PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 6, June 1998, Pages 1655–1659
S 0002-9939(98)04189-6

LACUNARY CONVERGENCE OF SERIES IN L_0

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(Communicated by Dale Alspach)

Abstract. For a finite measure λ, let $L_0(\lambda)$ denote the space of λ-measurable functions equipped with the topology of convergence in measure. We prove that a series in $L_0(\lambda)$ is subseries (or unconditionally) convergent provided each of its lacunary subseries converges.

A series in a topological vector space is said to be *subseries convergent* if each of its subseries converges. As is well known, a subseries convergent series is unconditionally convergent, and the converse holds in sequentially complete spaces. A strictly increasing sequence (n_k) in $\mathbb{N} = \{1, 2, \dots\}$, or the set $\{n_1, n_2, \dots\}$, is called *lacunary* if $\lim_k(n_{k+1} - n_k) = \infty$, and *of density zero* if $\lim_k(k/n_k) = 0$. The *lacunary subseries* (resp. *zero-density subseries*) of a given series are those corresponding to lacunary sequences of indices (resp. sequences of indices of density zero). In our use of the term 'lacunary', we follow [SF]; its Hadamard meaning, e.g. in the theory of trigonometric series, is more restrictive.

In 1930, Auerbach [A, Hilfsatz], published the following result which, according to Footnote 1 in his paper, he had already obtained in 1923:

(**A**) *A scalar series is subseries (or unconditionally, or absolutely) convergent provided each of its zero-density subseries converges.*

Without mentioning Auerbach, this result reappeared in a 1986 paper by Estrada and Kanwal [EK, Thm. 1], a 1989 paper by Noll and Stadler [NS, Lemma on p. 116] and, in a stronger form, in a 1980 paper by Sember and Freedman [SF, Prop. 2]. The latter reads as follows:

(**B**) *A scalar series is subseries convergent provided each of its lacunary subseries converges.*

We note that the proofs given by the authors mentioned above are essentially variations of Auerbach's; indeed, Auerbach's original proof is also a proof of (**B**).

A natural question arises as to what extent results of types (**A**) or (**B**) are valid for series in more general spaces.

Received by the editors February 18, 1996.

1991 *Mathematics Subject Classification.* Primary 46E30, 40A30, 28A20.

Key words and phrases. Space of all measurable functions, subseries convergence in measure, lacunary subseries.

The final version of this paper was written while the first author held a visiting position in the Department of Mathematics, University of Mississippi, in the Spring Semester of 1996. The first author was also partially supported by the State Committee for Scientific Research (Poland), grant no. 2 P301 003 07.

In fact, it was Orlicz who first, already in 1955, studied the convergence of those series in a Banach space whose zero-density subseries were convergent. (Orlicz's paper contains no reference to Auerbach.) One of his results was an exact analogue of (**A**) for weakly sequentially complete Banach spaces.

We examine the above question in detail in our forthcoming paper [DL]. In particular, we prove there that an analogue of (**A**) or (**B**) holds for a Banach space if and only if it contains no isomorphic copy of c_0. Actually, we establish this equivalence for sequentially complete locally pseudoconvex spaces, and then focus our attention on the general case. As it turns out, the fundamental space in this respect—at least in the realm of spaces (representable as spaces) of measurable functions—is the space L_0 of all measurable functions. We treat this basic case in the present paper and prove the following theorem.

Let (S, Σ, λ) be a finite positive measure space, and $L_0(\lambda) = L_0(S, \Sigma, \lambda)$ the F-space of all (λ-equivalence classes of) measurable scalar functions on S, equipped with the topology of convergence in measure.

Theorem. *A series in $L_0(\lambda)$ is subseries convergent provided each of its lacunary subseries converges.*

Let us first give a brief outline of the proof. Assuming the assertion fails, there is a nonconvergent series $\sum_n f_n$ in $L_0(\lambda)$ all of whose lacunary subseries are convergent. Then we construct a lacunary subseries $\sum_k f_{n_k}$ such that, for a suitable sequence (g_n) of its 'blocks', $\sum_n |g_n(s)|^2 = \infty$ on a set of positive λ measure. However, since the series $\sum_n g_n$ is unconditionally convergent, this contradicts the following result due to Orlicz (see [O1, Hilfsatz], [O2, Thm. 8] and [MO, Thm. 1]).

Orlicz's Theorem. *If a series $\sum_n g_n$ in $L_0(\lambda)$ is unconditionally convergent, then*

$$\sum_n |g_n(s)|^2 < \infty \quad a.e.$$

Before starting the proof proper, we introduce some technical terminology and state some elementary facts.

Given $r \in \mathbb{N}$, let us say that a set $A \subset \mathbb{N}$ is *r-rare* if $|a - a'| \geqslant r$ for all distinct $a, a' \in A$. Clearly, a set $A \subset \mathbb{N}$ is lacunary if for every $r \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that the 'tail' $A \cap \{n, n+1, \dots\}$ of A is r-rare.

Fact 1. *For every $A \subset \mathbb{N}$ and $r \in \mathbb{N}$, the sets*

$$A_j = A \cap \{j + (k-1)r \,:\, k \in \mathbb{N}\}, \qquad j = 1, \dots, r,$$

form a partition of A into r-rare subsets.

Fact 2. *An infinite subset A of $\mathbb{N}$ is lacunary if (and only if) it is the union of a sequence (A_r) of finite subsets of $\mathbb{N}$ such that*

$$\textit{each } A_r \textit{ is } r\textit{-rare} \quad \textit{and} \quad \max A_r + r \leqslant \min A_{r+1}.$$

Let $K \subset \mathbb{N}$. An *interval* in K is a subset $I = \{k \in K \,:\, m \leqslant k \leqslant n\}$, where $m, n \in K$ and $m \leqslant n$. An *interval partition* of K is a (finite or infinite) partition of K consisting of intervals in K.

Let $\sum_n f_n$ be a series in $L_0(\lambda)$. Given $r \in \mathbb{N}$, we shall say that a set $B \in \Sigma$ has

◇ *Property* (a_r) if there exists an infinite r-rare subset K of $\mathbb{N}$ and an interval partition (I_n) of K such that

$$\sum_{n=1}^{\infty}\Big|\sum_{i\in I_n} f_i(s)\Big|^2=\infty \qquad \text{for a.e. } s\in B;$$

◇ *Finite Property* (a_r) if there exists a finite r-rare subset K of $\mathbb{N}$ and an interval partition $I_1,\dots,I_q$ of K such that

$$\sum_{n=1}^{q}\Big|\sum_{i\in I_n} f_i(s)\Big|^2\geqslant 1 \qquad \text{for a.e. } s\in B.$$

The key ingredient of our proof of the Theorem are the following two lemmas.

Lemma 1. *Assume that a series $\sum_n f_n$ in $L_0(\lambda)$ is not convergent in measure on a set A with $\lambda(A)>0$. Then for every $r\in\mathbb{N}$ there exists $B\subset A$ with $\lambda(B)>0$ such that B has Property (a_r).*

Proof. There exist $\varepsilon>0$ and a sequence (J_n) of intervals in $\mathbb{N}$ such that for all n,

$$\max J_n + r\leqslant \min J_{n+1} \qquad \text{and} \qquad \lambda\bigg(\bigg\{s\in A\,:\Big|\sum_{i\in J_n} f_i(s)\Big|\geqslant\varepsilon\bigg\}\bigg)\geqslant\varepsilon.$$

For every n and $m=1,\dots,r$ denote

$$J_n(m)=J_n\cap\{m+(k-1)r\,:\,k\in\mathbb{N}\}$$

and

$$A_n(m)=\bigg\{s\in A\,:\Big|\sum_{i\in J_n(m)} f_i(s)\Big|\geqslant\frac{\varepsilon}{r}\bigg\}.$$

As easily seen, for every n there exists $m_n\in\{1,\dots,r\}$ such that if $B_n:=A_n(m_n)$, then $\lambda(B_n)\geqslant\varepsilon/r$. Hence also the set

$$B:=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}B_k$$

is of λ-measure $\geqslant\varepsilon/r$. Denote $I_n=J_n(m_n)$. Then the set $K:=\bigcup_{n=1}^{\infty}I_n$ is r-rare, and (I_n) is an interval partition of K. Moreover, if $s\in B$, then $|\sum_{i\in I_n} f_i(s)|\geqslant\varepsilon/r$ for infinitely many n. Hence $\sum_{n=1}^{\infty}|\sum_{i\in I_n} f_i(s)|^2=\infty$ for all $s\in B$. Thus B has Property (a_r). □

Lemma 2. *Let $E\in\Sigma$ and assume that a series $\sum_n f_n$ in $L_0(\lambda)$ is not convergent in measure on every subset A of E with $\lambda(A)>0$. Then for every $r\in\mathbb{N}$ and every $\eta>0$ there exists $F\subset E$ with $\lambda(E\setminus F)<\eta$ such that F has Finite Property (a_r).*

Proof. Let $\mathcal{B}$ be a maximal disjoint family of subsets B of E such that $\lambda(B)>0$ and B has Property (a_r). Then $\mathcal{B}$ is countable, say $\mathcal{B}=\{B_1,B_2,\dots\}$. Moreover, by Lemma 1, $\lambda(E\setminus\bigcup_j B_j)=0$. Fix k such that $\lambda(E\setminus B)<\eta/2$, where $B:=\bigcup_{j=1}^{k}B_j$. Now, each B_j has Property (a_r), hence there exists an infinite r-rare subset K_j of $\mathbb{N}$, and an interval partition $\big(I_n(j)\big)$ of K_j such that

$$\sum_{n=N}^{\infty}\Big|\sum_{i\in I_n(j)} f_i(s)\Big|^2=\infty \qquad \text{for all } s\in B_j \text{ and } N\geqslant 1.$$

By applying Egoroff's theorem and proceeding by an easy induction, we can find for $j = 1, \ldots, k$ a set $C_j \subset B_j$ and a block $\{I_n(j) : m_j \leqslant n \leqslant n_j\}$ in the sequence $\big(I_n(j)\big)$, where $m_j \leqslant n_j$, such that

$$\lambda(B_j \setminus C_j) < \eta/(2k), \qquad \text{for } j = 1, \ldots, k,$$
$$\max I_{n_j}(j) + r \leqslant \min I_{m_{j+1}}(j+1) \qquad \text{for } j = 1, \ldots, k-1,$$
$$\sum_{n=m_j}^{n_j} \Big| \sum_{i \in I_n(j)} f_i(s) \Big|^2 \geqslant 1 \qquad \text{for all } s \in C_j, \quad j = 1, \ldots, k.$$

Denote

$$F = \bigcup_{j=1}^{k} C_j \qquad \text{and} \qquad K = \bigcup_{j=1}^{k} \bigcup_{n=m_j}^{n_j} I_n(j).$$

Then $\lambda(E \setminus F) < \eta$, the set K is finite and r-rare, the family

$$\{I_n(j) : 1 \leqslant j \leqslant k, \, m_j \leqslant n \leqslant n_j\} = \{I_1, \ldots, I_q\}$$

(properly arranged) is an interval partition of K, and

$$\sum_{n=1}^{q} \Big| \sum_{i \in I_n} f_i(s) \Big|^2 \geqslant 1 \qquad \text{for all } s \in F.$$

□

Proof of the Theorem. Let $\sum_n f_n$ be a series in $L_0(\lambda)$ each of whose lacunary subseries converges. Suppose it is not subseries convergent in $L_0(\lambda)$. Then, without loss of generality, we may assume that the series $\sum_n f_n$ itself is not convergent. Let $\mathcal{A}$ be a maximal disjoint family consisting of sets A with $\lambda(A) > 0$ such that our series converges in measure on A. Then $\mathcal{A}$ is countable, and it is clear that the series converges in measure on the union A_0 of $\mathcal{A}$. From this and the maximality of $\mathcal{A}$ it follows that the set $E := S \setminus A_0$ is of positive λ-measure and satisfies the assumptions of Lemma 2. Now, applying Lemma 2, first with $r = 1$ to the whole series $\sum_{n=1}^{\infty} f_n$ and then, subsequently, with $r = 2, 3, \ldots$ to its remainders $\sum_{n=N_r}^{\infty} f_n$ with N_r increasing sufficiently fast, we construct

sequences (K_r) and (I_n) of finite subsets of $\mathbb{N}$,
a sequence $1 = p_1 < p_2 < \ldots$ in $\mathbb{N}$, and
a sequence (F_r) of measurable subsets of E

such that for every r:

K_r is r-rare and $\max K_r + r + 1 \leqslant \min K_{r+1}$,
the family $\{I_n : p_r \leqslant n < p_{r+1}\}$ is an interval partition of K_r,
$\lambda(F_r) > (1 - 2^{-r})\lambda(E)$

and

$$\sum_{n=p_r}^{p_{r+1}-1} |g_n(s)|^2 \geqslant 1 \quad \text{for all } s \in F_r, \qquad \text{where} \quad g_n := \sum_{i \in I_n} f_i.$$

Set $F = \bigcap_{r=1}^{\infty} F_r$. Then $\lambda(F) > 0$ and

$$\sum_{n=1}^{\infty} |g_n(s)|^2 = \infty \qquad \text{for all } s \in F.$$

Moreover, by Fact 2, the set $K := \bigcup_{r=1}^{\infty} K_r$ is lacunary, and the sequence (I_n) is an interval partition of K. By assumption, the series $\sum_{i\in K} f_i$ is subseries (or unconditionally) convergent in $L_0(\lambda)$. It follows that also the series $\sum_n g_n$ is subseries convergent in $L_0(\lambda)$. Therefore, by Orlicz's Theorem, $\sum_n |g_n(s)|^2 < \infty$ for λ-a.e. $s \in S$. However, as we have just seen, $\sum_n |g_n(s)|^2 = \infty$ for all $s \in F$, where $\lambda(F) > 0$. A contradiction. □

Remark. The theorem remains valid if λ is an arbitrary positive measure such that $L_0(\lambda)$ is a sequentially complete Hausdorff topological vector space when equipped with the topology of convergence in measure on all sets of finite measure (see [DL] for more details).

Added in proof

Prof. Pedro J. Paúl pointed out to us that both the results (**A**) and (**B**) were also obtained by Agnew [Ag].

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