PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 5, May 1998, Pages 1445–1451
S 0002-9939(98)04523-7

ON THE REDUCIBILITY OF LINEAR DIFFERENTIAL EQUATIONS WITH QUASIPERIODIC COEFFICIENTS WHICH ARE DEGENERATE

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(Communicated by Hal L. Smith)

ABSTRACT. This paper proves the reducibility of a class of linear differential equations with quasiperiodic coefficients which are degenerate with respect to a small perturbation parameter. Our results generalize some that were obtained by Jorba and Simó.

1. Introduction and main results

Notation and definitions. The function $f(t)$ is called a quasiperiodic function of t with frequencies $\omega_1,\ \omega_2,\ \cdots,\ \omega_r$, if there is a function $F(\theta_1,\theta_2,\cdots,\theta_r)$, which is 2π-periodic in all its arguments θ_i $(i=1,2,\cdots,r)$, such that $f(t)=F(\omega_1 t,\omega_2 t,\cdots,\omega_r t)$. If $F(\theta)=F(\theta_1,\theta_2,\cdots,\theta_r)$ $\quad(\theta=(\theta_1,\theta_2,\cdots,\theta_r))$ is analytic on a strip $D_\rho=\{\theta\in C^r \mid |Im\theta_j|\le\rho,\ j=1,2,\cdots,r\}$, we say that $f(t)$ is analytic quasiperiodic in D_ρ. Denote the sup-norm of f on D_ρ by $\|f\|_\rho=\sup_{\theta\in D_\rho}|F(\theta)|$.

An $n\times n$ matrix $Q(t)=(q_{ij}(t))_{1\le i,j\le n}$ is called analytic quasiperiodic on D_ρ with frequencies $\omega_1,\omega_2,\cdots,\omega_r$ if q_{ij} $(i,j=1,2,\cdots,n)$ are all analytic quasiperiodic on D_ρ with the frequencies $\omega_1,\omega_2,\cdots,\omega_r$. Define a matrix norm of Q by $\|Q\|_\rho=n\times\max_{1\le i,j\le n}\|q_{ij}\|_\rho$. It is easy to see that $\|Q_1\ Q_2\|_\rho\le\|Q_1\|_\rho\cdot\|Q_2\|_\rho$. If Q is a constant matrix, set $\|Q\|=\|Q\|_\rho$ for simplicity. Write the avarage of $Q(t)$ as $\bar{Q}=(\bar{q}_{ij})_{1\le i,j\le n}$, where $\bar{q}_{ij}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}q_{ij}(t)\,dt$; for the existence of the limit, see [1]. Let $A(t)$ be an $n\times n$ quasiperiodic matrix. The differential equations $\dot{x}=A(t)x,\ x\in R^n$, are called reducible if there exists a nonsingular quasiperiodic change of variables $x=\Phi(t)y$ such that $\Phi(t)$ and $\Phi^{-1}(t)$ are quasiperiodic and bounded, and such that it changes the equations to $\dot{y}=By$, where B is a constant matrix.

Problems. In this paper we consider the reducibility of the following linear differential equations:

$$\dot{x}=(A+\epsilon Q(t))x,\quad x\in R^n, \tag{1.1}$$

Received by the editors October 22, 1996.
1991 *Mathematics Subject Classification.* Primary 34D20; Secondary 34C05.
Key words and phrases. Linear differential equations, reducibility, quasiperiodic, KAM iteration.

where A is an $n \times n$ constant matrix with different eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, $Q(t)$ is an $n \times n$ quasiperiodic matrix of time t with frequencies $\omega_1, \omega_2, \cdots, \omega_r$, and ϵ is a small perturbation parameter.

This problem was considered by Jorba and Simó in [2]. They proved that if

$$|\langle \omega, k\rangle \sqrt{-1} + \lambda_i - \lambda_j| \geq \frac{\alpha}{|k|^\tau}, \forall 0 \neq k \in Z^r, \forall i, j = 1, 2, \cdots, n, \tag{1.2}$$

and

$$\frac{d}{d\epsilon}(\bar{\lambda}_i(\epsilon) - \bar{\lambda}_j(\epsilon))|_{\epsilon=0} \neq 0, \quad i \neq j, \tag{1.3}$$

where $\alpha > 0$ and $\tau > r - 1$ are constants and $\bar{\lambda}_i(\epsilon)$ $(i = 1, 2, \cdots, n)$ are eigenvalues of $A + \epsilon\bar{Q}$, then, for sufficiently small $\epsilon_0 > 0$, there exists a nonempty Cantor subset $E \subset (0, \epsilon_0)$ such that for $\epsilon \in (0, \epsilon_0)$ the equations (1.1) are reducible.

The method used in [2] is a kind of KAM iteration. In the KAM iteration the difficulty is caused by a small divisor $\langle \omega, k\rangle + \lambda_i - \lambda_j$. The small divisor conditions (1.2) are necessary to overcome the difficulty of the small divisor. Since the frequencies ω are invariant in the iteration, which is different from the usual KAM iteration [4], one must adjust the small parameter $\epsilon \in (0, \epsilon_0)$ to guarantee the small divisor conditions. So Jorba and Simó needed the nondegeneracy conditions (1.3) to guarantee existence of the nonempty Cantor subset E, on which all the small divisor conditions in the KAM iterations hold.

If the nondegeneracy conditions (1.3) do not hold, we say $\bar{\lambda}_i(\epsilon) - \bar{\lambda}_j(\epsilon)$ are degenerate. This degenerate case is mentioned in [2], but there is no any result given. Like the motivation of [2], in this paper we want to consider this degenerate case. We will prove a similar result under weaker nondegeneracy conditions. Moreover, in the situation of the usual nondegeneracy conditions (1.3), our result is just the same as that of [2], but the proof is simpler.

Main result.

Theorem. *Suppose* $A = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ *with* $\lambda_i \neq \lambda_j$ *for* $i \neq j, 1 \leq i, j \leq n$, *and* $Q(t) = (q_{ij}(t))_{1\leq i,j\leq n} = \sum_{k\in Z^r} Q_k e^{i\langle k,\omega\rangle t}$ *is an analytic quasiperiodic matrix on* D_ρ *with frequencies* $\omega_1, \omega_2, \cdots, \omega_r$, *where the Fourier coefficients* Q_k *depend analytically on the small parameter* ϵ. *Let* $Q_0^d = diag(\bar{q}_{11}, \bar{q}_{22}, \cdots, \bar{q}_{nn})$, *where* $\bar{q}_{ii}$ *is the average of* $q_{ii}(t)$. *Suppose that for* $i \neq j$, $\epsilon(\bar{q}_{ii} - \bar{q}_{jj})$ *has one of the following forms*:

$$\mu_1\epsilon^{l_1} + o(\epsilon^{l_1}),\ \mu_2\epsilon^{l_2} + o(\epsilon^{l_2}), \cdots, \mu_p\epsilon^{l_p} + o(\epsilon^{l_p}),$$

where $\mu_i \neq 0, i = 1, 2, \cdots, p$, $1 \leq l_1 < l_2 < \cdots < l_p$, *and* $o(\epsilon^l)$ *is of order smaller than* ϵ^l *as* $\epsilon \to 0$. *Suppose*

(1) (nonresonance conditions) $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ *and* $\omega = (\omega_1, \omega_2, \cdots, \omega_r)$ *satisfy*

$$|\langle k, \omega\rangle + \lambda_i - \lambda_j| \geq \frac{\alpha}{|k|^\tau}, \forall 0 \neq k \in Z^r, \forall 1 \leq i, j \leq n,$$

where $\alpha > 0$, $\tau > r - 1$.

(2) Q *is* l_p *order continuously differentiable with respect to sufficiently small* ϵ *and*

$$\|\frac{d^l\tilde{Q}}{d\epsilon^l}\|_{D_\rho} \leq M_l,\ l = 0, 1, 2, \cdots, l_p,$$

where $\tilde{Q} = Q - Q_0^d$. *Then, there exist* $N_1,\ N_2,\ \cdots, N_p, N_i$ *depending on* $M_1,\ M_2, \cdots, M_{l_j}, \alpha, \tau, n, l_p$ *and* $\mu_1\epsilon^{l_1} + o(\epsilon^{l_1}), \cdots, \mu_j\epsilon^{l_j} + o(\epsilon^{l_j})$, *where* $l_j \leq l_i - 2$, *such that if* $|\mu_i| > N_i, i = 1, 2, \cdots, p$, *then for sufficiently small* $\epsilon_0 > 0$, *there exists a nonempty Cantor subset* $E \subset (0, \epsilon_0)$ *with positive Lebesgue measure such that for* $\epsilon \in E$ *the equations (1.1) are reducible, i.e., there exists a nonsingular quasiperiodic transformation* $x = \Phi(t)y$ *that changes (1.1) to* $\dot{y} = By$, *where* B *is a constant matrix. If* ϵ_0 *is small enough, the relative measure of* E *in* $(0, \epsilon_0)$ *is close to 1. Moreover, the quasiperiodic matrix* $\Phi(t)$ *has the same frequencies as* $Q(t)$.

Remark 1. If $l_1 = 1$, we can choose $N_1 = 0$. If $\epsilon(\bar{q}_{ii} - \bar{q}_{jj})$ take only the form $\mu\epsilon + o(\epsilon)$, this corresponds to the nondegenerate case of [2].

Remark 2. There are many ω and λ satisfying the nonresonance conditions in the theorem. We refer to [2], [3] for detailed discussions about nonresonance conditions or small divisor conditions.

2. Proof of the main result

In this section we prove the theorem by the same idea of [2] with a little modification, which can simplify the proof in the nondegenerate case.

A. Outline of the proof. Write the equations (1.1) as

$$\dot{x} = (A^+ + \epsilon\tilde{Q}(t))x, \tag{2.1}$$

where $A^+ = diag(\lambda_1^+, \lambda_2^+, \cdots, \lambda_n^+) = A + \epsilon Q_0^d, \tilde{Q} = Q(t) - \epsilon Q_0^d$. From [2] we know that under the change of variables $x = (I + \epsilon P(t))y$, where P satisfies

$$\dot{P} = A^+P - PA^+ + \tilde{Q}, \tag{2.2}$$

the equations (2.1) are changed to

$$\dot{y} = (A^+ + \epsilon^2 Q_+(t)P)y,$$

where $Q_+ = (I + \epsilon P)^{-1}\tilde{Q}P$. If the above process can go on, then the perturbation term $\epsilon^2 Q_+(t)$ becomes smaller and smaller and the equations converge to constant coefficient equations.

The key to the iteration is to solve the equation (2.2) for P. Denote by $P = (p_{ij}(t))_{1\leq i,j\leq n},\ \ \tilde{Q} = (\tilde{q}_{ij}(t))_{1\leq i,j\leq n}$. Expanding them in Fourier series and substituting them into the equations (2.2), by comparing the coefficients of both sides of the equations, we see formally that

$$p_{ij}^k = \frac{\tilde{q}_{ij}^k}{\langle k, \omega\rangle\sqrt{-1} + \lambda_i^+ - \lambda_j^+}, \forall k \in Z^r, |i-j| + |k| \neq 0.$$

If

$$|\langle \omega, k\rangle\sqrt{-1} + \lambda_i^+ - \lambda_j^+| \geq \frac{\alpha_+}{|k|^{\tau'}}, \forall 0 \neq k \in Z^r, \forall i, j = 1, 2, \cdots n,$$

where $\tau' = 3\tau$, then $|p_{ij}^k| \leq (|k|^{\tau'}/\alpha_+)|\tilde{q}_{ij}^k|$. Since $\tilde{Q}$ is analytic on D_ρ, we have $||\tilde{Q}_k|| \leq ||\tilde{Q}||_{D_\rho}e^{-|k|\rho}$. So

$$||P||_{D_{\rho-s}} \leq \sum_{k\in Z^r} ||P_k||e^{|k|(\rho-s)} \leq (\frac{1}{\delta} + \sum_{0\neq k\in Z^r}\frac{|k|^{\tau'}e^{-s|k|}}{\alpha_+})||\tilde{Q}||_{D_\rho} \leq \frac{c}{\alpha_+ s^v}||Q||_{D_\rho}, \tag{2.3}$$

where $v=\tau'+r-1$, $0<s\le\frac{1}{2}\rho$, $\delta=\min_{i\neq j}|\lambda_i^+-\lambda_j^+|$, and c depends on τ',r,n. Thus , If ϵ is sufficiently small that $||(I+\epsilon P)^{-1}||_{D_{\rho-s}}\le 2$, then

$$||Q_+||_{D_{\rho-s}}\le\frac{2c}{\alpha_+ s^v}||Q||^2_{D_\rho}. \tag{2.4}$$

B. Iteration step. Consider the following equations:

$$\dot{x}_m=(A_m+\epsilon^{2^m}Q_m(t))x_m,\quad m\ge 0 \tag{2.5}$$

where $A_m=diag(\lambda_1^m,\lambda_2^m,\cdots,\lambda_n^m)$, and $Q_m(t)$ is an analytic quasiperiodic matrix on D_{ρ_m} with the frequencies $\omega_1,\omega_2,\cdots,\omega_r$. Let

$$A_{m+1}=diag(\lambda_1^{m+1},\lambda_2^{m+1},\cdots,\lambda_n^{m+1}),$$

where $\lambda_i^{m+1}=\lambda_i^m+\epsilon^{2^m}\bar{q}_{ii}^m, i=1,2,\cdots,n$, with $\bar{q}_{ii}^m$ being the average of $q_{ii}^m(t)$. Let $\tilde{Q}_m(t)=Q_m(t)-\epsilon^{2^m}diag(\bar{q}_{11}^m,\bar{q}_{22}^m,\cdots,\bar{q}_{nn}^m)$.

If

$$|\langle\omega,k\rangle\sqrt{-1}+\lambda_i^{m+1}-\lambda_j^{m+1}|\ge\frac{\alpha_m}{|k|^{3\tau}},\ |k|+|i-j|\neq 0, \tag{2.6}$$

then by the above discussions we have an analytic quasiperiodic matrix P_m on $D_{\rho_m-s_m}$ with the frequencies $\omega_1,\omega_2,\cdots,\omega_r$, such that under the change of variables $x_m=(I+\epsilon^{2^m}P_m)x_{m+1}$, the equations (2.5) are changed to

$$\dot{x}_{m+1}=(A_{m+1}+\epsilon^{2^{m+1}}Q_{m+1}(t))x_{m+1}.$$

Moreover, we have

$$\|P_m\|_{D_{\rho_m-s_m}}\le\frac{c}{\alpha_m s_m^v}\|Q_m\|^2_{D_{\rho_m}}. \tag{2.7}$$

If ϵ is sufficiently small that $\|(I+\epsilon^{2^m}P_m)^{-1}\|_{D_{\rho_m-s_m}}\le 2$, we have

$$\|Q_{m+1}\|_{D_{\rho_m-s_m}}\le\frac{2c}{\alpha_m s_m^v}\|Q_m\|^2_{D_{\rho_m}}. \tag{2.8}$$

Now we prove this iteration is convergent. At the first step, let $A_0=A, Q_0(t)=Q(t),\alpha_0=\alpha/2,\rho_0=\rho,s_0=\rho/4,D_0=D_{\rho_0},F_0=(\epsilon\|Q_0\|_{D_0})/(\alpha_0 s_0^v)$. At the mth step we choose $\alpha_m=\alpha_0/(1+m)^2$, $s_m=s_{m-1}/2$, $\rho_{m+1}=\rho_m-s_m, D_m=D_{\rho_m}, F_m=(\epsilon^{2^m}||Q_m||_{D_m})/(\alpha_m s_m^v)$. If

$$\|(I+\epsilon^{2^m}P_m)^{-1}\|_{D_{m+1}}\le 2,\forall m\ge 0, \tag{2.9}$$

then by (2.8) we have $F_{m+1}\le\bar{c}F_m^2$, where $\bar{c}=2^{v+1}c$. So $\bar{c}F_{m+1}\le(\bar{c}F_m)^2$. If $|\bar{c}F_0|<\frac{1}{2}$, then $\bar{c}F_m\le(\frac{1}{2})^{2^m}$. By (2.7) it follows that $\|\epsilon^{2^m}P_m\|_{D_{m+1}}\le\bar{c}F_m\le(\frac{1}{2})^{2^m}$. So, we have

$$||(I+\epsilon^{2^m}P_m)^{-1}\|_{D_{m+1}}\le 1+\sum_{l=1}^{\infty}\|\epsilon^{2^m}P_m\|^l_{D_{m+1}}\le 2.$$

Let $\epsilon_1>0$ be so small that $\bar{c}F_0\le\frac{1}{2}$ for $\epsilon\in(0,\epsilon_1)$. Thus, for $\epsilon\in(0,\epsilon_1)$ satisfying all the small divisor conditions (2.6), all the above estimates hold.

Let $D_*=\bigcap_{m=0}^{\infty}D_m$, $P^m=(I+\epsilon^{2^m}P_m)(I+\epsilon^{2^{m-1}}P_{m-1})\cdots(I+\epsilon P_0)$. Obviously, $D_{\frac{1}{2}\rho}\subset D_*$ and P^m is convergent as $m\to\infty$ under the norm $\|\cdot\|_{D_*}$. Let $\Phi=\lim_{m\to\infty}P^m$.

From the above discussion, it follows that $\|A_m-A_{m-1}\|\le\epsilon^{2^m}\|Q_m\|_{D_m}$. So, A_m is also convergent. Let $\lim_{m\to\infty}A_m=B$. Obviously, $\lim_{m\to\infty}\|\epsilon^{2^m}Q_m\|_{D_{\frac{1}{2}\rho}}=0$.

Thus, if $\epsilon \in (0, \epsilon_1)$ satisfy the small divisor conditions (2.6) for all $m \geq 0$, then, under the change of variables $x = \Phi y$, the equations (1.1) become $\dot{y} = By$. Moreover, Φ is an analytic quasiperiodic matrix on $D_{\frac{1}{2}\rho}$ with the frequencies $\omega_1, \omega_2, \cdots, \omega_r$.

To finish the proof of the theorem, it remains to prove that there exist $N_1, N_2, \cdots, N_p$ such that if $|\mu_i| \geq N_i$ $(i = 1, 2, \cdots, p)$, then there exist $0 < \epsilon_0 < \epsilon_1$ and a nonempty Cantor subset $E \subset (0, \epsilon_0)$ such that for $\epsilon \in E$, the small divisor conditions (2.6) hold for all $m \geq 0$. Now we first prove that there exist $N_1, N_2, \cdots, N_p$ such that if $|\mu_i| \geq N_i, i = 1, 2, \cdots, p$, then for all $m \geq 0$ and $i \neq j$, there exist $l \in \{l_1, l_2, \cdots, l_p\}$ such that the lth derivative of $\lambda_i^{m+1} - \lambda_j^{m+1}$ with respect to ϵ at $\epsilon = 0$ does not vanish.

Suppose that $\lambda_i^1 - \lambda_j^1 = \mu_1 \epsilon^{l_1} + o(\epsilon^{l_1})$. Let $N_1 \geq 0$ be an integer such that $2^{\bar{N}_1} \leq l_1 \leq 2^{\bar{N}_1+1}$. Let

$$\bar{M}_i^1 = \left\| \frac{d^{l_1-2^i} Q_i}{d\epsilon^{l_1-2^i}} |_{\epsilon=0} \right\|_{D_{\rho_i}}, i = 1, 2, \cdots, \bar{N}_1.$$

Then $\bar{M}_i^1$ $(i = 1, 2, \cdots, \bar{N}_1)$ only depends on τ, n, α, ρ, and $M_0, M_1, \cdots, M_{l_1-2^i}$. By the construction of the transformation, $\bar{M}_i^1$ should depend on all derivatives up to the $l_1 - 2^i$-th of $\lambda_{j_1}^i(\epsilon) - \lambda_{j_2}^i(\epsilon)$ with respect to ϵ at $\epsilon = 0$. But these derivatives only depend on $M_0, M_1, \cdots, M_{l_1-2^i}$ and are independent of $\mu_1, \mu_2, \cdots, \mu_p$. Let

$$N_1 = \frac{\bar{M}_1^1 + \bar{M}_2^1 + \cdots + \bar{M}_{\bar{N}_1}^1}{l_1!}.$$

If $|\mu_1| > N_1$, we have at $\epsilon = 0$

$$\left| \frac{d^{l_1}(\lambda_i^{m+1} - \lambda_j^{m+1})}{d\epsilon^{l_1}} \right| \geq l_1! |\mu_1| - (\bar{M}_1 + \bar{M}_2 + \cdots + \bar{M}_{\bar{N}_1}) > 0.$$

Similarly, let $\bar{N}_2 \geq \bar{N}_1$ be an integer such that $2^{\bar{N}_2} \leq l_2 < 2^{\bar{N}_2+1}$. Let

$$\bar{M}_i^2 = \left\| \frac{d^{l_2-2^i} Q_i}{d\epsilon^{l_2-2^i}} |_{\epsilon=0} \right\|_{D_{\rho_i}}, i = 1, 2, \cdots, \bar{N}_2.$$

Then $\bar{M}_i^2$ $(i = 1, 2, \cdots, \bar{N}_2)$ only depends on $\tau, n, \alpha, \rho, M_0, M_1, \cdots, M_{l_2-2^i}$ and the l-th order derivatives of $\mu_1 \epsilon^{l_1} + o(\epsilon^{l_1})$ with respect to ϵ, where $l \leq l_2 - 2^i$.

Let

$$N_2 = \frac{\bar{M}_1^2 + \bar{M}_2^2 + \cdots + \bar{M}_{\bar{N}_2}^2}{l_2!}.$$

If $|\mu_2| > N_2$, then, when $\lambda_i^1 - \lambda_j^1 = \mu_2 \epsilon^{l_2} + o(\epsilon^{l_2})$, for all $m \geq 0$, the l_2-th derivative of $\lambda_i^{m+1} - \lambda_j^{m+1}$ with respect to ϵ at $\epsilon = 0$ does not vanish. From the above we see that $\bar{N}_2$ may depend on $\bar{N}_1$, but $\bar{N}_1$ is independent of $\bar{N}_2$. In the same way we can obtain $\bar{N}_p$, which depends on $\tau, n, \alpha, \rho, M_0, M_1, \cdots, M_{l_p-2}$ and the l-th derivatives of $\mu_i \epsilon^{l_i} + o(\epsilon^{l_i})$ with respect to ϵ, $i = 1, 2, \cdots, p-1$, where $l \leq l_p - 2$. If $|\mu_p| > N_p$, then, when $\lambda_i^1 - \lambda_j^1$ consists of $\mu_p \epsilon^{l_p}$, for all $m \geq 0$, the l_p-th derivative of $\lambda_i^{m+1} - \lambda_j^{m+1}$ with respect to ϵ at $\epsilon = 0$ is not zero. It is easy to see that N_p

may depend on $N_1, N_2, \cdots, N_{p-1}$. Thus, if $\lambda_i^1 - \lambda_j^1 = \mu_i \epsilon^{l_i} + o(\epsilon^{l_i})$, then

$$\frac{d^{l_i}(\lambda_i^{m+1} - \lambda_j^{m+1})}{d\epsilon^{l_i}}|_{\epsilon=0} \neq 0, \forall m \geq 0.$$

From the above iteration we see that the first step can only be done for all $\epsilon \in E_0 \subset (0, \epsilon_0)$, where E_0 is the set on which the small divisor conditions of the first step hold . Let E_{m-1} be the set on which the small divisor conditions of the m-th step hold. Then the m-th step can only be done for $\epsilon \in E_0 \cap E_1 \cap \cdots \cap E_{m-1}$. Thus, the above iteration can only be convegent on the set $E = \bigcap_{m=0}^{\infty} E_m$. Since the small divisor conditions hold on a Cantor subset, in all the iteration steps the differentiations with respect to ϵ are understood in Whitney's sense [5]. By the Whitney extension theorem in [5], all differentiable functions on a closed set in Whitney's sense can be extended to the usual differentiable function on $(-\infty, +\infty)$ and the differentiation in Whitney's sense can be treated as the usual differentiation. Thus all the functions in the iteration step can be regarded as regular differentiable functions on $(-\infty, +\infty)$, and so all the estimates can be obtained without any difficulty . However, the iteration makes sense only for $\epsilon \in E$.

Now we prove E is a nonempty set. For this we prove that for most sufficiently small ϵ, the small divisor conditions (2.6) hold for all $m \geq 0$. Let $\lambda_i^{m+1} - \lambda_j^{m+1} = \mu_l \epsilon^l + o(\epsilon^l)$, where $l \leq l_p$. It is easy to see that the term $\mu_l \epsilon^l$ is unvariant when $m \geq \bar{N}_p$. So there exists a sufficiently small $\epsilon_0 > 0$ such that if $|\epsilon| \leq \epsilon_0$, then $|\lambda_i^{m+1} - \lambda_j^{m+1}| \leq 2|\mu_l|\epsilon^l$ and $|\frac{d(\lambda_i^{m+1} - \lambda_j^{m+1})}{d\epsilon}| \geq \frac{1}{2}|\mu_l|\epsilon^{l-1}$ for all $m \geq 0$.

Let $f(\epsilon) = \langle \omega, k \rangle + \lambda_i^{m+1} - \lambda_j^{m+1}$, $i \neq j$, and

$$O_{ijm}^k = \{\epsilon \in (0, \epsilon_0) \ \mid \ |f(\epsilon)| < \frac{\alpha_m}{|k|^{3\tau}}\}.$$

Since $\lambda_i^0 - \lambda_j^0 \neq 0, \forall i \neq j$, we choose ϵ_0 so small that if $|\epsilon| \leq \epsilon_0$, $|\lambda_i^{m+1} - \lambda_j^{m+1}| \geq \bar{\alpha} > 0$ holds for all $i \neq j$ and $m \geq 1$. So we only consider $k \neq 0$. Since

$$|\langle \omega, k \rangle + \lambda_i^{m+1} - \lambda_j^{m+1}| \geq |\langle \omega, k \rangle \sqrt{-1} + \lambda_i^0 - \lambda_j^0| - 2\mu_l \epsilon^l,$$

by the nonresonance conditions of the theorem, if $1/|k|^\tau > 4\mu_l \epsilon^l / \alpha_0$, then $|f(\epsilon)| \geq \alpha_0 / 2|k|^\tau > \alpha_m / |k|^{3\tau}$.

Suppose $\alpha_0 / 4\mu_l |k|^\tau < \epsilon^l \leq \epsilon_0$. Since

$$\left|\frac{df(\epsilon)}{d\epsilon}\right| \geq \frac{1}{2}|\mu_l|\epsilon^{l-1},$$

by the differentiation mean value theorem, we have

$$\text{meas}(O_{ijm}^k) \leq \frac{2\alpha_m}{|k|^{3\tau}} \frac{2}{|\mu_l|\epsilon^{l-1}} \leq \frac{\epsilon_0^{l+1}}{|k|^\tau (m+1)^2} \frac{4|\mu_l|}{\alpha_0} \leq \frac{8|\mu_l|\epsilon_0^{l+1}}{\alpha(m+1)^2 |k|^\tau}.$$

So

$$\text{meas}(\bigcup_{i \neq j} \bigcup_{0 \neq k \in Z^r} O_{ijm}^k) \leq \frac{8n^2 \max_l |\mu_l| \epsilon_0^2}{\alpha} \sum_{m=0}^{\infty} \frac{1}{(m+1)^2} \sum_{0 \neq k \in Z^r} \frac{1}{|k|^\tau} = c\epsilon_0^2,$$

where c depends on n, τ, α and μ_l. Let E be the subset of $(0, \epsilon_0)$ where the small divisor conditions (2.6) hold for all $m \geq 0$. Then $E = (0, \epsilon_0) - \bigcup_{m,i,j,k} O_{ijm}^k$. Thus $\text{meas}(E) \geq \epsilon_0 - c\epsilon_0^2 = \epsilon_0(1 - c\epsilon_0)$. If ϵ_0 is so small that $1 - c\epsilon_0 > 0$. then E is a nonempty set with positive Lebesgue measure. Noticing that $\{\frac{k}{|k|} \mid 0 \neq k \in Z^r\}$ is dense on the unit ball of R^r, we conclude that E is a Cantor set.

Acknowledgement

We would like to thank the referee for much helpful advice which we used in our revision.

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