PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 6, June 1998, Pages 1637–1643
S 0002-9939(98)04228-2

A CHARACTERIZATION OF GORENSTEIN RINGS IN CHARACTERISTIC p (>0)

S. P. DUTTA

(Communicated by Wolmer V. Vasconcelos)

Abstract. A new characterization of Gorenstein rings in characteristic p (>0) is proved. It involves asymptotic behaviour of lengths of modules under the Frobenius map.

In this note we are going to study how the Gorenstein property for a local ring $(R, m, k = R/m)$ of characteristic p (> 0), dimension d, is connected with the behaviour of $\lim_{n\to\infty} \ell(F^n(M))/p^{nd}$ and $\lim_{n\to\infty} \ell(F^n(M^v))/p^{nd}$, when M is a module of finite length and finite projective dimension over R, $M^v = \mathrm{Hom}_R(M, E)$, E is the injective hull of k over R and $F^n(M)$ is obtained from M by applying the Frobenius map n times $((M \otimes_R f_R^n$ [P-S]; Notation). It is well known that for a module M of finite length over any local ring R of dim d and ch. p (> 0) $\lim_{n\to\infty} \ell(F^n(M))/p^{nd}$ is positive, and if R is Cohen-Macaulay and $M = R/(x_1, \ldots, x_d)$, where $x_1, \ldots, x_d$ is a system of parameters in R, the above limit is $\ell(R/(x_1, \ldots, x_d))$. Recall that when R is Gorenstein $M^v \simeq \mathrm{Ext}^d_R(M, R)$ for any module M of finite length. Moreover when $pd_R M < \infty$, we have $(F^n(M))^v \simeq \mathrm{Ext}^d(F^n(M), R) \simeq F^n(\mathrm{Ext}^d(M, R)) \simeq F^n(M^v)$. Thus $\ell(F^n(M)) = \ell(F^n(M^v))$, and obviously the above limits are same. This leads us to raise the same question for Cohen-Macaulay rings which are not Gorenstein. Our main theorem shows that the answer, in this case, is in the negative even for cyclic modules of the form $R/(x_1, \ldots, x_d)$, where $x_1, \ldots, x_d$ is a system of parameters of R. In fact our main theorem is the following:

Theorem. *A Cohen-Macaulay local ring R of dimension d and* ch. p (> 0) *is Gorenstein if and only if there exists a system of parameters $x_1, \ldots, x_d$ such that*

$$\lim_{n\to\infty} \ell(F^n((R/\underline{x})^v))/p^{nd} = \ell(R/\underline{x}).$$

Motivation for the study of the above result came from the study of the Strong Intersection Conjecture due to Peskine and Szpiro [P-S]. In this conjecture they assert that, given a pair of finitely generated modules M and N over a local ring R such that $\ell(M \otimes_R N) < \infty$ and $pd_R M < \infty$, $dim N \leq$ grade M. While studying the conjecture over a Cohen-Macaulay ring of dimension d and positive characteristic, with a canonical module Ω $(\neq R)$, this author was led to inquire whether $\lim_{n\to\infty} \ell(H^0_m(F^n(\Omega/(x_2, \ldots, x_d)\Omega)))/p^{nd}$ is 0. Recall that, though $\Omega/(x_2, \ldots, x_d)\Omega$ is a Cohen-Macaulay module of dimension 1, $F^n(\Omega/(x_2, \ldots, x_2)\Omega)$

Received by the editors March 15, 1996 and, in revised form, November 26, 1996.
1991 *Mathematics Subject Classification.* Primary 13D02, 13H10.
This research was partially supported by an NSF grant and an NSA grant.

©1998 American Mathematical Society

is not necessarily Cohen-Macaulay. In the course of our proof of the main theorem, we show that the above limit is never zero. If the limit were zero, one could venture for the construction of a counterexample to the above conjecture.

I am thankful to the referee for comments and suggestions, some of which have been incorporated into the body of this paper.

Notations. Throughout this work (R, m, k) denotes a local ring of characteristic $p\ (>0)$, dimension d, m its maximal ideal; $k = R/m$ and k is perfect. The Frobenius map $f : R \to R$, given by $f(x) = x^p$ for all $x \in R$, is a ring homomorphism. We denote by f_R^n the bi-algebra R, having the structure of an R-algebra from the left by f^n and from the right by the identity map, i.e. if $\alpha \in R,\ x \in f_R^n,\ \alpha \cdot x = \alpha^{p^n} x$ and $x \cdot \alpha = x\alpha$. For any module M, we write $F_R^n(M)$ for $M \otimes_R f_R^n$ (we usually drop R from the notation when there is no chance of confusion) and when $\ell(M)$ (= length of M) $< \infty$, M^v for $\mathrm{Hom}_R(M, E)$, where E is the injective hull of k over R. If $pd_R M < \infty$, $\chi_i(M, N)$ will denote $\sum_{j=0}^{pdM} (-1)^j \ell(\mathrm{Tor}_{i+j}^R(M, N))$. For any finitely generated module M, $\mu(M)$ denotes the minimal number of generators of M over R. We write lim to denote $\lim_{n\to\infty}$ and Ω to denote a canonical module of R. For any system of parameters $x_1, \dots, x_d$ of R and for any module M, $\underline{x}_i M$ denotes the submodule $(x_i, \dots, x_d)M$ and $\overline{M}$ denotes M/x_1M. If $\theta = (\theta_{ij})$ denotes an $s \times r$ matrix, then $\theta^{[p^n]}$ stands for the $s \times r$ matrix $(\theta_{ij}^{p^n})$. Also for any module M, and for any element $x \in R$, $(0 : x)M = \{\alpha \in M \mid x\alpha = 0\}$. If I denotes the ideal $(y_1, \dots, y_t)$ in R, then $I^{[p^n]}$ denotes the ideal $(y_1^{p^n}, \dots, y_t^{p^n})$ in R.

Now we state our main theorem.

Theorem. *Let R be a Cohen-Macaulay local ring of dimension d and characteristic $p\ (>0)$. Then R is Gorenstein if and only if there exists a system of parameters $x_1, \dots, x_d$ such that*

$$\lim_{n\to\infty} \ell(F^n((R/\underline{x})^v))/p^{nd} = \ell(R/\underline{x}).$$

Proof. We note that without any loss of generality one can assume R is complete. Recall that for any system of parameters $x_1, \dots, x_d$ of R

$$\lim \ell(F^n(R/\underline{x}))/p^{nd} = \ell(R/\underline{x}).$$

Now let R be Gorenstein. Then we have

$$(F^n(R/\underline{x}))^v \simeq \mathrm{Ext}_R^d(F^n(R/\underline{x}), R) \simeq F^n(\mathrm{Ext}^d(R/\underline{x}, R)) \simeq F^n((R/\underline{x})^v).$$

Hence

$$\lim \ell(F^n((R/\underline{x})^v))/p^{nd} = \lim \ell((F^n(R/\underline{x}))^v)/p^{nd} = \lim \ell(F^n(R/\underline{x}))/p^{nd} = \ell(R/\underline{x}).$$

Next, let us assume that R is not Gorenstein. Our proof is completed in two parts.

In Part 1, we will show that it is possible to construct a system of parameters $x_1, \dots x_d$ of R such that

$$\lim \ell(F^n((R/\underline{x})^v))/p^{nd} \neq \ell(R/\underline{x});$$

and in Part 2 we will prove the above result for any system of parameters.

Part 1. We first note a rather trivial fact in dimension 0, which will be used in our proof:

Fact. Let R be a local ring of dimension zero. Let M be a module of finite length. Let $\mu(M)$ denote the number of minimal generators of M. Then $\ell(F^n(M)) = \mu(M)\ell(R)$ for $n >> 0$.

Proof of the Fact. Consider a minimal representation of M:

$$R^r \xrightarrow{\theta} R^s \to M \to 0$$

where the entries of θ are in m. Applying $\otimes f_R^n$, we get an exact sequence

$$R^r \xrightarrow{\theta^{[p^n]}} R^s \to F^n(M) \to 0.$$

Since $\dim R = 0$, for $n >> 0$, entries of $\theta^{[p^n]}$ are all 0. Hence the result.

We note that $(R/\underline{x})^v \simeq \mathrm{Ext}^d(R/\underline{x}, \Omega) = \Omega/\underline{x}\Omega$. So we need to establish that for a certain system of parameters $x_1, \ldots, x_d$

$$\ell(R/\underline{x})(= \lim \ell(F^n(R/\underline{x}))/p^{nd} = \lim \ell(R/\underline{x}^{p^n})/p^{nd}) \neq \lim \ell(F^n(\Omega/\underline{x}\Omega))/p^{nd}.$$

($\underline{x}^{p^n}$ stands for $(x_1^{p^n}, \ldots, x_d^{p^n})$ and $\underline{x}_i^{p^n}$ stands for $(x_i^{p^n}, \ldots, x_d^{p^n})$). Note that $\ell(\Omega/\underline{x}\Omega) = \ell(R/\underline{x})$ (for any module M of finite length, $\ell(M) = \ell(M^v)$).

Step 0. $\dim R = 0$. In this case $\ell(F^n(\Omega)) = \mu(\Omega)\ell(R)$ for $n >> 0$ (Fact).

Thus the above limits are equal if and only if $\mu(\Omega) = 1$ i.e., R is Gorenstein.

Now suppose $x_1, \ldots, x_d$ $(d > 0)$ is a system of parameters of R such that

$$\lim \ell(R/\underline{x}^{p^n})/p^{nd} = \lim \ell(F^n(\Omega/\underline{x}\Omega))/p^{nd}. \tag{1}$$

Note that the left-hand side is $\ell(R/\underline{x})$. We also recall that $F^n(M \otimes_R N) \simeq F^n(M) \otimes_R F^n(N)$. So $F^n(\Omega/\underline{x}\Omega) \simeq F^n(\Omega) \otimes F^n(R/\underline{x}) = F^n(\Omega) \otimes R/\underline{x}^{p^n}$. Since R is Cohen-Macaulay, $pd_R R/\underline{x} < \infty$.

Step 1. *(1) holds if and only if* $\Omega \subset R$ *and* $\chi_1(R/\underline{x}^{p^n},\ F^n(\Omega))/p^{nd} \to 0$ *as* $n \to \infty$.

Since support Ω = support R and $F^n(\Omega)_p = F^n_{R_p}(\Omega_p)$ [P-S], we have

$$\begin{aligned}\chi(R/\underline{x}^{p^n}, F^n(\Omega)) &= p^{nd}\chi(R/\underline{x}, F^n(\Omega))(\chi \text{ is additive})\\ &= p^{nd} \sum_{P \in \mathrm{Ass}(R)} \ell(F^n_{R_P}(\Omega_p))\chi(R/\underline{x},\ R/P).\end{aligned}$$

We have

$$\chi(R/\underline{x}^{p^n}, F^n(\Omega)) = p^{nd} \sum_{P \in \mathrm{Ass}(R)} \mu_{R_p}(\Omega_p)\ell(R_P)\chi(R/\underline{x}, R/P)$$

for $n >> 0$ (Fact).

Recall ([L]) that for any finitely generated module T such that $\ell(T/\underline{x}T) < \infty, \chi(R/\underline{x}, T) \geq 0$ and equality holds if and only if $\dim T < d$. Moreover $\chi_i(R/\underline{x}, T) \geq 0$. Thus, if $\mu_{R_p}(\Omega_p) > 1$ for any prime $P \in \mathrm{Ass}(R)$, it follows from the above that

$$\lim \chi(R/\underline{x}^{p^n},\ F^n(\Omega))/p^{nd} > \sum_{P \in \mathrm{Ass}(R)} \ell(R_P)\chi(R/\underline{x}, R/P).$$

But the right-hand side of the above inequality is nothing but $\chi(R/\underline{x}, R) = \ell(R/\underline{x})$. Since $\chi_1(R/\underline{x}^{pn},\ F^n(\Omega)) \geq 0$, when $\lim \ell(F^n(\Omega/\underline{x}\Omega))/p^{nd} = \ell(R/\underline{x})$, the above inequality forces us to have

$$\mu_{R_p}(\Omega_p) = 1, \forall P \in \mathrm{Ass}(R) \quad \text{and} \quad \lim \chi_1(R/\underline{x}^{p^n}, F^n(\Omega))/p^{nd} = 0. \tag{2}$$

This can easily be checked to be both necessary and sufficient in order that (1) may hold. Note that Ω_p is a canonical module of R_p, and thus (2) implies that R is generically Gorenstein. So, we can assume that Ω is an ideal of height 1 in R. Recall that in such a case R/Ω is Gorenstein.

Now we need to study what it means to have $\lim \chi_1(R/\underline{x}^{p^n}, F^n(\Omega))/p^{nd} = 0$.

Step 2. *We can choose* x_1 *so that* $\lim \chi_1(R/\underline{x}^{p^n}, F^n(\Omega))/p^{nd} = 0$ *if and only if*

$$\lim \ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega)))/p^{nd} = 0.$$

From Step 1, $\Omega \subset R$ is a height one ideal of R. Take $x_1 \in \Omega$.

Consider the short exact sequence

$$0 \to \Omega \to R \to R/\Omega \to 0.$$

Applying $\otimes f^n_R$, we obtain the following short exact sequences:

$$0 \to \operatorname{Tor}_1(R/\Omega, f^n_R) \to F^n(\Omega) \to \Omega^{[p^n]} \to 0,\ 0 \to \Omega^{[p^n]} \to R \to R/\Omega^{[p^n]} \to 0.$$

Note that $\ell(R/(\Omega + \underline{x}_2)) < \infty$ and hence $\ell(\operatorname{Tor}_i(R/\underline{x}_2, F^n(\Omega)) < \infty$. Now consider the exact sequence

$$0 \to R/\underline{x}_2^{p^n} \xrightarrow{x_1^{p^n}} R/\underline{x}_2^{p^n} \to R/\underline{x}^{p^n} \to 0.$$

Applying $\otimes F^n(\Omega)$ to the above sequence, from the long exact sequence of Tor's we obtain

$$\chi_1(F^n(\Omega),\ R/\underline{x}^{p^n}) = \ell((0 : x_1^{p^n})F^n(\Omega/\underline{x}_2\Omega)).$$

Again from the exact sequence

$$0 \to \Omega/\underline{x}_2\Omega \to R/\underline{x}_2 \to R/(\Omega + \underline{x}_2) \to 0,$$

applying $\otimes f^n_R$, we get the following exact sequence:

$$0 \to \operatorname{Tor}_1(R/(\Omega + \underline{x}_2),\ f^n_R) \to F^n(\Omega/\underline{x}_2\Omega) \to R/\underline{x}_2^{p^n} \to R/(\Omega + \underline{x}_2)^{[p^n]} \to 0.$$

Since $x_1 \in \Omega$, it follows from the above sequence that

$$(0 : x_1^{p^n})F^n(\Omega/\underline{x}_2\Omega) = \operatorname{Tor}_1(R/(\Omega + \underline{x}_2), f^n_R) = H^0_m(F^n(\Omega/\underline{x}_2\Omega)).$$

Thus $\lim \chi_1(R/\underline{x}^{pn},\ F^n(\Omega))/p^{nd} = 0$ if and only if

$$\lim \ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega)))/p^{nd} = 0. \tag{3}$$

We have now two cases to consider:

(a) $\ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega))) \leq Kp^{n(d-2)}$, where K is a constant.
(b) $\ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega))) = \mathbf{O}(p^{n(d-1)})$.

Step 3. *Case* (a). We consider the case when $\ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega))) \leq Kp^{n(d-2)}$. We use induction on d (≥ 0). For $d = 0$, see Step 0. Consider the exact sequence

$$0 \to H^0_m(F^n(\Omega/\underline{x}_2\Omega)) \to F^n(\Omega/\underline{x}_2\Omega) \to N_n \to 0. \tag{4}$$

Because of (1), it follows that $\lim \ell(N_n/x_1^{pn}N_n)/p^{nd} = \ell(R/\underline{x})$. But x_1 is not a zero-divisor on N_n, so we have $\lim \ell(N_n/x_1N_n)/p^{n(d-1)} = \ell(R/\underline{x})$. Tensoring (4) by R/x_1R, we get the following exact sequence:

$$0 \to \overline{H^0_m(F^n(\Omega/\underline{x}_2\Omega))} \to F^n_{\overline{R}}(\overline{\Omega}/\underline{x}_2\overline{\Omega}) \to \overline{N}_n \to 0. \tag{5}$$

(For any module M, we write $\overline{M}$ to denote M/x_1M.) Since $\ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega))) \leq Kp^{n(d-2)}$, $\lim \ell(\overline{H^0_m(F^n(\Omega/\underline{x}_2\Omega))})/p^{n(d-1)} = 0$. This shows that

$$\lim \ell(F^n_{\overline{R}}(\overline{\Omega}/\underline{x}_2\overline{\Omega}))/p^{n(d-1)} = \lim \ell(\overline{N}_n)/p^{n(d-1)} = \ell(R/\underline{x}) = \ell(\overline{R}/\underline{x}_2\overline{R}).$$

Since the dimension of $\overline{R}$ is $d-1$, we are done by induction on $d = \dim R$ ($\overline{\Omega} = \Omega/x_1\Omega$ is a canonical module for $\overline{R} = R/x_1R$).

Step 4. *Case* (b). Now we consider the case when $\ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega)))$ is strictly $\mathbf{O}(p^{n(d-1)})$. If $\ell(\overline{H^0_m(F^n(\Omega/\underline{x}_2\Omega))})$ is $\mathbf{O}(p^{n(d-2)})$, we are done by Step 3. So assume that $\ell(\overline{H^0_m(F^n(\Omega/\underline{x}_2\Omega))}) = \mathbf{O}(p^{n(d-1)})$. Recall from Step 2, that $H^0_m(F^n(\Omega/\underline{x}_2\Omega))$ $= \mathrm{Tor}_1(R/(\Omega+\underline{x}_2), f^n_R)$. We have

$$\ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega))) \leq c\ p^{n(d-1)}, \tag{6}$$

where c is a constant. Suppose that

$$\ell(\overline{H^0_m(F^n(\Omega/\underline{x}_2\Omega))}) = c_1p^{n(d-1)} + \mathbf{O}(p^{n(d-2)}) \tag{7}$$

(the existence of c_1 follows from (5)) and

$$\ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega)) \otimes R/x_1^iR) = c_ip^{n(d-1)} + \mathbf{O}(p^{n(d-2)}).$$

Let t be an integer such that $t > [c/c_1]$. Applying $\otimes R/x_1^iR$ to (4), we obtain the following exact sequence:

$$0 \to H^0_m(F^n(\Omega/\underline{x}_2\Omega)) \otimes R/x_1^iR \to F^n(\Omega/\underline{x}_2\Omega) \otimes R/x_1^iR \to N_n/x_1^iN_n \to 0. \tag{8}$$

Since x_1 is a non-zero-divisor on N_n, $\ell(N_n/x_1^iN_n) = i\ell(\overline{N}_n)$. Again, from the exact sequence

$$0 \to R/x_1R \to R/x_1^2R \to R/x_1R \to 0$$

we get

$$\ell(F^n(\Omega/x_1\Omega) \otimes R/x_1^2R) = 2\ell(F^n(\Omega/\underline{x}_2\Omega) \otimes R/x_1R) + \mathbf{O}(p^{n(d-2)})$$

(by applying Proposition 1, [S]). Hence for any fixed i,

$$\ell(F^n(\Omega/x_1\Omega) \otimes R/x_1^iR) = i\ell(F^n_{\overline{R}}(\overline{\Omega}/\underline{x}_2\overline{\Omega})) + \mathbf{O}(p^{n(d-2)}).$$

Now choose an $i > t$. It then follows from (8) and the above equality that, for $n >> 0$,

$$\begin{aligned}\ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega)) \otimes R/x_1^iR) &= i\ell(F^n_{\overline{R}}(\overline{\Omega}/\underline{x}_2\overline{\Omega})) - i\ell(\overline{N}_n) + \mathbf{O}(p^{n(d-2)})\\ &= i\ell(\overline{H^0_m(F^n(\Omega/\underline{x}_2\Omega))}) + \mathbf{O}(p^{n(d-2)}) \quad \text{from (5)}\\ &> t\ell(\overline{H^0_m(F^n(\Omega/\underline{x}_2\Omega))}) + \mathbf{O}(p^{n(d-2)}).\end{aligned}$$

Thus, (6), (7), (8) and the above inequality imply that

$$\ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega))) \otimes R/x_1^iR) > \ell(H^0_m(F^n(\Omega/\underline{x}_2\Omega)))$$

for $n >> 0$, which is impossible. (The above arguments also show that c cannot be equal to c_1.) Hence, by Step 3 and Step 4, $\lim \ell(F^n(H^0_m(\Omega/\underline{x}_2\Omega)))/p^{nd}$ cannot be 0. Thus for any system of parameters $x_1, \ldots, x_d$ of R, with $x_1 \in \Omega$,

$$\ell(R/\underline{x}) = \lim \ell(F^n(R/\underline{x}))/p^{nd} \neq \lim \ell(F^n((R/\underline{x})^v))/p^{nd}.$$

Part 2. If possible, let there be a system of parameters $x_1, \dots, x_d$ such that

$$\ell(R/\underline{x}) = \lim \ell(F^n(R/\underline{x}))/p^{nd} = \lim \ell(F^n((R/\underline{x})^v))/p^{nd}.$$

Then as in Part 1, we conclude that Ω is a height 1 ideal of R and

$$\lim \chi_1(R/\underline{x}^{pn},\ F^n(\Omega))/p^{nd} = 0.$$

If any $x_i \in \Omega$, we arrive at a contradiction by Theorem 1. So we can assume that not a single x_i is in Ω. Now consider $x_1^{t_1}, x_2^{t_2}, \dots, x_d^{t_d}$, where $t_1, \dots, t_d$ are positive integers. Then

$$\chi_1(R/(x_1^{t_1p^n}, \dots, x_d^{t_dp^n}), F^n(\Omega)) \leq t_1t_2, ..., t_d\chi_1(R/\underline{x}^{pn},\ F^n(\Omega)).$$

Hence

$$\lim \chi_1(R/(x_1^{t_1p^n}, \dots, x_d^{t_dp^n}),\ F^n(\Omega))/p^{nd} = 0.$$

Thus $x_1^t, \dots, x_d^t$ also satisfy the above equality, i.e.

$$\ell(R/(x_1^{t_1}, \dots, x_d^{t_d})) = \lim \ell(F^n(R/(x_1^{t_1}, \dots, x_d^{t_d}))^v)/p^{nd} \quad \text{(Step 1, Theorem 1)}.$$

Now, since $\ell(R/(\Omega + \underline{x})) < \infty$, we can reorganize the ideal $(x_1, \dots, x_d)$ in such a way that $x_1, \dots, x_{d-1}$ is a system of parameters on R/Ω (Theorem 124, [K]). Choose t so that $x_d^t \in \Omega + (x_1, \dots, x_{d-1})$. Then, by our observation above, the system of parameters $x_1, \dots, x_{d-1}, x_d^t$ also satisfies the following:

$$\ell(R/(x_1, \dots, x_{d-1}, x_d^t)) = \lim \ell(F^n(\Omega/(x_1, \dots, x_{d-1}, x_d^t)\Omega))/p^{nd}.$$

But this contradicts the result proved in Part 1, since

$$(x_1, \dots, x_{d-1}, x_d^t) = (x_1, \dots, x_{d-1}, x_d'),$$

where $x_d' \in \Omega$.

Remark. On a complete Cohen-Macaulay local ring R of dimension d, if M is a module of finite length and finite projective dimension, $\ell(M)$ may be different from $\ell(\tilde{M})$, where $\tilde{M} = \mathrm{Ext}^d(M, R)$. (See [R], for such an example.) Still, one can show, by using Theorem 1.5 of [D2], that

$$\lim \ell(F^n(M))/p^{nd} = \lim \ell(F^n(\tilde{M}))/p^{nd}.$$

Actually, a more general statement is valid: for any module M of finite length

$$\lim \ell(\mathrm{Ext}^d(F^n(M), R))/p^{nd} = \lim \ell(F^n(M))/p^{nd}.$$

References

[D1] S. P. Dutta, *Frobenius and Multiplicities*, J. Algebra No. 2 **85** (1983), 424–448. MR **85f:**13022

[D2] S. P. Dutta, Ext *and Frobenius*, J. Algebra No. 1 **127** (1989), 163–177. MR **91g:**13018

[K] I. Kaplansky, *Commutative Rings, Revised Edition*, Univ. of Chicago Press (1974). MR **49:**10674

[L] S. Lichtenbaum, *On the vanishing of* Tor *in regular local rings*, Illinois J. Math. **10** (1966), 220–226. MR **32:**5688

[P-S] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie local*, Publ. I.H.E.S. Paris **42** (1973), 47–119. MR **51:**10330

[R] P. Roberts, *Intersection theorems,* Commutative algebra, Hochster, Huneke, Sally Editors, MSRI Publications, 15, Springer-Verlag (1989), 417–437. MR **90j:**13024

[S] G. Seibert, *Complexes with homology of finite length and Frobenius functions*, J. Algebra **125** (1989), 278–287. MR **90j:**13012

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801

E-mail address: dutta@math.uiuc.edu