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A CHARACTERIZATION OF THE HILBERT TRANSFORM

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ABSTRACT. In this note the Hilbert transform is characterized in terms of function algebras with respect to pointwise multiplication.

Let H be the Hilbert transform on the real line,

$$Hf(x) = \frac{1}{\pi}\text{P.V.}\int_{-\infty}^{\infty} f(x-y)\frac{\mathrm{d}y}{y}$$

for $x \in \mathbb{R}$. H extends to a bounded linear operator on $L^p(\mathbb{R})$ for $1 < p < \infty$.

There are several ways to characterize H. For instance, if $F \in H^p$, the analytic Hardy space, with $f = \Re F|_{\mathbb{R}}$, $g = \Im F|_{\mathbb{R}}$, then $g = Hf$. In this context, the Hilbert transform can be extended as the operator that maps the real part u of a function $F = u + iv$ in H^p, $0 < p \leq \infty$, to the imaginary part v. Notice that $\bigcup_{0<p} H^p$ is an algebra with respect to pointwise multiplication. From this we can easily obtain the following equality:

$$H(f^2 - (Hf)^2) = 2fHf. \qquad (*)$$

In fact, the restriction to $\mathbb{R}$ of F^2 is

$$f^2 - (Hf)^2 + i2fHf,$$

which proves $(*)$. Relation $(*)$ was used by Cótlar [Co] to prove the boundedness of H on L^p and by Gokhberg and Krupnik [GK] to find the exact value of $\|H\|_p$, when $p = 2^n$, which is a special case of the later complete result by Pichorides [Pi].

Formula $(*)$ is essentially an "algebra" condition and it is remarkable that it characterizes the Hibert transform. In fact, let T be a bounded linear operator on $L^2(\mathbb{R})$ satisfying

(i) T maps real valued functions into real valued functions,
(ii) T commutes with translations,
(iii) $-T^2 = I$, the identity operator.

Let $\mathcal{A}$ be the space of functions $F = f + iTf$, where $f \in L^2(\mathbb{R})$ is real valued. Then, the following is true.

Theorem 1. *If $\mathcal{A}$ has the property that $F^2 \in \mathcal{A}$ whenever $F \in \mathcal{A}$ and $f^2 \in L^2$, then $T = \pm H$ and $\mathcal{A} = H^2$.*

The proof relies on the following Lemma.

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Lemma 1. *Let $E \subset \mathbb{R}$ be a measurable set and χ_E be its characteristic function. Suppose that*

(a) $\operatorname{supp}(\chi_{E\cap[-R,R]} * \chi_{E\cap[-R,R]}) \subseteq E$, *for all $R > 0$,*
(b) $-E = E^c$, *the complement of E.*

Then either $\chi_E = \chi_{(0,\infty)}$ a.e., or $\chi_E = \chi_{(-\infty,0)}$ a.e..

Proof. Suppose $|E\cap(0,\infty)| > 0$. Then there exists $R > 0$ such that $|E\cap[0,R]| > 0$. Since $\chi_{E\cap[0,R]} * \chi_{E\cap[0,R]}$ is continuous and supported in $[0,\infty)$, (a) implies that E contains an interval (α,β) with $0 < \alpha < \beta$. By (a) again, $E \supseteq \bigcup_{N=1}^{\infty}(N\alpha, N\beta) \supseteq (\gamma,\infty)$, for some $\gamma > 0$. Likewise, if $|E \cap (-\infty,0)| > 0$,we have $E \supset (-\infty,\delta)$ for some δ. By (b), these conditions cannot both hold. Thus, either $E \subset (0,\infty)$ or $E \subset (-\infty,0)$ modulo a nullset, and then (b) implies the desired result. □

Proof of Theorem 1. By properties (ii) and (iii) of T, we have that $(\hat{Tf})(\xi) = m(\xi)\hat{f}(\xi)$, for all $f \in L^2(\mathbb{R})$, where, for some measurable set $E \subset \mathbb{R}$, $m(\xi) = -i$ if $\xi \in E$ and $m(\xi) = i$ if $\xi \in E^c$. (i) implies that $\chi_{E^c} = \chi_{-E}$ a.e., and we can assume that $E^c = -E$ by modifying E on a nullset.

It is easy to see that $F = f + ig \in \mathcal{A}$ if and only if $\hat{F}(\xi) = 0$ for $\xi \in E^c$. Let F be such that $\hat{F} = \chi_{E\cap[-R,R]}$, $R > 0$. Then $F \in \mathcal{A}$. $F^2 \in L^2(\mathbb{R})$ because $\hat{F} * \hat{F}$ is a continuous function with compact support. By the hypothesis, $F^2 \in \mathcal{A}$, and therefore $\operatorname{supp}(\chi_{E\cap[-R,R]} * \chi_{E\cap[-R,R]}) = \operatorname{supp}(F^2)\hat{} \subseteq E$.

By the Lemma, either $E = (0,\infty)$, or $E = (-\infty,0)$, modulo a set of measure zero, hence $T = H$ or $T = -H$. □

The theorem has analogues if we replace $\mathbb{R}$ with $\mathbb{S}^1$ or $\mathbb{Z}$.

Theorem 2. (a) *Let T be a bounded linear operator on $L^2(\mathbb{S}^1)$ that satisfies (i)-(iii) and let $\mathcal{A}$ be the linear space of functions $F = f + iTf$ with $f \in L^2(\mathbb{S}^1)$, real valued, such that $\hat{f}(0) = 0$. Suppose that $\mathcal{A}$ enjoys the same hypothesis as in Theorem 1. Then $T = \pm H$, where H is now the conjugate function operator.*

(b) *Let T be a bounded linear operator on $L^2(\mathbb{Z})$ that satisfies (i)-(iii) and let $\mathcal{A}$ be the linear space of sequences $F = f + iTf$ with $f \in L^2(\mathbb{Z})$, real valued. Then $\mathcal{A}$ cannot satisfy the same hypothesis as in Theorem 1. Namely, there exists $F \in \mathcal{A}$ such that $f^2 \in L^2(\mathbb{Z})$, but $F^2 \notin \mathcal{A}$.*

The proof of (a) and (b) follows the same lines as the proof of Theorem 1. In particular, in case (b) we get a contradiction by requiring that a subset E of $\mathbb{S}^1$ satisfy assumptions (a) and (b) of Lemma 1.

There are non translation invariant operators T on $L^2(\mathbb{R})$ such that T and the associated space $\mathcal{A}$ satisfy (i), (iii) and the hypothesis of Theorem 1. It suffices to consider spaces $\mathcal{A}$ of holomorphic functions on suitable domains, T being the conjugate function operator. We do not know whether all the operators satisfying the above properties can be obtained in this way.

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