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# ON ASYMMETRY OF TOPOLOGICAL CENTERS OF THE SECOND DUALS OF BANACH ALGEBRAS

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Abstract. Let $\mathfrak{A}$ be a Banach algebra with a bounded approximate identity and let $Z_1(\mathfrak{A}^{**})$ and $Z_2(\mathfrak{A}^{**})$ be the left and right topological centers of $\mathfrak{A}^{**}$. It is shown that i) $\mathfrak{A}^*\mathfrak{A} = \mathfrak{A}\mathfrak{A}^*$ is not sufficient for $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**})$; ii) the inclusion $\hat{\mathfrak{A}}Z_1(\mathfrak{A}^{**}) \subseteq \hat{\mathfrak{A}}$ is not sufficient for $Z_2(\mathfrak{A}^{**})\hat{\mathfrak{A}} \subseteq \hat{\mathfrak{A}}$; iii) $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$ is not sufficient for $\mathfrak{A}$ to be weakly sequentially complete. These results answer three questions of Anthony To-Ming Lau and Ali Ülger.

Suppose that $\mathfrak{A}$ is a Banach algebra and let $\mathfrak{A}^*$ be the dual space of $\mathfrak{A}$. Then $\mathfrak{A}^*$ can be made into a Banach $\mathfrak{A}$-bimodule as follows: for $f \in \mathfrak{A}^*, a \in \mathfrak{A}, fa$ and $af$ are defined by

$$\langle fa, b\rangle = \langle f, ab\rangle,$$
$$\langle af, b\rangle = \langle f, ba\rangle \qquad (b \in \mathfrak{A})$$

where $\langle , \rangle$ is used for the dual pairing between elements of $\mathfrak{A}^*$ and $\mathfrak{A}$.

Let $\mathfrak{A}^*\mathfrak{A} = \{fa : f \in \mathfrak{A}^*, a \in \mathfrak{A}\}$ and $\mathfrak{A}\mathfrak{A}^* = \{af : a \in \mathfrak{A}, f \in \mathfrak{A}^*\}$. The dual space $\mathfrak{A}^*$ is said to factor on the left (resp. right) if $\mathfrak{A}^* = \mathfrak{A}^*\mathfrak{A}$ (resp. $\mathfrak{A}^* = \mathfrak{A}\mathfrak{A}^*$). In a recent paper [5] Anthony To-Ming Lau and Ali Ülger have obtained various necessary and sufficient conditions for factoring of $\mathfrak{A}^*$. Here we answer three questions left open in [5].

The second dual space $\mathfrak{A}^{**}$ of a Banach algebra $\mathfrak{A}$ admits two Banach algebra products known as first (left) and second (right) Arens products. Each of these products extends the product of $\mathfrak{A}$ as canonically embedded in $\mathfrak{A}^{**}$ (see [1] or [3]). We briefly recall the definition of these products. For $m, n \in \mathfrak{A}^{**}$, their first (left) Arens product indicated by $m \,\Box\, n$ is given by

$$\langle m \,\Box\, n, f\rangle = \langle m, nf\rangle \quad (f \in \mathfrak{A}^*),$$

where $nf \in \mathfrak{A}^*$ is defined by

$$\langle nf, a\rangle = \langle n, fa\rangle \quad (a \in \mathfrak{A}),$$

where $fa$ is as defined earlier.

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The second Arens product of elements $m, n \in \mathfrak{A}^{**}$ indicated by $m \diamondsuit n$ is defined by

$$\langle m \diamondsuit n, f \rangle = \langle n, fm \rangle \quad (f \in \mathfrak{A}^*),$$

where $fm$ is an element of $\mathfrak{A}^*$ defined by

$$\langle fm, a \rangle = \langle m, af \rangle \quad (a \in \mathfrak{A}).$$

Again we note that $af$ is as defined earlier.

For each $n \in \mathfrak{A}^{**}$, the mapping $m \mapsto m \,\Box\, n$ (resp. $m \mapsto n \diamondsuit m$) is weak*-weak* continuous. However for certain $n$, the mapping $m \mapsto n \,\Box\, m$ (resp. $m \mapsto n \diamondsuit m$) may fail to be weak* continuous. Due to this lack of symmetry the left (resp. right) topological center $Z_1(\mathfrak{A}^{**})$ (resp. $Z_2(\mathfrak{A}^{**})$) of $\mathfrak{A}^{**}$ is defined by

$$Z_1(\mathfrak{A}^{**}) = \{m \in \mathfrak{A}^{**} : n \mapsto m \,\Box\, n \text{ is weak}^*\text{-weak}^* \text{ continuous}\},$$

$$Z_2(\mathfrak{A}^{**}) = \{m \in \mathfrak{A}^{**} : n \mapsto n \diamondsuit m \text{ is weak}^*\text{-weak}^* \text{ continuous}\}.$$

It follows easily from the definition of $Z_1(\mathfrak{A}^{**})$(resp. $Z_2(\mathfrak{A}^{**})$) that $m \in Z_1(\mathfrak{A}^{**})$ (resp. $m \in Z_2(\mathfrak{A}^{**})$) if and only if $m \Box n = m \diamondsuit n$ (resp. $n \diamondsuit m = n \Box m$) for all $n \in \mathfrak{A}^{**}$. In general $Z_1(\mathfrak{A}^{**})$ and $Z_2(\mathfrak{A}^{**})$ need not be equal (see [5], Example 2.5 and Remark 6.j, p.1211), but they both contain $\hat{\mathfrak{A}}$ ( = the image of $\mathfrak{A}$ in $\mathfrak{A}^{**}$ under the canonical mapping).

Anthony To-Ming Lau and Ali Ülger have shown in [5] that if $\mathfrak{A}^*$ factors on one side but not on the other, then $Z_1(\mathfrak{A}^{**}) \neq Z_2(\mathfrak{A}^{**})$. In connection to this result they have asked:

**Question.** Suppose that $\mathfrak{A}$ has a bounded approximate identity. Does the equality $\mathfrak{A}^*\mathfrak{A} = \mathfrak{A}\mathfrak{A}^*$ imply that $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**})$ ?

We answer this question in the negative. In fact our counter example is a modification of one of the examples in [5].

For a Banach algebra $\mathfrak{A}$ we let $\mathfrak{A}^\sharp$ be the unitization of $\mathfrak{A}$. Thus $\mathfrak{A}^\sharp = \mathfrak{A} \oplus \mathbf{C}$, where $\mathbf{C}$ is the field of complex numbers. For $a, b \in \mathfrak{A}, \alpha, \beta \in \mathbf{C}$ we have $\| (a, \alpha) \| = \| a \| + | \alpha |$ and $(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$.

**1. Lemma.** *For any Banach algebra* $\mathfrak{A}, Z_1((\mathfrak{A}^\sharp)^{**}) = Z_1(\mathfrak{A}^{**}) \oplus \mathbf{C}$ *and a similar result holds for the right topological centers.*

*Proof.* First we note that the two Banach algebras $(\mathfrak{A}^\sharp)^{**}$ and $(\mathfrak{A}^{**})^\sharp$ are isomorphic, when they both have the first Arens product. Then the rest is straightforward from the definition of topological centers. □

The following answers question (6d) in [5].

**2. Theorem.** *There exists a Banach algebra* $\mathfrak{A}$ *possessing a bounded approximate identity and satisfying* $\mathfrak{A}^*\mathfrak{A} = \mathfrak{A}\mathfrak{A}^* = \mathfrak{A}^*$, *but* $Z_1(\mathfrak{A}^{**}) \neq Z_2(\mathfrak{A}^{**})$.

*Proof.* It is known that there exists a Banach algebra $\mathfrak{B}$ having a bounded approximate identity for which $Z_1(\mathfrak{B}^{**}) \neq Z_2(\mathfrak{B}^{**})$ (see [5]). Then for $\mathfrak{A} = \mathfrak{B}^\sharp$, by Lemma 1 we have $Z_1(\mathfrak{A}^{**}) \neq Z_2(\mathfrak{A}^{**})$. However, $\mathfrak{A}^*\mathfrak{A} = \mathfrak{A}\mathfrak{A}^* = \mathfrak{A}^*$, since $\mathfrak{A}$ is unital. □

For the origin of the following question see [5, question 6e].

**Question.** Does the inclusion $\hat{\mathfrak{A}} Z_1(\mathfrak{A}^{**}) \subseteq \hat{\mathfrak{A}}$ imply $Z_2(\mathfrak{A}^{**})\hat{\mathfrak{A}} \subseteq \hat{\mathfrak{A}}$ ?

To answer this question, we first recall that if $\mathfrak{A}$ is a Banach algebra then $\mathfrak{A}^{op}$ is the algebra obtained by reversing the order of multiplication in $\mathfrak{A}$; i.e., $\mathfrak{A}^{op}$ has the product $\circ$ given by $a \circ b = ba$, for every $a, b \in \mathfrak{A}$.

**3. Theorem.** *There exists a Banach algebra $\mathfrak{A}$ with a bounded approximate identity for which $\hat{\mathfrak{A}} Z_1(\mathfrak{A}^{**}) \subset \hat{\mathfrak{A}}$, but $Z_2(\mathfrak{A}^{**})\hat{\mathfrak{A}}$ is not contained in $\hat{\mathfrak{A}}$.*

*Proof.* There exists a Banach algebra $\mathfrak{B}$, with a bounded approximate identity, such that $Z_2(\mathfrak{B}^{**}) = \hat{\mathfrak{B}}$ but $Z_1(\mathfrak{B}^{**})$ is larger than $\hat{\mathfrak{B}}$ (see [5]). It is well known that switching to the opposite multiplication in a Banach algebra results in interchanging the first and the second Arens products in its second dual space. From this and by Lemma 1, for $\mathfrak{A} = (\mathfrak{B}^{op})^\sharp$ we have

$$Z_1(\mathfrak{A}^{**}) = \hat{\mathfrak{B}^{op}} \oplus \mathbf{C} = \hat{\mathfrak{A}}$$

and

$$Z_2(\mathfrak{A}^{**}) = Z_1(\mathfrak{B}^{**}) \oplus \mathbf{C}.$$

Hence

$$\hat{\mathfrak{A}} Z_1(\mathfrak{A}^{**}) = (\hat{\mathfrak{B}^{op}} \oplus \mathbf{C})(\hat{\mathfrak{B}^{op}} \oplus \mathbf{C}) = \hat{\mathfrak{B}^{op}} \oplus \mathbf{C} = \hat{\mathfrak{A}},$$

and

$$Z_2(\mathfrak{A}^{**})\hat{\mathfrak{A}} = (Z_1(\mathfrak{B}^{**}) \oplus \mathbf{C})(\hat{\mathfrak{B}^{op}} \oplus \mathbf{C}) = Z_1(\mathfrak{B}^{**}) \oplus \mathbf{C}.$$

But $Z_1(\mathfrak{B}^{**}) \oplus \mathbf{C}$ is larger than $\hat{\mathfrak{A}}$, since $Z_1(\mathfrak{B}^{**})$ is larger than $\hat{\mathfrak{B}}$. □

In [5], Lau and Ülger ask (question 6j): if $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$, must $\mathfrak{A}$ be weakly sequentially complete?

We give an example of a Banach algebra $\mathfrak{A}$ with identity, and such that $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$, but with $\mathfrak{A}$ not weakly sequentially complete. First we describe a general construction; this construction is known and has been used by workers in the area of Automatic Continuity (see [2], p. 647).

Let $A$ be a Banach algebra, and let $M$ be a Banach $\mathfrak{A}$-bimodule. Then $\mathfrak{A} = M \oplus A$, with norm $\| (m, a) \| = \| m \| + \| a \|$, and product $(m, a)(n, b) = (mb + an, ab)$ is a Banach algebra. If $A$ has a bounded approximate identity $(e_j)$ and if $M$ factors $(M = MA = AM)$, then $(0, e_j)$ is a bounded approximate identity for $\mathfrak{A}$. In particular, if $A$ has an identity $e$, and $M$ is unital, then $(0, e)$ is an identity for $\mathfrak{A}$. Note that $\mathfrak{A}^* = M^* \times A^*$ (with norm $\| (s, f) \| = \max \{\| s \|, \| f \|\}$), and that $\mathfrak{A}^{**} = M^{**} \oplus A^{**}$. It is routine to check that $\mathfrak{A}$ is weakly sequentially complete if and only if each of $A$ and $M$ is weakly sequentially complete. Also, it can be checked that the first Arens product on $\mathfrak{A}^{**}$ satisfies

(1) $(\mu, \alpha) \,\Box\, (\nu, \beta) = (\mu \,\Box\, \beta + \alpha \,\Box\, \nu, \alpha \,\Box\, \beta)$, for $\mu, \nu \in M^{**}$ and $\alpha, \beta \in A^{**}$.

Here, in all cases, $\Box$ has been used to denote that extension of a bilinear operation to second duals, which is weak*-weak* continuous in the left-hand argument; for instance, $(\mu, \beta) \mapsto \mu \,\Box\, \beta$ denotes the extension to $M^{**} \times A^{**}$ of the right module action $(m, b) \mapsto mb$ of $A$ on $M$ which is weak*-weak* continuous in $\mu$.

Now we identify $Z_1(\mathfrak{A}^{**})$. Since a net $(\nu_j, \beta_j)$ in $\mathfrak{A}^{**}$ converges weak* to $(\nu, \beta)$ if and only if $\nu_j \to \nu$ weak* in $M^{**}$ and $\beta_j \to \beta$ weak* in $A^{**}$, it is straightforward to check that $(\mu, \alpha) \in Z_1(\mathfrak{A}^{**})$ if and only if

(2) $\alpha \in Z_1(A^{**})$;

(3) $\nu \mapsto \alpha \,\Box\, \nu$ is weak*-weak* continuous on $M^{**}$; and

(4) $\beta \mapsto \mu \,\Box\, \beta$ is weak*-weak* continuous $A^{**} \to M^{**}$.

Now suppose $Z_1(A^{**}) = \hat{A}$. Then (2) implies $\alpha \in \hat{A}$, and it is then straightforward to verify that (3) holds. So in this case, $(\mu, \alpha) \in Z_1(\mathfrak{A}^{**})$ if and only if $\alpha \in \hat{A}$ and (4) holds.

Finally, we consider the particular case $M = A^*$. Recall that we are assuming $A$ has a bounded approximate identity. To ensure that $\mathfrak{A}^{**}$ has a bounded approximate identity, we suppose that $A^*$ factors, in the sense of [5]. Note that the case that $A$ has an identity is included.

Now we have $M^* = A^{**}$ and $M^{**} = A^{***}$, and the meanings of the various bilinear operations have to be carefully distinguished; however, it can be shown that, for $\mu \in M^{**}, \beta \in A^{**}$, and $\alpha \in M^* = A^{**}$,

$$\langle \mu \,\Box\, \beta, \alpha \rangle = \langle \mu, \beta \,\Box\, \alpha \rangle;$$

that is, the bilinear map $(\mu, \beta) \mapsto \mu \,\Box\, \beta$ on $M^{**} \times A^{**}$ to $M^{**}$ is the right dual module action of $A^{**}$ on $A^{***} = M^{**}$ corresponding to the first Arens product on $A^{**}$. Suppose (4) holds, and take $\alpha = E$, a right identity for the first Arens product in $A^{**} = M^*$. If $\beta_j \to \beta$ weak* in $A^{**}$, then by (4),

$$\langle \mu, \beta_j \rangle = \langle \mu, \beta_j \,\Box\, E \rangle = \langle \mu \,\Box\, \beta_j, E \rangle \to \langle \mu \,\Box\, \beta, E \rangle = \langle \mu, \beta \rangle.$$

That is, (4) implies that $\mu$ is a weak*- continuous linear functional on $A^{**}$, so that $\mu \in (A^*)\hat{} = \hat{M}$. The converse is easy, so we have proved the conclusion $Z_1(\mathfrak{A}^{**}) = \hat{M} \oplus \hat{A} = \hat{\mathfrak{A}}$ in the following proposition; the arguments for $Z_2(\mathfrak{A}^{**})$ are quite similar.

**6. Proposition.** *Let $A$ be a Banach algebra with a bounded approximate identity.*

*(1) Assume $A^*$ factors (in the sense of* [5]*). Then $\mathfrak{A} = A^* \oplus A$ has a bounded approximate identity.*

*(2) Assume $Z_1(A^{**}) = Z_2(A^{**}) = \hat{A}$. Then $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$.*

*(3) $\mathfrak{A}$ is weakly sequentially complete if and only if each of $A$ and $A^*$ is weakly sequentially complete.*

If we take $A = l^1(\mathbb{Z})$ (the group algebra of the discrete group of integers), then $Z_1(A^{**}) = Z_2(A^{**}) = \hat{A}$ (see [4]), and all the assumptions above are satisfied; in particular, $A$ has an identity. Also, $A^* = l^\infty(\mathbb{Z})$ is not weakly sequentially complete. So $\mathfrak{A} = A^* \oplus A$ has an identity, satisfies $Z_1(\mathfrak{A}^{**}) = Z_2(\mathfrak{A}^{**}) = \hat{\mathfrak{A}}$, and $\mathfrak{A}$ is not weakly sequentially complete. That answers (negatively) question 6j of [5].

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