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QUASI-FLATS IN SEMIHYPERBOLIC GROUPS

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Abstract. We prove that the Cayley graph of a group which is semihyperbolic but not hyperbolic contains a subset quasi-isometric to $\mathbb{R}^2$.

§0. Introduction

Gromov in his influential article [Gr] defined what it means for a metric space to be negatively curved (hyperbolic) in the large scale and generalized results known for Riemmanian manifolds of negative curvature to groups which are negatively curved in the large scale. In that paper Gromov asked if it is possible to develop an adequate notion of non-positive curvature in the large (semihyperbolicity), thus enlarging the class of groups that can be studied using geometric techniques. This problem has attracted a lot of attention (see [A-B], [G-S], [S], [Ep], [M-O]). Alonso and Bridson in [A-B] suggest that semihyperbolic groups are groups that admit a quasi-geodesic bicombing. In this note we show that if one defines semihyperbolic to mean combable then one important result in the class of non-positively curved manifolds has a "large scale" analog.

Eberlein in [Eb] showed that if M is a compact, non-positively curved manifold which is not hyperbolic as a metric space, then there is an isometric embeding $f : \mathbb{R}^2 \to \tilde{M}$. Gromov in [Gr, 4.2C] (see also [Br]) showed that this generalizes to metric polyhedra satisfying the CAT(0)-inequality and which are not hyperbolic.

Schroeder in [Sc] strengthened Eberlein's result by showing that $\tilde{M}$ is not hyperbolic if and only if it contains a quasi-flat.

In analogy to these results we show that if a group is semihyperbolic but not hyperbolic then its Cayley graph contains a subset quasi-isometric to $\mathbb{R}^2$. The class of semihyperbolic groups includes the fundamental groups of compact manifolds of non-positive curvature. It is not known if there is an algebraic characterization of semihyperbolic groups that are not hyperbolic. In particular, do such groups always contain free abelian subgroups of rank 2? We remark that abelian subgroups of semihyperbolic groups give rise to quasi-flats in the Cayley graph of the group (see [S], [A-B]). These questions parallel the hyperbolization conjecture of Thurston which says that closed irreducible 3-manifolds with infinite fundamental group which are not hyperbolic have an immersed incompressible torus.

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§1. Preliminaries

A metric space X is called a geodesic metric space if for all points x, y in X there is an isometry $f : [0, d(x,y)] \to X$ with $f(0) = x, f(d(x,y)) = y$. We call such a map a geodesic. A geodesic triangle in a geodesic metric space X consists of three geodesics a, b, c whose endpoints match. A geodesic metric space X is called (δ)-hyperbolic if there is a $\delta \geq 0$ such that for all triangles a, b, c in X any point on one side is in the δ-neighborhood of the two other sides. If G is a finitely generated group then its Cayley graph can be made a geodesic metric space by assigning length 1 to each edge. A finitely generated group is called (Gromov) hyperbolic if its Cayley graph is a (δ)-hyperbolic geodesic metric space. A path $\alpha : [0, l] \to X$ is called a (K, L)-*quasi-geodesic* if there are $K \geq 1, L \geq 0$ such that $length(\alpha|_{[t,s]}) \leq Kd(\alpha(t), \alpha(s)) + L$ for all t, s in $[0, l]$. In what follows we will always assume paths to be parametrized with respect to arc length. A (not necessarily continuous) map $f : X \to X'$ is called a (K, L) *quasi-isometry* if every point of X' is in the L-neighborhood of the image of f and, for all $x, y \in X$,

$$\frac{1}{K}d(x,y) - L \leq (f(x), f(y)) \leq Kd(x,y) + L.$$

Let $G = \langle S|R \rangle$ be a finitely presented group and let $C_S(G)$ be its Cayley graph. A *combing* of G is a map assigning to each $g \in G$ a set of paths P_g in $C_S(G)$ such that

1. If $p_g : [0, l] \to C_S(G)$ is in P_g then $p_g(0) = e$ and $p_g(l) = g$.
2. There is an $\epsilon > 0$ such that if $s \in S$, $p_g \in P_g$ and $p_{gs} \in P_{gs}$, then $d(p_g(t), p_{gs}(t)) \leq \epsilon$ for all t.

One usually refers to ϵ as the fellow traveller constant of the combing. A group is called *combable* if it admits a combing.

Remark. This definition of combing is slightly more general than the usual definition where one assigns excactly one path to each element of the group. The advantage of our definition is that according to it automatic groups admit prefix-closed quasi-geodesic combings (see [Ep], Thm 2.5.9).

A combing is called *quasi-geodesic* if there are constants A, B so that all the combing paths are (A, B) quasi-geodesics. A combing is called *prefix-closed* if for any combing path $p : [0, l] \to C_S(G)$ and any $n \in \mathbb{N}$, $n \in [0, l]$, $p|_{[0,n]}$ is also a combing path. We remark that there are three constants ϵ, A, B associated to a quasi-geodesic combing. On the other hand, if $K = max\{\epsilon, A, B\}$ an (ϵ, A, B)-quasi-geodesic combing is a (K, K, K) quasi-geodesic combing, so we can associate a unique constant K to a quasigeodesic combing. In the rest of the paper we follow this convention; by a K-quasi-geodesic combing we mean a combing whose fellow traveller constant is K and whose combing lines are (K, K) quasi-geodesics.

Let $G = \langle S|R \rangle$ be a finitely presented group. Let D be a Van Kampen diagram over $\langle S|R \rangle$ (see [L-S], ch.6). There is a continuous map f from the 1-skeleton of D to the Cayley graph of G, which preserves the labelling of the edges of $D^{(1)}$. We define the radius of D, $r(D)$, to be:

$$r(D) = max\{d(f(x), f(\partial D)) : x \in D^{(0)}\}.$$

Let w be a word on S representing the identity in G. We define its radius $r(w)$ by

$$r(w) = min\{r(D) : \partial D = w\}.$$

We define the filling radius function of G, $FR : \mathbb{N} \to \mathbb{N}$, by:

$$FR(n) = max\{r(w) : |w| \leq n\},$$

where we denote by $|w|$ the length of w.

Proposition 1. *Let $G = \langle S|R \rangle$ be a finitely presented group and let FR denote its filling radius function. Assume that there is an $n_0 \in \mathbb{N}$ such that $FR(n) \leq n/73$ for all $n \geq n_0$. Then G is hyperbolic.*

Remarks. 1. If G is hyperbolic it's easy to see that $FR(n) \sim log(n)$, unless G is virtually free, in which case $FR(n)$ is bounded.

2. The number 73 appearing in the statement is not "best possible"; it seems likely that the best possible constant is 8.

Proof. Assume that G is not hyperbolic. Then in the Cayley graph $C_S(G)$ of G there are two geodesics c, c' with common endpoints such that

$$d(c(t), c') > 100n_0 > 100\, max\,\{|r|; r \in R\}$$

for some t (see [P]). Let $n \in \mathbb{N}$ be such that

$$d(c(n), c') = max\{d(c(t), c') : t \in \mathbb{N}\}.$$

Let $d(c(n), c') = m$. Let $n_1 = max\{t \in \mathbb{N},\, t < n : d(c(t), c') < m/3 \;\; or \;\; t = n - 2m\}$ and let $n_2 = min\{t \in \mathbb{N},\, t > n : d(c(t), c') < m/3 \;\; or \;\; t = n + 2m\}$. Let $n - n_1 = x, n_2 - n = y$. It is easy to see that

$$2m/3 \leq x, y \leq 2m.$$

Let n_1', n_2' be such that

$$d(c(n_1), c'(n_1')) = d(c(n_1), c'),\;\; d(c(n_2), c'(n_2')) = d(c(n_2), c').$$

Let w be the word corresponding to the quadrilateral $(c(n_1), c(n_2), c'(n_2'), c'(n_1'))$ (see Figure 1).

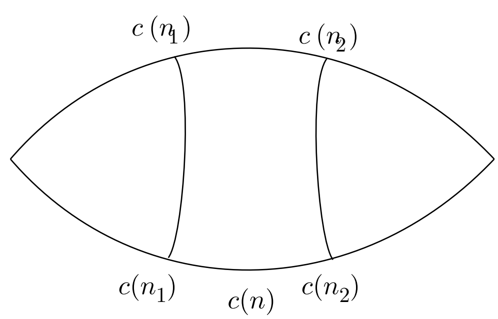

FIGURE 1

We claim that $r(w) > |w|/73$. Indeed let D be a Van Kampen diagram for w. Let f be the obvious map, $f : D^{(1)} \to C_S(G)$. Let

$$A = \{v \in D^{(0)} : d(f(v), f(c(n_1), c'(n_1')) \leq \frac{m}{6}\},$$

$$B = \{v \in D^{(0)} : d(f(v), f(c(n_2), c'(n_2')) \leq \frac{m}{6}\}.$$

It is clear that if $v_1 \in A$, $v_2 \in B$ then $d(f(v_1), f(v_2)) \geq \frac{m}{6}$. It follows that there is a path p in $D^{(1)}$ joining $c(n)$ to a point of $[c'(n_1'), c'(n_2')]$ which lies outside $A \cup B$. If for all vertices $v \in p$ we have $d(f(v), f(w)) \leq m/6 - 1$, then clearly there is a vertex v in $(c(n_1), c(n_2))$ with $d(v, [c'(n_1'), c'(n_2')]) < m/3$, which is a contradiction. We conclude that $r(w) \geq m/6 - 1$. On the other hand,

$$|w| = length([c(n_1), c(n_2)]) + length([c(n_1), c'(n_1')]) + length([c(n_2), c'(n_2')]$$

$$+length([c'(n_1'), c'(n_2')]) \leq 4m + m + m + 6m = 12m;$$

therefore $r(w) > |w|/73$. □

§2. Quasi-flats in semihyperbolic groups

In what follows we assume that $G = \langle S|R \rangle$ is a group admitting a prefix-closed quasi-geodesic combing. We denote by $C = C_S(G)$ the Cayley graph of G.

Lemma 1. *Assume that G is not hyperbolic. Then for any $k, n \in \mathbb{N}$ with $k > n$ there are two combing lines $\alpha_1 : [0, l_1] \to C$ $\alpha_2 : [0, l_2] \to C$ such that:*

(1) *$\alpha_1(0) = \alpha_2(0) = e$, $d(\alpha_1(l_1), \alpha_2(l_2)) \leq n$.*
(2) *$d(\alpha_1(t), \alpha_2) \geq \frac{n}{100K}$, $d(\alpha_2(t), \alpha_1) \geq \frac{n}{100K}$ for all $t \geq min\{l_1 - k, l_2 - k\}$, $t \in \mathbb{N}$, where K is the constant associated to the combing of G.*

Proof. We will prove this by contradiction. In particular, we show that if no pair of combing lines as above exists, then for every sufficiently long word w representing the identity in G we have $r(w) < |w|/73$; hence, by proposition 1, G is hyperbolic.

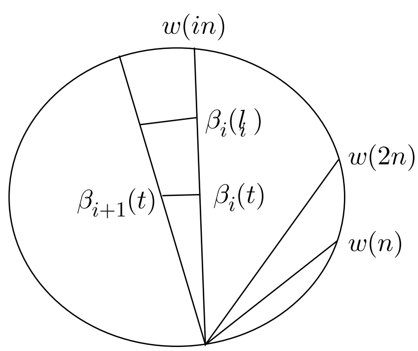

Figure 2

Indeed, let w be a word representing the identity. We will denote also by w the path in C based at e corresponding to w. Let $\beta_1, ..., \beta_r$ be combing lines from e to $w(n), w(2n), ..., w(rn)$, where $|w| - n < rn \leq |w|$. We denote by l_i the length of β_i. Assume that no pair β_i, β_{i+1} satisfies the conclusion of the lemma. Then we can join $\beta_i(t_1)$ to $\beta_{i+1}(t_2)$ by a geodesic path of length less than $\frac{n}{100K}$, where either t_1 or t_2 is greater than or equal to $m = min\{l_i - k, l_{i+1} - k\}$. The quasi-geodesic property of combing lines implies then that both t_1, t_2 are greater than $min\{l_i - 3kK, l_{i+1} - 3kK\} = l_i'$: Indeed, suppose, without restriction of generality, that $t_1 \geq m > t_2$. We have then

$$d(\beta_{i+1}(t_2), \beta_{i+1}(t_1)) < d(\beta_{i+1}(t_2), \beta_i(t_1)) + d(\beta_i(t_1), \beta_{i+1}(t_1)).$$

Now $d(\beta_i(t_1), \beta_{i+1}(t_1)) < \frac{n}{100}$ (this follows from the fellow traveller property of the combing and the fact that $d(\beta_i(t_1), \beta_{i+1}(t_2)) < \frac{n}{100K}$). Hence

$$d(\beta_{i+1}(t_2), \beta_{i+1}(t_1)) < \frac{n}{100K} + \frac{n}{100} < 2k.$$

The quasi-geodesic property of geodesics, applied to β_{i+1}, implies now that $t_1 - t_2 \leq 2kK + K < 3kK$; therefore $t_2 > l_i'$, proving our assertion.

For each $t, t+1 \leq l_i'$ we can join $\beta_i(t)$ to $\beta_{i+1}(t)$ and $\beta_i(t+1)$ to $\beta_{i+1}(t+1)$ by arcs of length less than $n/100$. This follows from the fellow traveler property of combing lines using the fact that we can join $\beta_i(t_1)$ to $\beta_{i+1}(t_2)$ for some $t_1, t_2 > l_i'$ by a path of length less than $\frac{n}{100K}$.

We fill in the loop created in this way by a Van Kampen diagram of minimal radius. We join $\beta_i(l_i')$ to $\beta_{i+1}(l_i')$ by an arc p_i of length less than $n/100$ and we fill in the loop

$$\beta_i|_{[l_i', l_i]} \cup w|_{[in,(i+1)n]} \cup \beta_{i+1}|_{[l_i', l_{i+1}]} \cup p_i$$

by a Van Kampen diagram of minimal radius. Note that all the Van Kampen diagrams that we have considered so far have boundary length less than $6kK + 2n$. By pasting all these Van Kampen diagrams (for all i) we get a Van Kampen diagram for w. If

$$D = max\{r(u) : |u| \leq 6kK + 2n\},$$

we claim that $r(w) \leq \frac{|w|}{100} + D + n + 3kK$. Indeed, if v is a vertex of the Van Kampen diagram that we constructed for w, then v is at distance at most $D + n$ from a vertex $\beta_i(t)$. If $t > l_i - 3kK$ then $d(v, w) < 3kK + D + n$. Otherwise, since $d(\beta_i(t), \beta_{i-1}) \leq n/100$, we have that, for some t, $d(v, \beta_{i-1}(t)) \leq n/100 + D + n$. So by induction

$$d(v, w) \leq \frac{rn}{100} + 3kK + n + D \leq \frac{|w|}{100} + D + n + 3kK$$

□

Lemma 2. *Assume that G is not hyperbolic. Then for any $n \in \mathbb{N}$ there are combing lines $\beta_i : [0, l_i] \to C$, $i = 1, ..., n$, such that:*

1. $\beta_i(0) = e$, $d(\beta_i(l_i), \beta_{i+1}(l_{i+1})) \leq K$, $d(\beta_i(t), \beta_{i+1}(t)) \leq K$ *for all* i, t.
2. $d(\beta_i(t), \beta_j(s)) \geq |i - j| - 1$ *for all* $s, t \geq min\{l_1 - n, ..., l_n - n\}$, $s, t \in \mathbb{N}$, *and for all* $1 \leq i, j \leq n$.

Proof. We apply lemma 1, where we take k to be $(100Kn)^2$ and n to be $100Kn$. Let $\alpha_1 : [0, k_1] \to C, \alpha_2 : [0, k_2] \to C$ be the combing lines satisfying the required properties. Let p be a geodesic path joining $\alpha_1(k_1)$ to $\alpha_2(k_2)$ and let $\beta_i' : [0, l_i'] \to C$ be combing lines from e to $p(i)$ for all i. If for all β_i' we have $d(\beta_i'(t), \beta_j'(s)) \geq |i-j|-1$ for all $s, t \geq min\{l_i' - n\}$, then clearly we are done. Suppose that this is not the case. Then there are $i < j$ such that $d(\beta_i'(t), \beta_j'(s)) < |i - j| - 1$ with $s, t \geq min\{l_i' - n\}$. Let q be a geodesic path joining $\beta_i'(t)$ to $\beta_j'(s)$. We consider the combing lines $c_1, \ldots, c_l$, $l = length(q) - 2$, from e to the interior vertices of q. We replace the combing lines $\beta_{i+1}', \ldots, \beta_{j-1}'$ by $c_1, \ldots, c_l$ and we remark that the combing lines

$$\alpha_1, \beta_1', \ldots, \beta_i', c_1, \ldots, c_l, \beta_j', \beta_{j+1}', \ldots, \alpha_2,$$

if we truncate $\alpha_1, \beta_1', \ldots, \beta_i'$ at t and $\beta_j', \beta_{j+1}', \ldots, \alpha_2$ at s, satisfy condition (1) of the lemma, while they are all defined on intervals of length greater than $min\{k_1, k_2\} -$

$100Kn$. Note that for the truncated combing lines to be themselves combing lines one needs the hypothesis that the combing is prefixed-closed.

If any two adjacent combing lines of this new set also satisfy the second condition of the lemma, then we are done. Otherwise we repeat, replacing a subset of consecutive combing lines by a set containing fewer combing lines so that condition (1) of the lemma is always satisfied. Every time we do this we get a set of consecutive combing lines between α_1 and α_2 satisfying (1). Since $d(\alpha_1(t), \alpha_2(s)) > n$ for t, s large enough, we cannot repeat this operation more than $100Kn$ times; therefore eventually we will find a set of combing lines satisfying both (1) and (2). Then we can take $\beta_i, i = 1, ..., n$, to be n consecutive combing lines among those. □

Theorem. *Assume that $G = \langle S|R\rangle$ is a group admitting a prefix-closed quasi-geodesic combing which is not hyperbolic.Then there is subset Q of $C = C_S(G)$ quasi-isometric to $\mathbb{R}^2$*

Proof . We show first that for every n that there exist a subset Q_n of C and a $(K(K+2), K(K+1))$ quasi-isometry $f_n : D_n \to Q_n$, where

$$D_n = \{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{Z}, |x|, |y| \leq n\}$$

(the points on the integer grid inside the square of side length $2n+1$ centered at the origin). Indeed let $\beta_i : [0, l_i] \to C, i = -n, -n+1, ..., n$, be combing lines (see lemma 2) such that

(1) $\beta_i(0) = e$, $d(\beta_i(l_i), \beta_{i+1}(l_{i+1})) \leq K$, for all i.
(2) $d(\beta_i(t), \beta_j(s)) \geq |i-j|-1$ for all $s, t \geq min\{l_{-n}-(2n+1), ..., l_n-2n+1\}$, $t \in \mathbb{N}$, and for all $-n \leq i, j \leq n$.

If $l = min\{l_{-n}, ..., l_n\}$ we take

$$Q_n = \{b_i(t), t \in \mathbb{N},\ 0 \leq t \leq l-2n,\ -n \leq i \leq n\}.$$

We define

$$f_n(t, i) = b_i(t + l - n).$$

It is clear that f_n is onto. We show now that f_n is a $(K(K+2), K(K+1))$-quasi-isometry if we use the Minkowski metric in D_n (i.e. $d((a_1, b_1), (a_2, b_2)) = |a_1 - b_1| + |a_2 - b_2|$). This fact follows easily from the inequalities

$$d(b_i(t), b_i(s)) \geq \frac{|t-s|}{K} - K, \tag{1}$$

$$d(b_i(t), b_j(s)) \geq |i-j| - 1. \tag{2}$$

Indeed, it is obvious that $d(f_n(i, s), f_n(j, t)) \leq K|i-j| + |s-t|$. On the other hand, by (1)

$$d(f_n(i, s), f_n(j, t)) \geq \frac{|t-s|}{K} - K - K|i-j|,$$

while by (2)

$$d(f_n(i, s), f_n(j, t)) \geq |i-j| - 1.$$

So

$$(K+2)d(f_n(i, s), f_n(j, t)) \geq |i-j| + \frac{|t-s|}{K} - 1 \geq \frac{|i-j| + |t-s|}{K} - K - 1,$$

i.e. f_n is a $(K(K+2), K(K+1))$-quasi-isometry. We can assume that $f_n(0,0) = e$, since we can translate Q_n to the origin by multiplying by $f_n(0,0)^{-1}$. We will use the f_n's to define a $(K(K+2), K(K+1))$-quasi-isometry $f : \mathbb{Z} \times \mathbb{Z} \to Q$ where $Q \subset C$. Since C is locally finite there is a subsequence f_{k_n} of f_n such that $f_{k_n}|_{D_n} = f_{k_m}|_{D_n}$ for all $m > n$. We define f by

$$f|_{D_n} = f_{k_n}|_{D_n}.$$

It is obvious that f is well defined and that it is a $(K(K+2), K(K+1))$-quasi-isometry. As $\mathbb{Z} \times \mathbb{Z}$ is quasi-isometric to $\mathbb{R}^2$, we have that Q is quasi-isometric to $\mathbb{R}^2$. □

Question. *Can one remove the condition that the combing is prefix-closed from the hypothesis of the theorem? It seems likely that the answer is yes.*

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