = X(\{|z| = r\})$ be the Jordan curve on A, where $|p_0| < r < R$, and let A_1 be the proper minimal annulus in A with boundary $A(1)$ and J. By our assumption, $A(1)$ is a straight line. Let $\mathcal{R}$ be the rotation around $A(1)$ of angle π. It is well known that $A_1 \cup \mathcal{R}(A_1)$ is a smooth proper minimal surface with boundary $J \cup \mathcal{R}(J)$. The conformal structure of $A_1 \cup S(A_1)$ is then $\{r < |z| < R^2/r\} - C \cap \{|z| = R\}$ (with the mapping $Y(z) = X(z)$ for $z \in A_R - C$ and $Y(z) = \mathcal{R}(X(R^2 z/|z|^2))$ for $z \in \{R < |z| < R^2/r\}$).

Since $\{|z| = R\} - C$ is homeomorphic to a straight line, $C \cap \{|z| = R\}$ is a point or an interval.

Let $D \subset \{r < |z| < R^2/r\}$ be a disk-like neighborhood of $C \cap \{|z| = R\}$ such that $z \in D$ if and only if $R^2 z/|z| \in D$ and ∂D is diffeomorphic to a circle. Then $Y(\partial D)$ is a Jordan curve on $A_1 \cup \mathcal{R}(A_1)$ which bounds a properly embedded minimal annulus $\tilde{A} = Y(D - C \cap \{|z| = R\})$. Since $A_1 \cap \mathcal{R}(A_1)$ is contained in the slab $-1 \le x_3 \le 3$, by the Cone Lemma in [8], $\tilde{A}$ has finite total curvature. Therefore, by Osserman's theorem, this annular end has the conformal type of a punctured disk, and the Gauss map of $\tilde{A}$ can be extended to the puncture. In particular, $C \cap \{|z| = R\}$ is a single point p and the Gauss map $g : D \to \mathbf{C}$ of Y can be extended to p, and $g(p)$ is either zero or ∞ since the flat end has vertical limit normal. Similarly, we can prove that $C \cap \{|z| = 1/R\} = \{q\}$ if $A(-1)$ is a straight line and $g(q)$ is either zero or ∞.

Since the flat end of the extended end at p has vertical limit normal, the third coordinate X^3 extends smoothly to $\{r < |z| < R^2/r\}$, and $X^3 = 1$ on $\{|z| = R\}$. Similarly, $X^3 = -1$ on $\{|z| = 1/R\}$. By the uniqueness for harmonic functions, we know that $X^3(z) = \log |z| / \log R$.

The level set $A(t) = X^{-1}(\{x_3 = t\})$, $-1 < t < 1$, is $A(t) = \{|z| = \exp(t \log R)\}$. It is a circle except at $t = \log |p_0| / \log R$, which is the height of the flat end at p_0. Thus along $A(t)$, $-1 < t < 1$, g has no zero or pole, otherwise $A(t)$ would not be a one dimensional manifold.

On the other hand, the Enneper-Weierstrass representation is

$$\omega_1 = \frac{1}{2}(1 - g^2)\eta, \quad \omega_2 = \frac{i}{2}(1 + g^2)\eta, \quad \omega_3 = g\eta.$$

Since $X^3(z) = \log |z| / \log R$, we have

$$\eta = f(z)dz = \frac{1}{zg(z) \log R} dz. \tag{1}$$

Since the surface has embedded flat ends of finite total curvature with vertical limit normals at p_0, p, and q, it is well known that at those points ω_1 and ω_2 should have an order 2 pole. Since z and dz do not have zero or pole in $\overline{A_R}$, g has order 2 pole or zero at p_0, p, and q. $\square$

Lemma 2. *The total curvature of A in Theorem 1 satisfies*

$$K(A) \ge -8\pi.$$

Equality holds if and only if both $A(1)$ and $A(-1)$ are straight lines.

Proof. By Lemma 1 and the Enneper-Weierstrass representation we know that the metric of A is given by $ds^2 = \Lambda^2 |dz|^2$ where

$$\Lambda = \frac{1}{2|z| \log R} \left(\frac{1}{|g(z)|} + |g(z)| \right).$$

Suppose that $A(1)$ and $A(-1)$ are both straight lines; then $C = \{p, q\}$. Let U, V, W be disks centered at p, q, p_0 with radii $\epsilon > 0$ small enough. By the

Gauss-Bonnet formula we have

$$\int_{A_R-(U\cup V\cup W)} KdA + \int_{\partial[A_R-(U\cup V\cup W)]} \kappa_g ds + \alpha_1 + \beta_1 + \alpha_2 + \beta_2 = -2\pi,$$

where α_1 and β_1 are the exterior angles between ∂U and ∂A_R; α_2 and β_2 are the exterior angles between ∂V and ∂A_R. Clearly,

$$\lim_{\epsilon\to 0}(\alpha_i+\beta_i) = \pi, \quad i = 1,\ 2.$$

Let $(\epsilon\cos t, -\epsilon\sin t)$, $0 \le t \le 2\pi$, be the parametrization of ∂U, ∂V, ∂W respectively. By Minding's formula

$$\kappa_g \Lambda = -\frac{1}{\epsilon} + \frac{\partial}{\partial \nu}\log\Lambda,$$

where ν is the interior normal $(-\cos t, \sin t)$; see [4], page 34. A little calculation using $\kappa_g ds = \kappa_g \Lambda\epsilon\, dt$ and $\Lambda \sim |z-p_0|^{-2}$ shows that (see [6] for details)

$$\lim_{\epsilon\to 0}\int_{\partial W}\kappa_g ds = 2\pi, \qquad \lim_{\epsilon\to 0}\int_{\partial U\cap A_R}\kappa_g ds = \lim_{\epsilon\to 0}\int_{\partial V\cap A_R}\kappa_g ds = \pi.$$

Since $\int_{\partial A_R}\kappa_g ds = 0$, we have

$$\int_{A_R} KdA = \lim_{\epsilon\to 0}\int_{A_R-(U\cup V\cup W)} KdA = -8\pi.$$

If $A(1)$ or $A(-1)$ is a circle, then

$$\left|\int_{A(1)}\kappa_g ds\right| < 2\pi \quad \text{or} \quad \left|\int_{A(-1)}\kappa_g ds\right| < 2\pi,$$

and we have

$$\int_{A_R} KdA > -8\pi.$$

Therefore, the total curvature is equal to -8π if and only if both $A(1)$ and $A(-1)$ are straight lines. □

Our proof of A being a part of a Riemann's example is to show that each level set $A(t)$ has constant plane curvature. We introduce a function u for this purpose. Let $z = re^{i\theta}$ and denote the plane curvature of $A(t)$ by $\kappa(r,\theta)$; then it is not hard to calculate that

$$u := r\Lambda\frac{\partial\kappa}{\partial\theta} = \operatorname{Im}\left[\frac{1}{2}\frac{|g|^2-1}{|g|^2+1}\left(z\frac{g'}{g}\right)^2 - z\frac{d}{dz}\left(z\frac{g'}{g}\right)\right]. \tag{2}$$

Since g can be extended to a neighborhood of $\overline{A_R}$, u can also be extended to the neighborhood, except possibly at p_0, p, and q. We now are going to prove that the exceptions will not occur.

Lemma 3. *The above function u can be smoothly extended to a neighborhood of $\overline{A_R}$.*

Proof. We only need to prove that u can be smoothly extended to p, q, and p_0. Let $z_0 = p$, q, or p_0.

By Lemma 1, the extended Gauss map $\tilde{g}$ has an order 2 zero or pole at z_0, and $g(z) \neq 0$ or ∞ for $z \in A_R - \{p_0\}$. By (2), $u = u(z,g) = -u(z,g^{-1})$; we can assume that $g(z_0) = 0$.

Let $\zeta = z - z_0$; then

$$\tilde{g}(z) = (z - z_0)^2 h(z) = \zeta^2 h(z_0 + \zeta),$$

where h is a holomorphic function and $h(z_0) \neq 0$.

For convenience, we will write g instead of $\tilde{g}$. Then

$$z\frac{g'(z)}{g(z)} = \frac{2z_0}{z - z_0} + 2 + z\frac{h'(z)}{h(z)} \text{ and } z\frac{g'(z)}{g(z)} = \frac{a_{-1}}{\zeta} + \sum_{k=0}^{\infty} a_k \zeta^k, \quad a_{-1} = 2z_0.$$

Thus,

$$\left(z\frac{g'(z)}{g(z)}\right)^2 = \frac{a_{-1}^2}{\zeta^2} + \frac{2a_{-1}a_0}{\zeta} + \sum_{k=0}^{\infty} b_k \zeta^k,$$

$$z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right) = -\frac{a_{-1}z_0}{\zeta^2} - \frac{a_{-1}}{\zeta} + (\zeta + z_0)\sum_{k=1}^{\infty} k a_k \zeta^{k-1}.$$

Since $a_{-1} = 2z_0$, we have $\frac{1}{2}a_{-1}^2 - a_{-1}z_0 = 0$. Since

$$z\frac{h'(z)}{h(z)} = (\zeta + z_0)\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{h'}{h}\right)^{(k)}(z_0)\zeta^k = z_0\frac{h'(z_0)}{h(z_0)} + \sum_{k=1}^{\infty} d_k \zeta^k,$$

it follows that

$$a_0 = 2 + z_0\frac{h'(z_0)}{h(z_0)}.$$

We would like to calculate a_0. The Weierstrass representation for the extended surface $\mathcal{S}$ is

$$\omega_1 = \frac{1}{\log R}\frac{1}{2z}\left(\frac{1}{\tilde{g}} - \tilde{g}\right)dz,\ \omega_2 = \frac{1}{\log R}\frac{i}{2z}\left(\frac{1}{\tilde{g}} + \tilde{g}\right)dz,\ \omega_3 = \frac{1}{\log R}\frac{1}{z}dz.$$

Again we will write g instead $\tilde{g}$. Let C be a loop around $z = z_0$ in a small disk. Then since $X : \{z : 1/R^3 < |z| < R^3\} - \{z_0\} \to \mathbb{R}^3$ (if $z_0 = q$, reflect around $A(-1)$) is well defined and

$$X(z) = \operatorname{Re}\int_{p_0}^{z} (\omega_1, \omega_2, \omega_3),$$

we must have

$$\operatorname{Re}\int_C \frac{1}{2z}\left(\frac{1}{g(z)} - g(z)\right)dz = 0,$$

$$-\operatorname{Im}\int_C \frac{1}{2z}\left(\frac{1}{g(z)} + g(z)\right)dz = 0,$$

$$\int_C \frac{1}{zg}dz = \overline{\int_C \frac{g}{z}dz} = 0,$$

since $g(z)/z$ is holomorphic at $z = z_0$. Hence we know that the residue of $1/zg(z)$ at $z = z_0$ is zero. Then we have

$$\begin{aligned} 0 &= \lim_{z\to z_0}\left(\frac{(z-z_0)^2}{zg(z)}\right)' = \lim_{z\to z_0}\left(\frac{1}{zh(z)}\right)' \\ &= \lim_{z\to z_0}\left(-\frac{1}{z^2h(z)} - \frac{h'(z)}{zh^2(z)}\right) \\ &= -\frac{1}{z_0^2h(z_0)} - \frac{h'(z_0)}{z_0h^2(z_0)}. \end{aligned}$$

Hence

$$\frac{h'(z_0)}{h(z_0)} = -\frac{1}{z_0} \quad \text{and} \quad a_0 = 2 + z_0\frac{h'(z_0)}{h(z_0)} = 1.$$

Then $a_{-1}a_0 - a_{-1} = 0$ and

$$\begin{aligned} \Phi(z) &:= -\frac{1}{2}\left(z\frac{g'(z)}{g(z)}\right)^2 - z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right) \\ &= -\frac{1}{2}\frac{a_{-1}^2 - 2a_{-1}z_0}{\zeta^2} - \frac{a_{-1}a_0 - a_{-1}}{\zeta} - \frac{1}{2}\sum_{k=0}^{\infty} b_k\zeta^k - (\zeta + z_0)\sum_{k=1}^{\infty} ka_k\zeta^{k-1} \\ &= \sum_{k=0}^{\infty} c_k\zeta^k \end{aligned}$$

is holomorphic near $z = z_0$. Now consider the function

$$U(z) = \frac{|g|^2 - 1}{2(1+|g|^2)}\left(z\frac{g'(z)}{g(z)}\right)^2 - z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right);$$

note that our function u is $\operatorname{Im} U$. $U(z)$ can be rewritten as

$$\begin{aligned} U(z) &= -\frac{1}{2}\left(z\frac{g'(z)}{g(z)}\right)^2 - z\frac{d}{dz}\left(z\frac{g'(z)}{g(z)}\right) + \left(1 - \frac{1}{1+|g|^2}\right)\left(z\frac{g'(z)}{g(z)}\right)^2 \\ &= \Phi(z) + \Psi(z). \end{aligned}$$

Since $|g|^2 = |z - z_0|^4|h(z)|^2 = |\zeta|^4|h(z)|^2$ and the function

$$\zeta^2\left(z\frac{g'(z)}{g(z)}\right)^2$$

is holomorphic, it follows that

$$\frac{1}{\zeta^2}\left(1 - \frac{1}{1+|g|^2}\right) = \frac{1}{\zeta^2}\sum_{k=1}^{\infty}(-1)^{k+1}|g|^{2k} = \overline{\zeta^2}\sum_{k=1}^{\infty}(-1)^{k+1}|\zeta|^{4(k-1)}|h(z)|^{2k}$$

is a C^∞ complex function in a neighborhood of z_0. Thus

$$\Psi(z) = \frac{1}{\zeta^2}\left(1 - \frac{1}{1+|g|^2}\right)\zeta^2\left(z\frac{g'(z)}{g(z)}\right)^2$$

is a C^∞ complex function near $z = z_0$, and so is $U(z)$. In particular, $u(z) = \operatorname{Im} U(z)$ is C^∞ near $z = z_0$. Hence u can be smoothly extended to $p = z_0$. □

Proof of Theorem 1. Simple calculation shows that

$$\triangle u = 2K\Lambda^2 u, \tag{3}$$

where K is the Gauss curvature. Since $A(1)$ and $A(-1)$ have constant plane curvature, by the definition of u and its continuity on $\overline{A_R}$, $u = 0$ on ∂A_R. Thus u satisfies

$$\begin{cases} \triangle u = 2K\Lambda^2 u, \\ u|_{\partial A_R} = 0. \end{cases} \tag{4}$$

Since A is properly embedded, $A(t)$ is a Jordan curve except for $t = -1, 1$, and $\log|p_0|/\log R$. If we can show that $u \equiv 0$ on A_R, then each level set $A(t)$ is a circle except when $t = -1,\ 1,\ \log|p_0|/\log R$, in which case they are straight lines. Thus A is either a part of a catenoid, or a part of a Riemann's example. Since A has a flat end, A must be a part of a Riemann's example; that will complete the proof.

We now prove that $u \equiv 0$.

Assume that $u \not\equiv 0$. By the four vertex theorem, see [9] for example, the nodal set of $u = r\Lambda\kappa_\theta$ divides each $|z| = r$ into at least four segments if $A(t)$, $t = \log r/\log R$, is not a circle. Thus the nodal set Z of u divides A_R into at least four components. On each component Ω of $A_R - Z$, u is either positive or negative. Thus X is unstable on Ω. A theorem of Barbosa and do Carmo [1] says that the total curvature of X on Ω satisfies

$$\int_\Omega K dA \le -2\pi.$$

Thus the total curvature of X must be

$$\int_{A_R} K dA \le -8\pi.$$

However, by Lemma 2,

$$\int_{A_R} K dA \ge -8\pi,$$

and equality holds if and only if both $A(-1)$ and $A(1)$ are straight lines. Thus if at least one of $A(-1)$ and $A(1)$ is a circle, we get a contradiction. This contradiction shows that $u \equiv 0$.

This only leaves the case that both $A(1)$ and $A(-1)$ are straight lines and $A_R - Z$ has exactly four components, say Ω_i, $i = 1, 2, 3, 4$. This can happen only if $Z \cap A_R$ consists of four curves, each connecting the two boundaries of A_R. Let $\{p_1, p_2, p_3, p_4\}$ be the end points of the four curves on $\{|z| = R\}$, and $\{q_1, q_2, q_3, q_4\}$ be the end points of the four curves on $\{|z| = 1/R\}$. Note that it is possible that $p_i = p_j$ for $i \ne j$, and similarly for the q_i's.

Since $\int_{\Omega_i} K dA \le -2\pi$ and $\int_{A_R} K dA = -8\pi$, we must have

$$\int_{\Omega_i} K dA = -2\pi, \quad i = 1,\ 2,\ 3,\ 4.$$

Now consider the image of the Gauss map N. Let $S_i := N(\Omega_i) \subset S^2$. Barbosa and do Carmo's theorem says that if the area of S_i is less than 2π, then X is stable on Ω_i. Therefore the area of S_i must be larger than or equal to 2π. Since the total curvature of X on Ω_i is the negative of the area of S_i counted by multiplicity, we conclude that N is one-to-one onto S_i from Ω_i. Since N is one-to-one, the fact that u is a Jacobi field on Ω_i and vanishes on $\partial\Omega_i$ is equivalent to the first eigenvalue of the sphere Laplacian on S_i being 2. Moreover, S_i has area 2π. By a classical theorem (for example, see [3], page 87, Theorem 2 and the remark after the proof), S_i must be a hemisphere. Thus the Gauss map N maps $\partial\Omega_i$ onto a great circle. Since each Ω_i has common boundary parts with at least two others, N maps $\partial\Omega_i$ into the same great circle, for $i = 1, 2, 3, 4$.

Since $A(1)$ and $A(-1)$ are straight lines, the images of N on $\{|z| = R\}$ or $\{|z| = 1/R\}$ are in planes perpendicular to $A(1)$ and $A(-1)$ respectively. Hence N sends $\{|z| = R\} - \{p\}$ and $\{|z| = 1/R\} - \{q\}$ into great circles. As mentioned above, N sends them into the same great circle. We conclude that $A(1)$ and $A(-1)$ are parallel. Otherwise N sends $\{|z| = R\} - \{p\}$ and $\{|z| = 1/R\} - \{q\}$ into different great circles.

Repeatedly rotating 180° around the parallel boundary straight lines of A and the lines resulting from the rotations gives a singly-periodic minimal surface $\mathcal{N}$ with a translation symmetry T such that $\mathcal{N}/T$ is the surface by rotating A once around $A(1)$, and $\mathcal{N}/T$ is conformally a torus T_1 with 4 punctures. The Gauss map $G = \tau \circ N$ is meromorphic on T_1. Since $\mathcal{N}/T$ has total curvature -16π, we know that G has degree 4; see [2] or [7].

Without loss of generality, we may assume that $A(1)$ is parallel to the y-axis; thus g is real along $\{|z| = R\}$. After the rotation, g is extended to $\tilde{g}(z)$ on $\{1/R < |z| < R^3\}$ by $\tilde{g}(z) = \overline{g(R^2/\overline{z})}$. It is easy to see that if we identify $\{|z| = 1/R\}$ and $\{|z| = R^3\}$ by $z_1 \sim z_2$ if and only if $z_2 = R^2/\overline{z_1}$, then $\mathcal{N}/T$ is conformally equivalent to the torus with 4 punctures

$$\left(\{1/R \leq |z| \leq R^3\} - \{p, q, p_0, R^2/\overline{p_0}\}\right) / \sim$$

and $\tilde{g} = G$.

Since $\deg G = 4$ and a torus has genus 1, by the Riemann-Hurwitz formula, the total branch order of G is 8. At the four ends $\tilde{g}$ has branch order 1. That leaves only four other branch points. But each p_i or q_i counts as one branch point (since at that point $\tilde{g}$ sends more than two curved rays into the real line, even if $p_i = p_j$, in that case the branch order is higher), we have at least 6 such points (since p and q may be among them). This contradiction proves that $u \equiv 0$. □

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