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ON THE SUSPENSION ORDER OF $(RP^{2m})^{[k]}$

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ABSTRACT. It is shown that the suspension order of the k-fold cartesian product $(RP^{2m})^{[k]}$ of real projective $2m$-space RP^{2m} is less than or equal to the suspension order of the k-fold symmetric product SP^kRP^{2m} of RP^{2m} and greater than or equal to 2^{r+s+1}, where k and m satisfy $2^r \leq 2m < 2^{r+1}$ and $2^s \leq k < 2^{s+1}$. In particular $RP^2 \times RP^2$ has suspension order 8, and for fixed $m \geq 1$ the suspension orders of the spaces $(RP^{2m})^{[k]}$ are unbounded while their stable suspension orders are constant and equal to $2^{\phi(2m)}$.

INTRODUCTION

For a space X the *suspension order* of X is the least integer l such that the degree l self-map of the suspension ΣX of X is null-homotopic. This is the order of the identity map of ΣX in the group of homotopy classes $[\Sigma X, \Sigma X]$. The *stable suspension order* of X is the order of the identity map of the m^{th} suspension of X, $\Sigma^m X$, in the group $[\Sigma^m X, \Sigma^m X]$ for m sufficiently large. The latter divides the former by the suspension homomorphism when X has finite suspension order.

It is natural to ask if the notions of suspension order and stable suspension order are really distinct, and if so, to provide simple examples illustrating this fact. Our approach to this question via cartesian products is motivated by the following observation. Let X and Y be localized at a prime p with suspension orders p^r and p^s, respectively, $r \geq s$. Then from Toda [9] the suspension order of $X \vee Y$ is p^r while that of $X \wedge Y$ divides p^s. From the suspension of the cofibration

$$X \vee Y \longrightarrow X \times Y \longrightarrow X \wedge Y$$

it follows that the suspension order of $X \times Y$ divides p^{r+s}.

The standard splitting $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$ is not as suspension spaces, since it is obtained by adding the suspensions of the projection maps and the quotient map $X \times Y \longrightarrow X \wedge Y$. So although $X \times Y$ and $X \vee Y \vee (X \wedge Y)$ have the same stable suspension order, they may have different suspension order.

In §1 we provide a homology calculation that establishes a lower bound for the suspension order of $(RP^{2m})^{[k]}$. This modifies and extends a similar calculation in [8], which showed that the suspension order of $RP^2 \times RP^2$ was a multiple of 8 (thus exhibiting a space with unequal suspension order and stable suspension order). On the positive side, the Palais Covering Homotopy Theorem is invoked in §2 to provide an upper bound in terms of the suspension order of SP^kRP^{2m}.

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Essentially nothing is known about the latter, except in the special case $m = 1$. Here by a result of Dupont-Lusztig [5] SP^kRP^2 is diffeomorphic to RP^{2k}, making some work (and a conjecture) of Mukai [6], [7] relevant.

Our notation is for the most part standard. H and $\tilde{H}$ denote ordinary and reduced (singular) homology. ΩX and $X^{[k]}$ stand for the loop space on X and the k-fold cartesian product of X. And RP^n is real projective n-space.

1. The Homomorphism l_*

As a Hopf algebra $H_*(\Omega\Sigma X;\mathbb{Z}/2)$ is isomorphic to the mod 2 tensor algebra $T\tilde{H}_*(X;\mathbb{Z}/2)$ on the $\mathbb{Z}/2$-vector space $\tilde{H}_*(X;\mathbb{Z}/2)$ [2]. The induced homomorphism of the standard inclusion $j : X \hookrightarrow \Omega\Sigma X$ embeds $H_*(X;\mathbb{Z}/2)$ as a summand of $H_*(\Omega\Sigma X;\mathbb{Z}/2)$. We sometimes write the same symbol for both an element of $H_*(X;\mathbb{Z}/2)$ and its j_*-image in $H_*(\Omega\Sigma X;\mathbb{Z}/2)$.

When $X = Y^{[k]}$, the k-fold cartesian product, we have two tensor products to account for. The Künneth tensor product structure is written here as usual, while that for the Bott-Samelson theorem for $H_*(\Omega\Sigma X;\mathbb{Z}/2)$ is written by juxtaposition. Thus we write $x_{P_1}x_{P_2}...x_{P_l} \in H_*(\Omega\Sigma Y^{[k]};\mathbb{Z}/2)$, where $x_{P_i} = x_{P_i(1)}\otimes...\otimes x_{P_i(k)}$ and $x_{P_i(j)} \in H_*(Y;Z/2)$. Also we abbreviate $x_{P_1}x_{P_2}...x_{P_l} = x_P$, where $P = (P_1,...,P_l)$.

Now consider the space $Y = RP^n$. The mod 2 cohomology ring $H^*(RP^n;\mathbb{Z}/2)$ is isomorphic to the truncated polynomial ring $R = \mathbb{Z}/2[\bar{x}]/(\bar{x}^{n+1})$, $\bar{x} \in H^1(RP^n;\mathbb{Z}/2)$ the nonzero class. And by the Künneth theorem the ring $H^*((RP^n)^{[k]};\mathbb{Z}/2)$ is given by the n-fold tensor product of R. The dual of the i^{th} power $\bar{x}^i \in H^i(RP^n;\mathbb{Z}/2)$ will be denoted $x_i \in H_i(RP^n;\mathbb{Z}/2)$.

Thus for $X = Y^{[k]} = (RP^n)^{[k]}$ our notation reads

$$x_{P_i} = x_{P_i(1)} \otimes ... \otimes x_{P_i(k)} \in H_{t_i}((RP^n)^{[k]};\mathbb{Z}/2),$$
$$x_P = x_{P_1}x_{P_2}...x_{P_r} \in H_*(\Omega\Sigma(RP^n)^{[k]};\mathbb{Z}/2),$$

where $t_i = \sum_j P_i(j)$, $P = (P_1, P_2, ..., P_r)$ and $P_i = (P_i(1), P_i(2), ..., P_i(k)) \in T_{n,k} = \{(n_1, n_2, ..., n_k) \in \mathbb{Z}^k | 0 \leq n_j \leq n\}$.

Proposition 1. *Let $X = (RP^n)^{[k]}$, let $x_i \in H_i(RP^n;\mathbb{Z}/2)$ denote the nonzero element, and $x_P = x_{P_1} \otimes x_{P_2} \otimes ... \otimes x_{P_k} \in H_s(X;\mathbb{Z}/2)$, where $\sum P_j = s$. If $l : \Omega\Sigma X \to \Omega\Sigma X$ is the power map that sends any loop to the integer l times that loop, then*

$$l_*j_*x_P = \sum_V x_{V_1}x_{V_2}...x_{V_l}$$

where $V = (V_1, V_2, ..., V_l)$ runs over all l-tuples such that $\sum V_i = P$ in $T_{n,k}$.

Proof. Set $Y = \Omega\Sigma(RP^n)^{[k]}$. Then the power map $l : Y \to Y$ can be viewed as the composite

$$Y \xrightarrow{\Delta_l} Y^{[l]} \xrightarrow{\mu_l} Y,$$

where Δ_l is the diagonal map and μ_l is the l-fold loop multiplication map. For the case of one factor $k = 1$, consider any element $\bar{x}_Q \in H^*((RP^n)^{[l]};\mathbb{Z}/2)$ of the form $\bar{x}_Q = \bar{x}_{Q_1} \otimes \bar{x}_{Q_2} \otimes ... \otimes \bar{x}_{Q_l}$, $Q = (Q_1, Q_2, ..., Q_l)$. An easy mod 2 cohomology argument shows that $\Delta_l^*\bar{x}_Q = \bar{x}^{s(Q)}$, $s(Q) = \sum_{j=1}^{l} Q_j$, where $Q = (Q_1, Q_2, ..., Q_l)$, and so dually

$$(\Delta_l)_* x_i = \sum_Q x_{Q_1} \otimes x_{Q_2} \otimes ... \otimes x_{Q_l}$$

summed over all $Q = (Q_1, Q_2, ..., Q_l)$ with $\sum_{j=1}^{l} Q_j = i$. Hence

$$(1) \qquad (\mu_l)_* (\Delta_l)_* x_i = \sum_Q (\mu_l)_* (x_{Q_1} \otimes x_{Q_2} \otimes ... \otimes x_{Q_k}) = \sum_Q x_Q.$$

Now for any k the above argument applies componentwise to give, for an element $x_P = x_{P_1} \otimes x_{P_2} \otimes ... \otimes x_{P_k} \in H_s(X; \mathbb{Z}/2) \subset H_s(\Omega\Sigma X; \mathbb{Z}/2)$, where $\sum P_j = s$,

$$(2) \qquad (\mu_l)_* (\Delta_l)_* x_P = \sum_V (\mu_l)_* (x_{V_1} \otimes x_{V_2} \otimes ... \otimes x_{V_l}) = \sum_V x_{V_1} x_{V_2} ... x_{V_l}$$

where $P, V_1, V_2, ..., V_l \in T_{n,k}$ and the summation is taken over all l-tuples $V = (V_1, V_2, ..., V_l)$ such that $\sum V_j = P$ in $T_{n,k}$. $\square$

Corollary 1. *Let $X = RP^n, k = 1$ and $x_P = x_{2^r}$ where $2^r \leq n < 2^{r+1}$. Then $(2^r)_* j_* x_{2^r} = x_1^{2^r} \neq 0$.*

Proof. In the expansion (1) there is exactly one term with no subscript equal to 0 (and hence all subscripts equal 1), namely $x_1^{2^r}$, and there are exactly $\binom{2^r}{j}$ terms having exactly j subscripts equal to 0 with the same ordered partition of 2^r for nonzero subscripts. Hence the summation reduces to the one nonzero term $x_1^{2^r}$. $\square$

Corollary 2. *Let $X = (RP^2)^{[k]}$, where $2^r \leq k < 2^{r+1}$, and let $P = (2, 2, ..., 2)$. Then $(2^{r+1})_* j_* x_P$ is nonzero.*

Proof. Let B_{2^r} be the standard basis for $\mathbb{Z}^{2^r}$, i.e., $e_i \in B_{2^r}$ is the 2^r-dimensional vector whose only nonzero component is 1 in the i^{th} position. Then each nonzero term in the expansion (2) for $(2^{r+1})_* j_* x_P$ is a monomial of the form $x_{V_1} x_{V_2} ... x_{V_k}$ where each element of B_{2^r} appears exactly twice as a subscript V_i. In fact this expansion consists of monomials, one for each ordered partition of an ordered set of 2^{r+1} elements into 2^r subsets each of cardinality 2. $\square$

The first interesting case of Corollary 2 is the expansion of $4_* x_{(2,2)} = 4_* x_2 \otimes x_2 \in H_4(\Omega\Sigma(RP^2)^{[2]}; \mathbb{Z}/2)$. Here the ordered set of subscripts is $S = \{1, 2, 3, 4\}$ and the six ordered partitions of S into 2 sets each of cardinality 2 are $[\{1, 2\}, \{3, 4\}]$, $[\{3, 4\}\{1, 2\}]$, $[\{1, 3\}\{2, 4\}]$, $[\{2, 4\}\{1, 3\}]$, $[\{1, 4\}\{2, 3\}]$, $[\{2, 3\}\{1, 4\}]$. In this case Corollary 2 asserts that

$$4_* x_{(2,2)} = a^2 b^2 + b^2 a^2 + abab + baba + ab^2 a + ba^2 b,$$

where $a = x_{(1,0)}$ and $b = x_{(0,1)}$. This result was obtained by the first author in [8] by a somewhat different method.

Corollary 3. *Let $X = (RP^n)^{[k]}$, $2^r \leq n < 2^{r+1}$, and $2^s \leq k < 2^{s+1}$ Then $(2^{r+s})_* x_{(2^r, 2^r, ..., 2^r)}$ is nonzero.*

Proof. As above, the monomial $x_{e_1}^{2^r} ... x_{e_{2^s}}^{2^r}$ appears as a nonzero term in the expansion of $(2^{r+s})_* x_{(2^r, 2^r, ..., 2^r)}$. $\square$

2. Suspension order

Toda in [9] computed the suspension order of a class of spaces which included the spaces $\Sigma^{n-1}RP^{2k}$. Here $n \geq 2$ for all k except $k = 1$, when n may be taken to be 1. In particular Toda's results determined the stable suspension order of RP^{2k}, $k \geq 1$.

Mukai [6] has proposed the following conjecture:

Conjecture. *The suspension order of RP^{2k} equals its stable suspension order.*

This conjecture arose from Mukai's simplified proof of Toda's result on the stable suspension order of RP^{2k}. The truth of the conjecture for $k = 1$ can be found in the early paper of Barratt [1] as well as in Toda [9]. Mukai [6], [7] has shown that

(i) the conjecture is true for $k = 2$;

(ii) the conjecture is true for all k if it is true for $k = 3$;

(iii) the suspension order of $\Sigma^3 RP^6$ is 8.

To relate this discussion to the study of the suspension order of the spaces $(RP^2)^{[k]}$ we recall two results.

Proposition (Dupont-Lusztig [5]). *The k-fold symmetric product of RP^2 is diffeomorphic to RP^{2k}.*

Covering Homotopy Theorem (Palais [3]). *Let G be a compact Lie group and let X and Y be G-spaces. Assume that every open subspace of X/G is paracompact. Let $f : X \to Y$ be equivariant and let $f' : X/G \to Y/G$ be the induced map. Let $F' : X/G \times I \to Y/G$ be a homotopy which preserves the orbit structure and starts at f'. (In particular, f' must preserve orbit structure.) Then there exists an equivariant homotopy $F : X \times I \to Y$ covering F' and starting at f.*

Define functions g and h by the requirement that the suspension order of $(RP^2)^{[k]}$ and RP^{2k} equal $2^{g(k)}$ and $2^{h(k)}$, respectively. Then the equality $h(k) = \phi(2k)$ is the assertion of Mukai's conjecture.

Since $\tilde{H}^*(X; \mathbb{Z})$ is all 2-torsion for $X = (RP^2)^k$ or $X = RP^{2k}$, the suspension orders of these spaces are powers of 2. Recall that the function $\phi(k)$ equals the number of integers l with $1 \leq l \leq k$ and $l \equiv 0, 1, 2$ or $4 \bmod 8$. Toda's result states that the stable suspension order of RP^{2k} is $2^{\phi(2k)}$.

Proposition 2. (i) *$h(k) \geq \phi(2k)$.*

(ii) *$h(k) \geq g(k) \geq r + 1$, where $2^r \leq 2k < 2^{r+1}$.*

(iii) *$h(2) = g(2) = \phi(4) = 3$.*

(iv) *The suspension order of the k-fold cartesian product $(RP^{2m})^{[k]}$ of real projective $2m$-space RP^{2m} is less than or equal to the suspension order of the k-fold symmetric product $SP^k RP^{2m}$ of RP^{2m} and greater than or equal to 2^{r+s+1}, where k and m satisfy $2^r \leq 2m < 2^{r+1}$ and $2^s \leq k < 2^{s+1}$.*

(iii) confirms the conjecture in [8] that the suspension order of $RP^2 \times RP^2$ is 8. Further, (ii) implies that $\lim_{k\to\infty} h(k)$ is infinite, answering a question also in [8]. Since the stable suspension order of $(RP^2)^{[k]}$ is 4 for all $k \geq 1$, (ii) also shows that the suspension order of $(RP^2)^{[k]}$ is greater than its stable suspension order for all $k \geq 2$.

Proof. (i) is an easy consequence of the fact that the order of a homomorphic image of an element is a divisor of the order of that element. For (ii) we have by adjointness that $l : \Sigma X \to \Sigma X$ is null-homotopic if and only if $l \circ j : X \hookrightarrow \Omega\Sigma X \to \Omega\Sigma X$ is null-homotopic. So for $X = RP^{2k}$ Corollary 1 implies that l must be a multiple of 2^{r+1}, where $2^r \leq 2k < 2^{r+1}$. Thus $g(k) \geq r+1$. To show that $h(k) \geq g(k)$, we invoke the Palais Covering Homotopy Theorem. First note that we have a commutative diagram

$$\begin{array}{ccc} \Sigma(RP^2)^{[k]} & \xrightarrow{l} & \Sigma(RP^2)^{[k]} \\ {\scriptstyle \Sigma\pi}\downarrow & & {\scriptstyle \Sigma\pi}\downarrow \\ \Sigma RP^{2k} & \xrightarrow{l} & \Sigma RP^{2k} \end{array}$$

Here

$$\pi : (RP^2)^{[k]} \to SP^k RP^2 = RP^{2k}$$

is the projection map. The G-action on the suspension variable is trivial, so the G-action on ΣRP^{2k} is trivial. The map $l : \Sigma(RP^2)^{[k]} \to \Sigma(RP^2)^{[k]}$ is $S(k)$-equivariant, so if there is a null-homotopy of $l : \Sigma RP^{2k} \to \Sigma RP^{2k}$, it is covered by a homotopy of $l : \Sigma(RP^2)^{[k]} \to \Sigma(RP^2)^{[k]}$. The latter must be null-homotopic as well, since the pre-image of a point under the map $\Sigma\pi : \Sigma(RP^2)^{[k]} \to \Sigma RP^{2k}$ is discrete. So for $l = 2^{h(k)}$ we obtain $2^{h(k)} \geq 2^{g(k)}$. Finally $\varphi(4) = 3$, $g(2) \geq 3$ since $2^2 \leq 2(2) < 2^3$, and Mukai [6] has shown $h(2) = 3$. (iv) The proof is similar to that of (ii). In particular, Corollary 3 to Proposition 2 implies the lower bound, while the upper bound again follows from an application of the Palais Covering Homotopy Theorem. $\square$

It seems reasonable to ask if $(RP^2)^{[k]}$ and RP^{2k} share the same suspension order. The truth of the Mukai Conjecture would give the inequality $\varphi(2k) \geq g(k)$. The other inequality $g(k) \geq \varphi(2k)$ might be attainable with a sharper functor than mod 2 homology theory.

3. Final remarks

1) Power maps $\Omega\Sigma X \xrightarrow{l} \Omega\Sigma X$ are almost never trivial [4], and so null-homotopies of composites of the form $X \overset{j}{\hookrightarrow} \Omega\Sigma X \xrightarrow{l} \Omega\Sigma X$ usually involve the map j.

2) Let $P^m(p^r)$ be the Moore space with integral homology Z_{p^r} in dimension m. For $(P^2(2))^{[2]} (= (\Sigma RP^2)^{[2]})$, 4_* vanishes on the top cell and so no conclusions can be deduced on the suspension order of $(P^2(2))^{[2]}$. Perhaps the existence of nontrivial cup-products in each factor of $X^{[2]}$ is related to its suspension order.

3) If X is a finite CW complex with $\tilde{H}_*(X;\mathbb{Q}) = 0$, then $\tilde{H}_*(SP^k X;\mathbb{Q}) = 0$. So both $X^{[k]}$ and $SP^k X$ have finite suspension orders. Are they related?

4) Here is another example of a space whose suspension order and stable suspension order differ. Let $X = \Omega P^m(p^r)$, with $m \geq 3$ and p an odd prime. If some degree p^k map on ΣX were null, then p^k would annihilate the group $[\Omega X, Y]$ for any Y, in particular the group $[\Omega X, P^m(p^r)]$. By adjointness p^k would then annihilate $[X, X]$, where the group operation is derived from the loop structure of X. But from [4] X has no loop-space exponent and so the suspension order of X is infinite. Now X splits after one suspension (by a theorem of Milnor) into a wedge of smash products of $P^m(p^r)$, so the stable suspension order of X is p^r. The James construction shows that this example is not unrelated to cartesian products.

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