

A MAXIMAL INEQUALITY FOR PARTIAL SUMS  
OF FINITE EXCHANGEABLE SEQUENCES  
OF RANDOM VARIABLES

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ABSTRACT. Let  $X_1, X_2, \dots, X_{2n}$  be a finite exchangeable sequence of Banach space valued random variables, i.e., a sequence such that all joint distributions are invariant under permutations of the variables. We prove that there is an absolute constant  $c$  such that if  $S_j = \sum_{i=1}^j X_i$ , then

$$P\left(\sup_{1 \leq j \leq 2n} \|S_j\| > \lambda\right) \leq cP(\|S_n\| > \lambda/c),$$

for all  $\lambda \geq 0$ . This generalizes an inequality of Montgomery-Smith and Latała for independent and identically distributed random variables. Our maximal inequality is apparently new even if  $X_1, X_2, \dots$  is an *infinite* exchangeable sequence of random variables. As a corollary of our result, we obtain a comparison inequality for tail probabilities of sums of arbitrary random variables over random subsets of the indices.

Montgomery-Smith [8] and Latała [7] have independently proved that if  $X_1, \dots, X_n$  are independent and identically distributed Banach space valued random variables, then

$$(1) \quad P\left(\sup_{1 \leq j \leq k} \left\| \sum_{i=1}^j X_i \right\| > \lambda\right) \leq cP\left(\left\| \sum_{i=1}^k X_i \right\| > \lambda/c\right),$$

whenever  $\lambda \geq 0$  and  $1 \leq k \leq n$ , where  $c$  is an absolute constant. It is obvious that this cannot hold for arbitrary independent random variables; as Montgomery-Smith [8] notes, we need only let  $k = n = 2$ ,  $X_1 \equiv 1$  and  $X_2 \equiv -1$  to see this. Lévy's inequality says that (1) also holds for arbitrary independent *symmetric* random variables  $X_i$  (not necessarily identically distributed). For positive random variables, (1) is trivial, of course.

A natural and much-studied extension of the concept of independent and identically distributed random variables is that of exchangeable random variables. We say that a finite sequence  $X_1, \dots, X_n$  of (not necessarily independent) random variables is *exchangeable* if the  $n$ -tuples  $(X_1, \dots, X_n)$  and  $(X_{\pi(1)}, \dots, X_{\pi(n)})$  both have the same distribution whenever  $\pi$  is a permutation of  $[n] \stackrel{\text{def}}{=} \{1, \dots, n\}$ . Evidently an exchangeable sequence of *independent* random variables is precisely a sequence of independent and identically distributed random variables.

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Note that given any sequence  $Y_1, \dots, Y_n$  of random variables (with no independence or identical distribution assumptions), if  $\tau$  is a random permutation of  $[n]$  (with all  $n!$  such permutations being equally likely, and with  $\tau$  independent of the  $Y_i$ ), and if we put  $X_i = Y_{\tau(i)}$ , then  $X_1, \dots, X_n$  is an exchangeable sequence. If the  $Y_i$  are constants, then this is just sampling without replacement from an urn, and moreover, in this case unless the  $Y_i$  are all the same, the sequence  $X_1, \dots, X_n$  cannot be extended to an exchangeable sequence of length  $n + 1$  (i.e., there is no exchangeable sequence  $X'_1, \dots, X'_n, X'_{n+1}$  such that the  $n$ -dimensional distribution of  $(X_i)_{1 \leq i \leq n}$  coincides with that of  $(X'_i)_{1 \leq i \leq n}$ ).

An infinite sequence of random variables is said to be *exchangeable* if every finite subsequence is exchangeable. De Finetti [2] has proved that the probability distribution of an infinite exchangeable sequence of random variables is a mixture of probability distributions of independent and identically distributed sequences. This, together with Montgomery-Smith and Latała's inequality (1), implies that

$$(2) \quad P\left(\left\|\sum_{i=1}^k X_i\right\| > \lambda\right) \leq cP\left(\left\|\sum_{i=1}^n X_i\right\| > \lambda/c\right),$$

for  $1 \leq k \leq n$ , providing that  $X_1, \dots, X_n$  can be extended to an infinite sequence of exchangeable random variables. (The reader may wish to see, e.g., [1, 3, 4, 5, 9, 10] for work on the question of extending finite exchangeable sequences to infinite ones or at least to longer finite ones.)

However, (2) need not hold if  $X_1, \dots, X_n$  are assumed to be merely exchangeable. For, let  $V_1, \dots, V_n$  be any sequence of vectors, not all zero, such that  $V_1 + \dots + V_n = 0$ . Let  $\tau$  be a random permutation of  $[n]$ , with all  $n!$  such permutations being equally likely. Let  $X_i = V_{\tau(i)}$  (this is just sampling the  $V_j$  from an urn without replacement). It is clear that  $X_1, \dots, X_n$  are exchangeable. But,  $X_1 + \dots + X_n = V_1 + \dots + V_n = 0$ , while  $P(\|X_1\| > 0) > 0$  since the  $V_i$  are not all zero, so that (2) cannot hold if  $k = 1$  and  $\lambda > 0$  is sufficiently small.

Hence, we see that (2) must hold if  $X_1, \dots, X_n$  can be extended to an infinite sequence of exchangeable random variables, but need not hold if  $X_1, \dots, X_n$  are merely assumed to be exchangeable. Professor Stephen J. Montgomery-Smith has asked the author whether perhaps (2) would hold if  $X_1, \dots, X_n$  can be extended to an exchangeable sequence of random variables of length  $2n$ . He also noted that

$$\begin{aligned} P(\|X\| > \lambda) &= P(\|X + Y + X + Z - Y - Z\| > 2\lambda) \\ &\leq P(\|X + Y\| > \frac{2}{3}\lambda) + P(\|X + Z\| > \frac{2}{3}\lambda) + P(\|Y + Z\| > \frac{2}{3}\lambda) \\ &= 3P(\|X + Y\| > \frac{2}{3}\lambda), \end{aligned}$$

whenever  $X, Y, Z$  are exchangeable, which shows that the answer is positive for  $n = 2$ .

We shall show that the answer to Professor Montgomery-Smith's question is positive in general, and in fact a length of  $2n$  is not needed, but any length at least as large as  $\gamma n$  will do, where  $\gamma$  is any fixed real number strictly greater than one; moreover we not only obtain (2), but also the maximal inequality (1).

**Theorem 1.** Fix  $\gamma > 1$ . Let  $n'$  be an integer at least as large as  $\gamma n$ . Let  $X_1, \dots, X_{n'}$  be an exchangeable sequence of random variables. Then,

$$(3) \quad P\left(\sup_{1 \leq j \leq k} \left\|\sum_{i=1}^j X_i\right\| > \lambda\right) \leq cP\left(\left\|\sum_{i=1}^k X_i\right\| > \lambda/c\right),$$

whenever  $\lambda \geq 0$  and  $1 \leq k \leq n$ , where  $c = c(\gamma) \in (0, \infty)$  is a constant depending only on  $\gamma$ .

This shows that (1) holds for  $1 \leq k \leq n$  providing  $X_1, \dots, X_n$  can be extended to an exchangeable sequence of random variables of length at least  $\gamma n$ . The proof of Theorem 1 will be given later on in this paper.

We immediately obtain the following corollary which is apparently new in itself.

**Corollary 1.** *Let  $X_1, X_2, \dots$  be an infinite exchangeable sequence of Banach valued random variables. Then, the Lévy-type maximal inequality (1) holds for all  $k \geq 1$ , where  $c$  is an absolute constant.*

We write  $[x]$  for the greatest integer less than or equal to  $x$ . Put  $[x] = -[-x]$ . We then have the following maximal inequality.

**Corollary 2.** *Fix  $\rho \in (0, 1)$ . Let  $X_1, \dots, X_n$  be an exchangeable sequence of random variables with  $\rho n \geq 1$ . Let  $m = [ \rho n ]$ . Then,*

$$P\left(\sup_{1 \leq j \leq n} \left\| \sum_{i=1}^j X_i \right\| > \lambda\right) \leq cP\left(\left\| \sum_{i=1}^m X_i \right\| > \lambda/c\right),$$

for all  $\lambda \geq 0$ , where  $c = c(\rho) \in (0, \infty)$  is an absolute constant depending only on  $\rho$ .

In particular, if  $X_1, \dots, X_{2n}$  are exchangeable, then

$$P\left(\sup_{1 \leq j \leq 2n} \left\| \sum_{i=1}^j X_i \right\| > \lambda\right) \leq cP\left(\left\| \sum_{i=1}^n X_i \right\| > \lambda/c\right),$$

where  $c = c(\frac{1}{2})$  is an absolute constant.

*Proof of Corollary 2.* Set  $S_j = \sum_{i=1}^j X_i$ . Let  $\gamma = 1/\rho$ . Then,  $n \geq \gamma m$ . Hence,

$$(4) \quad P\left(\sup_{1 \leq j \leq m} \|S_j\| > \lambda\right) \leq cP(\|S_m\| > \lambda/c),$$

by Theorem 1, where  $c$  depends only on  $\gamma = 1/\rho$ .

Note that  $n/m \leq 2/\rho$  (this uses the assumption that  $\rho n \geq 1$ ). Let  $N = [n/m]$ . Then,  $N \leq 1 + 2/\rho$ . For  $0 \leq k < N$ , consider the event

$$E_k = \left\{ \sup_{1 \leq j \leq m} \left\| \sum_{i=mk+1}^{\min(mk+j, n)} X_i \right\| > \lambda/N \right\}.$$

By exchangeability and (4),

$$P(E_k) = P\left(\sup_{1 \leq j \leq \min(m(k+1), n) - (mk+1)} \left\| \sum_{i=1}^j X_i \right\| > \lambda\right) \leq cP\left(\left\| \sum_{i=1}^m X_i \right\| > \lambda/c\right).$$

But is easy to see that if  $\omega \notin \bigcup_{k=0}^{N-1} E_k$ , then  $\|S_j(\omega)\| \leq \lambda$ , so that

$$\begin{aligned} P\left(\sup_{1 \leq j \leq n} \|S_j\| > \lambda\right) &\leq P\left(\bigcup_{k=0}^{N-1} E_k\right) \leq NcP(\|S_m\| > \lambda/Nc) \\ &\leq (1 + 2/\rho)cP\left(\|S_m\| > \frac{\lambda}{(1 + 2/\rho)c}\right), \end{aligned}$$

and the proof is complete. □

The following result then follows by the same method of proof as Corollary 5 of Montgomery-Smith [8] (cf. [6, Proposition 1.2.1]).

**Corollary 3.** Fix  $\rho \in (0, 1)$ . Let  $X_1, \dots, X_n$  be an exchangeable sequence of random variables with  $\rho n \geq 1$ . Let  $m = \lfloor \rho n \rfloor$ . Let  $\alpha_1, \dots, \alpha_n$  be an arbitrary sequence of complex constants with  $|\alpha_i| \leq 1$  for all  $i$ . Then,

$$P\left(\left\|\sum_{i=1}^n \alpha_i X_i\right\| > \lambda\right) \leq cP\left(\left\|\sum_{i=1}^m X_i\right\| > \lambda/c\right),$$

for all  $\lambda \geq 0$ , where  $c = c(\rho) \in (0, \infty)$  is an absolute constant depending only on  $\rho$ .

We also obtain the following comparison inequality for tail probabilities of random sums of arbitrary random variables.

**Corollary 4.** Fix  $\rho \in (0, 1)$ . Let  $X_1, \dots, X_n$  be arbitrary random variables. Assume that  $\rho n \geq 1$ . Put  $m = \lfloor \rho n \rfloor$ . For  $1 \leq k \leq n$ , let  $A_k$  be a random subset of  $[n]$  with cardinality  $k$ , where all such subsets have equal probability  $1/\binom{n}{k}$ , and where the  $A_k$  are independent of the  $X_i$ . Then,

$$(5) \quad P\left(\left\|\sum_{i \in A_k} X_i\right\| > \lambda\right) \leq cP\left(\left\|\sum_{i \in A_m} X_i\right\| > \lambda/c\right),$$

whenever  $1 \leq k \leq n$  and  $\lambda \geq 0$ , where  $c = c(\rho) \in (0, \infty)$  is a constant depending only on  $\rho$ . Moreover,

$$(6) \quad P\left(\left\|\sum_{i \in A_j} X_i\right\| > \lambda\right) \leq c'P\left(\left\|\sum_{i \in A_k} X_i\right\| > \lambda/c'\right),$$

whenever  $1 \leq j \leq k \leq m$  and  $\lambda \geq 0$ , where  $c' = c'(\rho) \in (0, \infty)$  is a constant depending only on  $\rho$ .

*Proof.* Let  $\tau$  be a random permutation of  $[n]$ , with all  $n!$  such permutations having equal probability and with  $\tau$  independent of the  $X_i$ . Let  $Y_i = X_{\tau(i)}$ . Then,  $Y_1, \dots, Y_n$  are exchangeable. Moreover,  $\sum_{i \in A_j} X_i$  has the same probability distribution as  $\sum_{i=1}^j Y_i$ , so that (5) and (6) follow from Corollary 2 and Theorem 1 (with  $\gamma = 1/\rho$ ), respectively.  $\square$

*Remark 1.* An examination of our proofs will show that the constant  $c = c(\gamma)$  in Theorem 1 may be taken to be

$$c(\gamma) = C_1 \frac{\gamma^2}{(\gamma - 1)^2},$$

where  $C_1$  is an absolute constant. Likewise, it can be seen by analyzing our proofs that the constant  $c = c(\rho)$  in Corollaries 2 and 3 can be taken to be

$$c(\rho) = C_2 \frac{1}{\rho(1 - \rho)^2},$$

where  $C_2$  is an absolute constant. The constant in (5) can be taken to be the constant of Corollary 2, while the constant in (6) can be chosen to just be the constant of Theorem 1 with  $\gamma = \rho^{-1}$ . It is not known whether the degrees of dependency on  $\gamma$  or  $\rho$  in any of the above constants can be improved.

Note also that the constant  $c$  appeared twice in (3). We could, thus, rewrite (3) with  $c_1(\gamma)$  in place of the first occurrence of  $c$  in (3) and  $c_2(\gamma)$  in place of the second. Our proofs then yield the validity of the rewritten (3) with

$$c_1(\gamma) = \frac{15\bar{\gamma}^2 + 3\bar{\gamma}}{(\bar{\gamma} - 1)^2}$$

and

$$c_2(\gamma) = \frac{24\bar{\gamma}^2}{(\bar{\gamma} - 1)^2},$$

where  $\bar{\gamma} = \min(\gamma, 2)$ . It is most unlikely that the numerical values here are optimal, and it is not known whether the degrees of  $\gamma$ -dependence are optimal in either  $c_1$  or  $c_2$ .

We now proceed to the proof of Theorem 1. First we need a certain definition of Montgomery-Smith [8]. We say that  $x$  is a  $\lambda$ -concentration point for a random variable  $X$  if  $P(\|X - x\| \leq \lambda) \geq \frac{2}{3}$ . The following result is crucial.

**Proposition 1** (Montgomery-Smith [8]). *Let  $X_1, \dots, X_n$  be exchangeable random variables. Fix  $1 \leq k \leq n$ , and suppose that the partial sums  $S_j = \sum_{i=1}^j X_i$  have  $\lambda$ -concentration points  $s_j$  whenever  $1 \leq j \leq k$ . Then  $\|ks_j - js_k\| \leq 3(k + j)\lambda$  whenever  $1 \leq j \leq k$ .*

Montgomery-Smith [8] states this in the independent and identically distributed case. However, the proof in the exchangeable case is identical.

A random variable  $U$  is said to be *symmetric* if  $U$  and  $-U$  have the same distribution. We say that a sequence  $U_1, \dots, U_n$  of random variables is *jointly symmetric* if  $(U_1, \dots, U_n)$  has the same joint distribution as  $(\varepsilon_1 U_1, \dots, \varepsilon_n U_n)$  for any choice of signs  $\varepsilon_i \in \{-1, 1\}$ . While Lévy's inequality is normally stated for sums of independent Banach space valued symmetric random variables, the same standard proof works just as well for jointly symmetric Banach space valued sequences of random variables and shows that if  $U_1, \dots, U_n$  are jointly symmetric, then

$$(7) \quad P\left(\left\|\sup_{1 \leq j \leq n} \sum_{i=1}^j U_i\right\| > \lambda\right) \leq 2P\left(\left\|\sum_{i=1}^n U_i\right\| > \lambda\right),$$

for every  $\lambda \geq 0$ .

Recall that we have defined  $[n] = \{1, \dots, n\}$  for any positive integer  $n$ .

**Lemma 1.** *Let  $X_1, \dots, X_n, Y_1, \dots, Y_n$  be a sequence of exchangeable Banach space valued random variables. Put  $S_k = \sum_{i=1}^k X_i$  and  $T_k = \sum_{i=1}^k Y_i$ . Then for any  $\lambda \geq 0$  we have*

$$P\left(\sup_{1 \leq k \leq n} \|S_k - T_k\| > \lambda\right) \leq 2P(\|S_n - T_n\| > \lambda).$$

*Proof.* Let  $U_i = X_i - Y_i$ . Exchangeability then shows that  $(X_i, Y_i)$  and  $(Y_i, X_i)$  have the same conditional distributions given the  $\sigma$ -field

$$\mathcal{F}_i \stackrel{\text{def}}{=} \sigma(U_1, \dots, U_{i-1}, U_{i+1}, \dots, U_n)$$

for any fixed  $i \in [n]$ . Therefore,  $U_i$  and  $-U_i$  have the same conditional distributions given  $\mathcal{F}_i$ , and it follows that  $U_1, \dots, U_n$  are jointly symmetric. The desired inequality then follows immediately from Lévy's inequality (7).  $\square$

**Lemma 2.** Fix  $\gamma \in (1, 2]$ . Let  $n' = \lceil \gamma n \rceil$ . Suppose that  $X_1, \dots, X_{n'}$  are exchangeable Banach space valued random variables. Then,

$$P\left(\sup_{1 \leq i \leq n'-n} \left\| \sum_{i=1}^j X_i \right\| > \lambda\right) \leq cP\left(\left\| \sum_{i=1}^n X_i \right\| > \lambda/c\right),$$

for all  $\lambda \geq 0$ , where  $c = c(\gamma) \in (0, \infty)$  depends only on  $\gamma$ .

If  $\gamma = 2$ , Theorem 1 is obviously equivalent to Lemma 2. For other values of  $\gamma$ , we need the simple argument below, which is the same argument as was used to show that Theorem 1 implies Corollary 2.

*Proof of Theorem 1.* Replacing  $\gamma$  by  $\min(\gamma, 2)$ , we may assume that  $\gamma \in (1, 2]$ . Without loss of generality,  $n' = \lceil \gamma n \rceil$ . Put  $S_j = \sum_{i=1}^j X_i$ . Let  $r = n' - n$ . Let  $N = \lceil n/r \rceil$ . Note that  $r \geq (\gamma - 1)n$  so that  $N \leq 1 + (\gamma - 1)^{-1}$ . Fix  $\lambda \geq 0$ . Let

$$E_k = \left\{ \sup_{1 \leq j \leq \max((k+1)r, n) - kr} \left\| \sum_{i=kr+1}^{kr+j} X_i \right\| > \frac{\lambda}{N} \right\},$$

for  $0 \leq k < N$ . By exchangeability and Lemma 2, we have

$$\begin{aligned} P(E_k) &= P\left(\sup_{1 \leq j \leq \max((k+1)r, n) - kr} \left\| \sum_{i=1}^j X_i \right\| > \frac{\lambda}{N}\right) \\ (8) \quad &\leq cP\left(\|S_n\| > \frac{\lambda}{Nc}\right), \end{aligned}$$

where  $c$  depends only on  $\gamma$ . It is easy to see that if  $\omega \notin \bigcup_{i=0}^{N-1} E_k$ , then  $\|S_j(\omega)\| \leq \lambda$  for all  $j \in [n]$ . Thus by (8) we have

$$\begin{aligned} P\left(\sup_{1 \leq j \leq n} \|S_j\| > \lambda\right) &\leq P\left(\bigcup_{k=0}^{N-1} E_k\right) \\ &\leq NcP\left(\left\| \sum_{i=1}^n X_i \right\| > \frac{\lambda}{Nc}\right) \\ &\leq (1 + (\gamma - 1)^{-1})cP\left(\left\| \sum_{i=1}^n X_i \right\| > \frac{\lambda}{(1 + (\gamma - 1)^{-1})c}\right). \end{aligned}$$

□

It remains to prove Lemma 2. The careful reader will note that the proof in the case  $\gamma = 2$  simplifies somewhat.

*Proof of Lemma 2.* Let  $r = n' - n$ . Note that  $r \leq n$  as  $\gamma \leq 2$ . Put  $S_j = \sum_{i=1}^j X_i$  for  $j \in [n]$ . Set  $Y_i = X_{n+i}$  for  $i \in [r]$ , and define  $T_j = \sum_{i=1}^j Y_i$ . Fix  $\lambda \geq 0$ . Let

$$(9) \quad p = P(\|S_n\| > \lambda).$$

Note that  $S_n = S_r + (S_n - S_r)$  and that exchangeability then shows that  $S_n$  has the same distribution as  $T_r + (S_n - S_r)$ . Hence we have

$$\begin{aligned} (10) \quad &P(\|S_r - T_r\| > 2\lambda) \\ &= P(\|S_r + (S_n - S_r) - [T_r + (S_n - S_r)]\| > 2\lambda) \\ &\leq P(\|S_n\| > \lambda) + P(\|T_r + (S_n - S_r)\| > \lambda) = 2p. \end{aligned}$$

By Lemma 1 we then have

$$(11) \quad P\left(\sup_{1 \leq j \leq r} \|S_j - T_j\| > 2\lambda\right) \leq 4p.$$

Let  $N = \lceil n/r \rceil$ . We have  $N \geq 1$  since  $r \leq n$ . Note that  $r \geq (\gamma - 1)n$  so that  $N \leq 1 + (\gamma - 1)^{-1}$ . For  $0 \leq k < N$  and  $rk < j \leq \min(r(k + 1), n)$ , let

$$T'_j = kT_r + T_{j-rk}.$$

Note that  $T'_{rk} = kT_r$ . Define  $T'_0 = S_0 = 0$ . Suppose that  $rk < j \leq \min(r(k + 1), n)$ . Then,

$$(12) \quad \begin{aligned} \|S_j - T'_j\| &= \left\| \sum_{l=1}^k (S_{rl} - S_{r(l-1)} - T_r) + \sum_{i=rk+1}^j (X_i - Y_{i-rk}) \right\| \\ &\leq \sum_{l=1}^N \|S_{rl} - S_{r(l-1)} - T_r\| + \left\| \sum_{i=rk+1}^j (X_i - Y_{i-rk}) \right\|. \end{aligned}$$

Let  $F_1 = \{\sup_{1 \leq j \leq n} \|S_j - T'_j\| > 4N\lambda\}$ . Because of (12) we then have  $P(F_1) \leq q_1 + q_2$ , where

$$\begin{aligned} q_1 &= P\left(\sum_{l=1}^{N-1} \|S_{rl} - S_{r(l-1)} - T_r\| > 2N\lambda\right) \\ &\leq \sum_{l=1}^{N-1} P(\|S_{rl} - S_{r(l-1)} - T_r\| > 2\lambda) \\ &= \sum_{l=1}^{N-1} P(\|S_r - T_r\| > 2\lambda) \\ &\leq 2(N - 1)p, \end{aligned}$$

by exchangeability and (10), and where

$$\begin{aligned} q_2 &\leq P\left(\sup_{\substack{0 \leq k < N \\ rk < j \leq \min(r(k+1), n)}} \left\| \sum_{i=rk+1}^j (X_i - Y_{i-rk}) \right\| > 2N\lambda\right) \\ &\leq \sum_{k=0}^{N-1} P\left(\sup_{rk < j \leq \min(r(k+1), n)} \left\| \sum_{i=rk+1}^j (X_i - Y_{i-rk}) \right\| > 2N\lambda\right) \\ &= \sum_{k=0}^{N-1} P\left(\sup_{1 \leq j \leq \min(r(k+1), n) - rk} \left\| \sum_{i=1}^j (X_i - Y_i) \right\| > 2N\lambda\right) \\ &\leq 4Np, \end{aligned}$$

by exchangeability together with (11) and as  $N \geq 1$ . Hence,  $P(F_1) \leq q_1 + q_2 \leq (6N - 2)p$ . Set

$$F_2 = \{\|S_n\| > \lambda\},$$

and note that  $P(F_2) = p$ . Put  $F = F_1 \cup F_2$  so that we will have  $P(F) \leq (6N - 1)p$ . Let  $\mathcal{G} = \sigma(Y_1, \dots, Y_r)$ . Let  $E$  be the event  $\{P(F | \mathcal{G}) \leq \frac{1}{3}\}$ .

Then,

$$(6N - 1)p \geq P(F) = E[P(F | \mathcal{G})] \geq \frac{1}{3}(1 - P(E)),$$

so that

$$(13) \quad P(E) \geq 1 - (18N - 3)p.$$

Note that  $E \in \mathcal{G}$ . But

$$(14) \quad P\left(\sup_{1 \leq j \leq n} \|S_j - T'_j\| > 4N\lambda \mid \mathcal{G}\right) \leq \frac{1}{3} \quad \text{on } E.$$

(Note of course that  $E$  could have measure zero, in which case this last assertion is trivial.) Hence, given  $\mathcal{G}$  and  $E$ , the vector  $T'_j$  is a  $4N\lambda$ -concentration point of  $S_i$  whenever  $1 \leq j \leq n$  since  $P(F_1 \mid \mathcal{G}) \leq \frac{1}{3}$  on  $E$ . Moreover, since  $P(F_2 \mid \mathcal{G}) \leq \frac{1}{3}$  on  $E$ , the vector 0 is a  $\lambda$ -concentration point of  $S_n$  given  $\mathcal{G}$  and  $E$ , and thus also a  $4N\lambda$ -concentration point. Since the variables  $X_1, \dots, X_n$  can easily be seen to be (conditionally) exchangeable given  $\mathcal{G}$ , it follows from Proposition 1 that almost surely on  $E$  we have

$$\|j \cdot 0 - nT'_j\| \leq 12(n + j)N\lambda,$$

for  $1 \leq j \leq n$ . But  $T'_j = T_j$  for  $1 \leq j \leq r$ . Moreover  $(n + j)/n \leq 2$  for  $1 \leq j \leq r$  since  $(n + r)/n \leq 2$  as  $\gamma \leq 2$ , so that on  $E$  we almost surely have  $\|T_j\| \leq 24N\lambda$  for  $1 \leq j \leq r$ . Therefore,

$$\begin{aligned} P\left(\sup_{1 \leq j \leq r} \|T_j\| > 24N\lambda\right) &\leq 1 - P(E) \leq (18N - 3)p \\ &= (18N - 3)P(\|S_n\| > \lambda), \end{aligned}$$

by (13) and (9). By exchangeability, we may replace  $T_j$  by  $S_j$  on the left-hand side of this inequality and then since  $\lambda \geq 0$  was arbitrary, it follows that

$$\begin{aligned} P\left(\sup_{1 \leq j \leq r} \|S_j\| > \lambda\right) &\leq (18N - 3)P(\|S_n\| > \lambda/24N) \\ &\leq (18(\gamma - 1)^{-1} + 15)P\left(\|S_n\| > \frac{\lambda}{24(1 + (\gamma - 1)^{-1})}\right), \end{aligned}$$

as  $N \leq 1 + (\gamma - 1)^{-1}$ . □

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