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ON A THEOREM BY DO CARMO AND DAJCZER

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Abstract. We give a new proof of a theorem by M.P. do Carmo and M. Dajczer on helicoidal surfaces of constant mean curvature.

1. Introduction

Let G be a one-parameter group of proper Euclidean motions of $\mathbb{R}^3$ of the form

$$g_t(x, y, z) = (x \cos t + y \sin t, -x \sin t + y \cos t, z + ht), t \in \mathbb{R},$$

i.e., G is a group of helicoidal transformations with pitch $h \in \mathbb{R}$. In the degenerate case $h = 0$, G becomes a group of pure rotations. Up to an affine change of coordinates and reparametrization, all one-parameter groups of Euclidean motions are either of this form or are groups of pure translations.

In 1982 do Carmo and Dajczer [5] investigated surfaces of constant mean curvature (CMC-surfaces) which are generated from a plane curve by the action of a helicoidal group. They proved the following theorem:

Theorem 1.1. *A complete immersed CMC-surface is helicoidal if and only if it is in the associated family of a Delaunay surface.*

They proved this result by introducing for each helicoidal CMC-immersion the 2-parameter family of helicoidal surfaces given by Bour's Lemma [2] and evaluating the constant curvature condition for the elements of these families. This approach on one hand gives an explicit parametrization of helicoidal CMC-immersions. On the other hand, it reaches its goal, the proof of Theorem 1.1, in a fairly indirect way.

Since helicoidal surfaces still spawn interest [4], [7], we want to show in this note how Theorem 1.1 can be obtained in a much simpler way using a more recent theorem of Smyth [8] and some results from [3]. We state Smyth's theorem in the language of [3]:

Theorem 1.2. *Let $\Phi : M \to \mathbb{R}^3$, M a Riemann surface, be a complete conformally immersed CMC-surface admitting a one-parameter group of self-isometries. Then the simply connected cover of M is the complex plane and the surface is either in the associated family of a Delaunay surface or its metric is rotationally invariant.*

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Here a self-isometry is an automorphism of the simply connected cover $\mathcal{D}$ of M, which preserves the metric of the universal covering immersion $\Psi : \mathcal{D} \to \mathbb{R}^3$ given by pulling back the immersion Φ to $\mathcal{D}$. For details see [3]. Those CMC-surfaces which have a rotationally invariant metric are now commonly called Smyth surfaces.

We also introduce the notion of a space symmetry of a CMC-immersion $\Phi : M \to \mathbb{R}^3$. A space symmetry of Φ is a Euclidean motion in $\mathbb{R}^3$ which preserves the image of Φ as a set. The relation between space symmetries and self-isometries was also studied exhaustively in [3]. By [3, Lemma 2.15] the group of space symmetries of a Smyth surface is discrete.

2. The proof of Theorem 1.1

Proof. For a given CMC-immersion $\Phi : M \to \mathbb{R}^3$ there exists (see e.g. [3, Theorem 2.2]) a conformal structure on M such that M becomes a Riemann surface and Φ becomes a conformal CMC-immersion. If Φ is also complete and in addition admits a one-parameter group of helicoidal space symmetries, then by [3, Prop. 2.12] and [3, Corollary 2.6], Φ admits also a one-parameter group of self-isometries. In particular it satisfies the assumptions of Theorem 1.2 above. Since a group of helicoidal Euclidean motions is never discrete, the surface cannot be a Smyth surface. It therefore has to be in the associated family of a Delaunay surface.

Conversely, by [3, Lemma 2.15] and [3, Prop. 3.4] each element of the associated family of a Delaunay surface admits a one-parameter group of space symmetries. Since the most general one-parameter group of Euclidean motions is a group of helicoidal transformations (with possibly degenerate pitch), all surfaces in the associated family of a Delaunay surface are helicoidal or rotational. □

It should also be noted that in the language of integrable systems (the metric of a conformal CMC-immersion without umbilics satisfies the integrable sinh-Gordon equation), Theorem 1.1 also implies that helicoidal CMC-surfaces are of finite type (see [6] and [1]). Thus for helicoidal surfaces, apart from the parametrizations given in [5] and [7], there is Bobenko's parametrization in terms of theta functions [1].

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