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ANGULAR DERIVATIVES AT BOUNDARY FIXED POINTS FOR SELF-MAPS OF THE DISK

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Abstract. Let ϕ be a one-to-one analytic function of the unit disk $\mathbb{D}$ into itself, with $\phi(0) = 0$. The origin is an attracting fixed point for ϕ, if ϕ is not a rotation. In addition, there can be fixed points on $\partial\mathbb{D}$ where ϕ has a finite angular derivative. These boundary fixed points must be repelling (abbreviated b.r.f.p.). The Kœnigs function of ϕ is a one-to-one analytic function σ defined on $\mathbb{D}$ such that $\phi = \sigma^{-1}(\lambda\sigma)$, where $\lambda = \phi'(0)$. If ϕ_K is the first iterate of ϕ that does have b.r.f.p., we compute the Hardy number of σ, $h(\sigma) = \sup\{p > 0 : \sigma \in H^p(\mathbb{D})\}$, in terms of the smallest angular derivative of ϕ_K at its b.r.f.p.. In the case when no iterate of ϕ has b.r.f.p., then $\sigma \in \bigcap_{p<\infty} H^p$, and vice versa. This has applications to composition operators, since σ is a formal eigenfunction of the operator $C_\phi(f) = f \circ \phi$. When C_ϕ acts on $H^2(\mathbb{D})$, by a result of C. Cowen and B. MacCluer, the spectrum of C_ϕ is determined by λ and the essential spectral radius of C_ϕ, $r_e(C_\phi)$. Also, by a result of P. Bourdon and J. Shapiro, and our earlier work, $r_e(C_\phi)$ can be computed in terms of $h(\sigma)$. Hence, our result implies that the spectrum of C_ϕ is determined by the derivative of ϕ at the fixed point $0 \in \mathbb{D}$ and the angular derivatives at b.r.f.p. of ϕ or some iterate of ϕ.

1. Introduction

Let ϕ be a one-to-one and analytic function of the unit disk $\mathbb{D}$ into itself, with $\phi(0) = 0$, and which is not a rotation. Then $0 < |\phi'(0)| < 1$ and, by Schwarz's Lemma, the iterates of ϕ converge uniformly on compact subsets of $\mathbb{D}$ to 0. A classical result of Kœnigs yields a one-to-one analytic function σ defined on $\mathbb{D}$, with image $G = \sigma(\mathbb{D})$, such that

$$\sigma \circ \phi = \lambda\sigma$$

where $\lambda = \phi'(0)$. Thus, $\lambda G \subset G$, and the action of ϕ on $\mathbb{D}$ is conjugated through σ to multiplication by λ on G. For this reason, the set G is called a **geometric model**. Kœnigs's theorem was introduced in the context of iteration theory to study the iterates of ϕ near the fixed point 0. Here we will relate the growth of σ near $\partial\mathbb{D}$ and the geometry of the image region G to the dynamics of ϕ near $\partial\mathbb{D}$. We will write ϕ_n to denote the nth iterate of ϕ.

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Note that under our assumptions, the function ϕ does not need to extend, even continuously, to $\partial\mathbb{D}$. However, one can consider classical generalized notions of limit and derivative at a point $\zeta \in \partial\mathbb{D}$, namely the notions of non-tangential limit and angular derivative. In this article, we will examine points $\zeta \in \partial\mathbb{D}$ where ϕ has non-tangential limit ζ, hence can be called fixed, and where ϕ has a finite angular derivative, which we will call the **multiplier** at ζ and write as $\phi'(\zeta)$. It follows from the Denjoy-Wolff theorem that $\phi'(\zeta) > 1$; see [9], p. 78, for a recent exposition. So, following the classification of fixed points in the theory of the iteration of rational maps, we can call such a point ζ a **boundary repelling fixed point** (b.r.f.p.) for ϕ.

Recall that, for $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ is the space of analytic functions f on $\mathbb{D}$ such that

$$\sup_{0<r<1} \int_0^\infty |f(re^{i\theta})|^p d\theta < \infty.$$

Our starting point is a result of [10] which states that if a map ϕ as above has a b.r.f.p., then the geometric model $G = \sigma(\mathbb{D})$ contains a twisted sector (see Definition 4.1), and σ does not belong to $H^p(\mathbb{D})$ for p large. In [6] and [7], we completely characterized the connection between the **Hardy class** of σ, i.e. the p's for which $\sigma \in H^p(\mathbb{D})$, and the geometry of $G = \sigma(\mathbb{D})$.

The main purpose of this paper is to relate the b.r.f.p. of ϕ or iterates of ϕ to the geometry of $G = \sigma(\mathbb{D})$ and to the "size" of the Kœnigs map σ. Namely, we will find an exact formula for the Hardy number of σ (see Definition 2.1) in terms of the multipliers at the b.r.f.p.. To do this we need to deal with a generalized notion of "basin of attraction" near each b.r.f.p..

Our study has application to the spectral theory of the composition operator C_ϕ acting on the analytic functions f defined on $\mathbb{D}$ by $C_\phi(f) = f \circ \phi$. In fact, we will show that the shape of the spectrum for the operator C_ϕ acting on the classical Hardy space $H^2(\mathbb{D})$ is completely determined by $\phi'(0)$ and the multipliers at the b.r.f.p. This result is in agreement with the work of H. Kamowitz [4], who, with entirely different techniques, analyzed the spectra of composition operators C_ϕ acting on $H^p(\mathbb{D})$, $1 \leq p < \infty$, under the assumptions that the symbol ϕ is analytic in a neighborhood of $\overline{\mathbb{D}}$.

2. Preliminaries

We first need to recall some notations that we introduced in [6] and [7]. In [6], Lemma 3.2, we considered the set $V = \text{int}(\bigcap_{n\geq 0} \lambda^n G) \subset G$, which we called the **invariant set** of G. Since V is open, we write $V = \bigcup_j V_j$, where V_j are the connected components of V. The invariant set V satisfies $\lambda V = V = \lambda^{-1} V$. Moreover, if V_j is a component of V, then $\lambda V_j = V_k$ is another component of V. Hence, given a component V_j, its orbit $\{\lambda^k V_j\}_{-\infty}^{+\infty}$ is either **wandering**, i.e. $\lambda^k V_j \cap V_j = \emptyset$ for all $k \neq 0$, or **periodic**, i.e. there is a smallest integer $K \geq 1$ such that $\lambda^K V_j = V_j$, and we called K the **period** of V_j.

In [7], Lemma 6.8, we showed that when V has some periodic components they all have the same period. Moreover, since for each periodic component V_0 of period $K \geq 1$ multiplication by λ^K is an automorphism of V_0, there is a conformal map ψ_0 of the upper half-plane $\mathbb{H}$ onto V_0 and a number $t_0 \in (0,1)$ such that $\psi_0(t_0 z) = \lambda^K \psi_0(z)$ ([7], Lemma 6.8). We say that ψ_0 is the **model** of V_0, and t_0 is the **step** of V_0. Also we call $\Gamma_0 = \psi_0(\{yi : \ y > 0\})$ the **axis** of V_0. Note that Γ_0 is the only geodesic in the hyperbolic metric of V_0 that is invariant under multiplication by λ^K; in particular, Γ_0 connects 0 to infinity.

Definition 2.1. If ψ is an analytic function on $\mathbb{D}$, we let $h(\psi) = \sup\{0 < p < \infty : \psi \in H^p(\mathbb{D})\} \in [0, \infty]$ be the **Hardy number** of ψ.

Recall that $H^p(\mathbb{D}) \subset H^{p'}(\mathbb{D})$ if $p > p'$. So the Hardy class of ψ is an interval: either $(0, h(\psi))$ or $(0, h(\psi)]$. If Ω is a simply connected set, then given any pair of one-to-one and analytic maps of $\mathbb{D}$ onto Ω, ψ and $\tilde{\psi}$, we have $\psi \in H^p(\mathbb{D})$ if and only if $\tilde{\psi} \in H^p(\mathbb{D})$. So in this case we can write that $\Omega \in H^p(\mathbb{D})$. Likewise, we set $h(\Omega) = \sup\{0 < p < \infty : \ \Omega \in H^p(\mathbb{D})\}$.

In [7], we showed that the Hardy number of the Kœnigs map σ can be computed in terms of the invariant set V of G:

$$h(\sigma) = \min h(V_j) \tag{2.1}$$

ranging over all the components V_j of V. Also, we showed that for every wandering component we have $h(V_j) = \infty$, and that for a periodic component V_0 with period K and step t_0 we have $h(V_0) = \log(t_0)/\log(|\lambda|^K)$. So (2.1) means that either V has no periodic components in which case $h(\sigma) = \infty$, or there is a periodic component V_0 such that $h(\sigma) = h(V_0) = \log(t_0)/\log(|\lambda|^K) < \infty$.

Recall that a composition operator with symbol ϕ is defined by $C_\phi(f) = f \circ \phi$, and is bounded on the Hardy spaces $H^p(\mathbb{D})$, $0 < p < \infty$. In [7], together with some recent results of P. Bourdon and J. Shapiro, we obtained a formula, in terms of $h(\sigma)$, for the essential spectral radius, $r_e(C_\phi)$, of the composition operator C_ϕ acting on $H^2(\mathbb{D})$:

$$r_e(C_\phi) = |\lambda|^{h(\sigma)/2}. \tag{2.2}$$

The case $r_e(C_\phi) = 0$ corresponds to $h(\sigma) = \infty$, and therefore to the case when the invariant set V does not have periodic components. When $r_e(C_\phi) > 0$, then V has periodic components, say of period $K \geq 1$, and there exists a periodic component V_0 with step t_0 such that $r_e(C_\phi) = |\lambda|^{h(V_0)/2} = t_0^K$.

Now let V_0 be a periodic component of the invariant set V of period $K \geq 1$, and let Γ_0 be its axis. Note that σ cannot be constant on an arc of $\partial\mathbb{D}$, because, being one-to-one, $\sigma \in H^p(\mathbb{D})$ for $p \in (0, 1/2)$ (Theorem 3.16 of [3]). Thus, $\sigma^{-1}(\Gamma_0)$ is a curve in $\mathbb{D}$ tending to a unique point $\zeta(V_0)$ on $\partial\mathbb{D}$. Hence, the correspondence $V_0 \mapsto \zeta(V_0)$, from periodic components of V to points of $\partial\mathbb{D}$, is well-defined. Our main theorem will show that this correspondence is injective and that its range consists of all the b.r.f.p. of ϕ_K. We call $W = \sigma^{-1}(V) = \operatorname{int}(\bigcap_{n\geq 0} \phi_n(\mathbb{D}))$ the **invariant set of** ϕ. Note that $W_0 = \sigma^{-1}(V_0)$ is a connected component of W such that $\phi_K(W_0) = W_0 = \phi_K^{-1}(W_0)$, i.e. ϕ_K is an automorphism of W_0. Also, 0 and $\zeta(V_0)$ are in ∂W_0. The curve $\sigma^{-1}(\Gamma_0)$, which connects 0 to $\zeta(V_0)$, is the unique geodesic in the hyperbolic metric of W_0 that is fixed by ϕ_K. We will show that $\sigma^{-1}(\Gamma_0)$ is perpendicular to $\partial\mathbb{D}$ at $\zeta(V_0)$, and that the set W_0 has an **inner tangent** at $\zeta(V_0)$. That is to say, for every $\beta \in (0, \pi/2)$ there is an $\epsilon = \epsilon(\beta) > 0$ such that the truncated cone

$$\{z \in \mathbb{D} : \ |z - \zeta(V_0)| < \epsilon, \ |\arg(\zeta(V_0)) - \arg(\zeta(V_0) - z)| < \beta\}$$

is contained in W_0. In particular, it follows that

$$W_0 = \{z \in \mathbb{D} : \ \phi_{nK}^{-1}(z) \to \zeta(V_0) \text{ non-tangentially as } n \to \infty\}.$$

So, W_0 can be thought of as a “basin of attraction” for ϕ_K^{-1} at $\zeta(V_0)$.

Finally, by a theorem of Ostrowski (see Theorem 11.5 of [8] or Theorem 3 of [5]), W_0 has an inner tangent at $\zeta(V_0)$ if and only if for any Riemann map F of $\mathbb{D}$ onto W_0 which fixes $\zeta(V_0)$, the argument of the derivative of F converges to 0 non-tangentially at $\zeta(V_0)$. In the last section of this paper, we give a negative answer to the following natural question: does W_0 have an angular derivative at $\zeta(V_0)$? In our example, the set W_0 is actually equal to $\operatorname{int}(\bigcap_{n\geq 0}\phi_n(\mathbb{D}))$ and has an inner tangent at 1. Every iterate ϕ_n has a boundary fixed point at 1 and has a finite angular derivative at 1. So, the sets $\phi_n(\mathbb{D})$ all have an angular derivative at 1 with respect to $\partial\mathbb{D}$. But, W_0 does not, i.e. any Riemann map F of $\mathbb{D}$ onto W_0 that fixes 1 does not have an angular derivative at 1.

3. Statement of the main result

We establish a one-to-one correspondence between the periodic components of V, and the b.r.f.p. of the iterate ϕ_K, where $K \geq 1$ is the common period of the periodic components. Also, we obtain a formula connecting the multiplier at a given b.r.f.p. and the Hardy number of the corresponding periodic component. This yields the following dichotomy. Either no iterate of ϕ has a finite angular derivative at a boundary fixed point—and this happens if and only if the invariant set V has no periodic components, hence if and only if $h(\sigma) = \infty$—or else the iterates of ϕ_K, where K is the common period of the periodic components of V, are the only iterates of ϕ that do have b.r.f.p. Moreover, in the latter case, in view of the result of [7] mentioned above, we obtain a formula for $h(\sigma)$ in terms of the smallest angular derivative of ϕ_K at its boundary fixed points.

Theorem 3.1. *Let $\phi:\ \mathbb{D} \to \phi(\mathbb{D}) \subset \mathbb{D}$ be analytic and one-to-one, with $\phi(0) = 0$, and ϕ not a rotation. Let σ be the Kœnigs function of ϕ, $G = \sigma(\mathbb{D})$, $\lambda = \phi'(0)$, and $V = \operatorname{int}(\bigcap_{n\geq 0}\lambda^n G)$. Assume that V has at least one periodic component. Let K denote the common period of the periodic components of V. For every periodic component V_j of V, let ψ_j, t_j and Γ_j denote respectively the model, the step and the axis of V_j. Finally, let $\zeta(V_j) = \overline{\sigma^{-1}(\Gamma_j)} \cap \partial\mathbb{D}$. Then:*

- (i) *$\zeta(V_0) = \zeta(V_1)$ implies $V_0 = V_1$.*
- (ii) *A point $\zeta \in \partial\mathbb{D}$ is a boundary fixed point of ϕ_n, $n \geq 1$, and ϕ_n has a finite angular derivative at ζ if and only if $\zeta = \zeta(V_0)$ for some periodic component V_0 of V and $n = mK$ for some $m = 1, 2, \ldots$.*
- (iii) *If $A > 1$ is the angular derivative of ϕ_K at $\zeta(V_0)$, and t_0 is the step of V_0, then $A = 1/t_0$.*
- (iv) *The curve $\sigma^{-1}(\Gamma_0)$ is perpendicular to $\partial\mathbb{D}$ at $\zeta(V_0)$, and $W_0 = \sigma^{-1}(V_0)$ has an inner tangent at $\zeta(V_0)$.*

The next corollary follows from Theorem 3.1 (ii).

Corollary 3.2. *Let $\phi:\ \mathbb{D} \to \phi(\mathbb{D}) \subset \mathbb{D}$ be analytic and one-to-one, with $\phi(0) = 0$, and ϕ not a rotation. Then, one of the following holds:*

- (I) *No iterate ϕ_n, $n = 1, 2\ldots$, has a finite angular derivative at a boundary fixed point.*
- (II) *There is an integer $K \geq 1$ such that ϕ_n has a finite angular derivative at a boundary fixed point if and only if $n = mK$ for some $m = 1, 2, \ldots$.*

Definition 3.3. We say that ϕ is of type (I) when ϕ is as in Corollary 3.2 (I), and of type (II) otherwise. If ϕ is of type (II), we set

$$A(\phi) = \min\{\phi_K'(\zeta) : \ \zeta \in \partial\mathbb{D}, \ \phi_K(\zeta) = \zeta\} > 1 \tag{3.1}$$

where K is given by Corollary 3.2 (II).

Note that the minimum in (3.1) is attained because it is attained in (2.1) and because of the one-to-one correspondence of Theorem 3.1 between periodic components and b.r.f.p. (see also Proposition 2.46 of [2]).

Corollary 3.4. *Let $\phi : \ \mathbb{D} \to \phi(\mathbb{D}) \subset \mathbb{D}$ be analytic and one-to-one, with $\phi(0) = 0$, and ϕ not a rotation. Let $\lambda = \phi'(0)$ and let σ be the Kœnigs map of ϕ. Put $h(\sigma) = \sup\{p > 0 : \ \sigma \in H^p(\mathbb{D})\}$.*

Then $h(\sigma) = \infty$ if ϕ is of type (I), while if ϕ is of type (II), then

$$h(\sigma) = \frac{\log A(\phi)}{\log(1/|\lambda|^K)}.$$

Proof. If ϕ is of type (I), V has no periodic components, by Theorem 3.1 (ii); hence $h(\sigma) = \infty$ by (2.1). Otherwise, the formula for $h(\sigma)$ follows from Theorem 3.1 (iii) by the remarks right after (2.1). □

Remark 3.5. Since σ is univalent, $h(\sigma) \geq 1/2$ (see Theorem 3.16 of [3]). So Corollary 3.4 implies that $A(\phi) \geq |\lambda|^{-K/2} > 1$.

Finally, C. Cowen and B. MacCluer (see [1] or Theorem 7.30 on p. 289 of [2]) showed that, under the same assumptions on ϕ as in Theorem 3.1, the spectrum of C_ϕ acting on $H^2(\mathbb{D})$ has the following form:

$$\{z \in \mathbb{C} : \ |z| \leq r_e(C_\phi)\} \cup \{\phi'(0)^n : \ n = 1, 2, \dots\} \cup \{1\}.$$

Our next result allows one to determine the shape of the spectrum of C_ϕ in terms of the derivative of the symbol ϕ at the origin and the possible angular derivatives at boundary fixed points of ϕ or some iterate of ϕ.

Corollary 3.6. *Let $\phi : \ \mathbb{D} \to \phi(\mathbb{D}) \subset \mathbb{D}$ be analytic and one-to-one, with $\phi(0) = 0$, and ϕ not a rotation. Let $C_\phi(f) = f \circ \phi$ be the composition operator with symbol ϕ acting on $H^2(\mathbb{D})$, and let $r_e(C_\phi)$ be the essential spectral radius of C_ϕ.*

Then $r_e(C_\phi) = 0$ if ϕ is of type (I), while if ϕ is of type (II), then

$$r_e(C_\phi) = \left(\frac{1}{A(\phi)}\right)^{\frac{1}{2K}}.$$

Proof. Use (2.2) and Corollary 3.4. □

4. Proof of the main theorem

We start by showing the injectivity, part (i) of Theorem 3.1. Consider two periodic components V_0, V_1 with $\zeta(V_0) = \zeta(V_1) = \zeta$, and assume that V_0 and V_1 are distinct. For $j = 0, 1$, let $W_j = \sigma^{-1}(V_j)$, let Γ_j be the axis of V_j and let $\gamma_j = \sigma^{-1}(\Gamma_j)$. As we said above, γ_0 and γ_1 are curves in $\mathbb{D}$ connecting 0 to ζ. So they bound a simply connected region $\Omega \subset \mathbb{D}$ such that $\partial\Omega = \{0\} \cup \gamma_0 \cup \gamma_1 \bigcup\{\zeta\}$, and in particular $\overline{\Omega} \cap \partial\mathbb{D} = \{\zeta\}$. Recall that V_0 and V_1 have the same period $K \geq 1$. Hence, $\phi_K(\gamma_j) = \gamma_j$, for $j = 0, 1$. Also, $\phi_K(\Omega)$ is a simply connected region in $\mathbb{D} \setminus (\gamma_0 \cup \gamma_1)$ whose boundary coincide with $\partial\Omega$. Therefore, $\phi_K(\Omega) = \Omega$. So Ω is contained in the invariant set of ϕ, i.e. $\Omega \subset W = \mathrm{int}(\bigcap_{n\geq 0} \phi_n(\mathbb{D}))$. On the other hand, $\Omega \cap W_j \neq \emptyset$ for $j = 0, 1$, which is a contradiction since W_0 and W_1 are distinct connected components of W. Thus, (i) is proved.

Now we prove part (ii) of Theorem 3.1. First, assume that V_0 is a periodic component. Note that $\phi_{mK}(z)$ tends to $\zeta(V_0)$ as z tends to $\zeta(V_0)$ along $\sigma^{-1}(\Gamma_0)$. So, by Lindelöf, ϕ_{mK} has non-tangential limit $\zeta(V_0)$ at $\zeta(V_0)$. Moreover, for $w \in \sigma^{-1}(\Gamma_0)$, if $\rho_{\mathbb{D}}$ and $\rho_{\mathbb{H}}$ denote respectively the hyperbolic distance on $\mathbb{D}$ and on $\mathbb{H}$,

$$\rho_{\mathbb{D}}(w, \phi_{mK}(w)) \leq \rho_{\mathbb{H}}(\psi_0^{-1} \circ \sigma(w), t_0^m(\psi_0^{-1} \circ \sigma(w))) = \log(1/t_0^m) < \infty$$

So, by the Julia-Carathéodory theorem, ϕ_{mK} has a finite angular derivative at $\zeta(V_0)$.

For the converse we introduce the notion of a twisted sector, which first appeared in [10]. Our definition of twisted sector is phrased differently, but is equivalent, to the one in [10]. For $z \in \mathbb{C}$ and $r > 0$, $B(z, r) = \{w \in \mathbb{C} : |z - w| < r\}$. If $E \subset \mathbb{C} \setminus \{0\}$, and $\epsilon > 0$, we set

$$S_\epsilon[E] = \bigcup_{z \in E} B(z, \epsilon|z|).$$

Definition 4.1. Let γ be a closed and connected subset of $\mathbb{C} \setminus \{0\}$ such that 0 and ∞ belong to the closure of γ in the Riemann sphere. Let $\epsilon > 0$. We say that $S_\epsilon[\gamma]$ is a **twisted sector of width ϵ supported on γ**.

Remark 4.2. Note that if $S_\epsilon[\gamma]$ is a twisted sector of width $\epsilon > 0$ supported on γ, then for every $c \in \mathbb{C} \setminus \{0\}$, $cS_\epsilon[\gamma]$ is also a twisted sector of width ϵ supported on $c\gamma$, since $cS_\epsilon[\gamma] = S_\epsilon[c\gamma]$.

Assume that $\zeta_0 \in \partial\mathbb{D}$ is a b.r.f.p. of ϕ_n, $n \geq 1$. We need to show that the set $V = \text{int}(\bigcap_{j \geq 0} \lambda^j G)$ does have periodic components, say of period $K \geq 1$, that there is one periodic component V_0 such that $\zeta(V_0) = \zeta_0$, and that $n = mK$ for some integer $m \geq 1$. Suppose that we can find a component V_0 of the set $V^{(n)} = \text{int}(\bigcap_{j \geq 0} \lambda^{jn} G)$ (the invariant set for ϕ_n) such that V_0 is of period one with respect to multiplication by λ^n, i.e. $\lambda^n V_0 = V_0$, and such that $\zeta(V_0) = \zeta_0$. Then, since $V = V^{(n)}$, V_0 must be a periodic component of V, and therefore we must have $n = mK$, where K is the period of the periodic components of V. So we only need to show that if $\zeta \in \partial\mathbb{D}$ is a b.r.f.p. for a map ϕ as in Theorem 3.1, then the corresponding invariant set V has a periodic component V_0 of period 1 such that $\zeta(V_0) = \zeta$.

Let $\zeta \in \partial\mathbb{D}$ be fixed by ϕ and assume that ϕ has a finite angular derivative $A > 1$ at ζ. For convenience, let's move to the upper-half plane $\mathbb{H}$ via the map $\tau(z) = \zeta(z - i)/(z + i)$; hence let $\Phi = \tau^{-1} \circ \phi \circ \tau$ and $\Sigma = \sigma \circ \tau$. Then, infinity is a boundary fixed point for Φ and Φ has a finite angular derivative there, i.e. $z/\Phi(z)$ converges to a number $A > 1$ as z tends to infinity non-tangentially. Let $\gamma = \Sigma(\{yi : y \geq 1\})$. Then, in the proof of Proposition 3.3 of [10] it is shown that γ supports a twisted sector of width ϵ that is contained in G, for some $\epsilon > 0$. For all $n \in \mathbb{N}$, let $\gamma_n = \lambda^n \gamma$. Recall that, by Schwarz's lemma, $0 < |\lambda| = |\phi'(0)| < 1$. By Remark 4.2

$$S_\epsilon[\gamma_n] = \lambda^n S_\epsilon[\gamma] \subset \lambda^n G. \tag{4.1}$$

Since γ_n connects 0 to ∞, for every $n \in \mathbb{N}$ we can find $\zeta_n \in \gamma_n \cap \partial\mathbb{D}$. Also, γ_n is parameterized by $\Sigma \circ \Phi_n(yi)$, for $y \geq 1$, so we can assume that $\zeta_n = \Sigma \circ \Phi_n(y_n i)$ is the last point of γ_n belonging to $\overline{\mathbb{D}}$. Set $\tilde{\gamma}_n = \Sigma \circ \Phi_n(\{yi : y \geq y_n\})$.

Claim 4.3. *There is a connected component V_0 of the invariant set V of G, such that for every integer $N_0 < \infty$ there is $N > N_0$ with $S_{\epsilon/2}[\tilde{\gamma}_N] \setminus \mathbb{D} \subset V_0$.*

Proof. By (4.1), $B(\zeta_n, \epsilon) \subset \lambda^n G$. Let ζ_{n_k} be a subsequence converging to $\zeta \in \partial\mathbb{D}$; then for some $k_0 \in \mathbb{N}$, and for every $k \geq k_0$, we have

$$B(\zeta, \epsilon/2) \subset B(\zeta_{n_k}, \epsilon) \subset \lambda^{n_k} G.$$

Hence $B(\zeta, \epsilon/2) \subset \bigcap_{n \geq 0} \lambda^n G$, i.e. $B(\zeta, \epsilon/2) \subset V$. Let V_0 be the connected component of V containing $B(\zeta, \epsilon/2)$. We can assume that $\tilde{\gamma}_{n_k} \bigcap V_0 \neq \emptyset$, for all $k \in \mathbb{N}$. Now, suppose that for infinitely many k we can find $z_k \in (S_{\epsilon/2}[\tilde{\gamma}_{n_k}] \setminus \mathbb{D}) \cap \partial V_0$. Since $z_k \in S_{\epsilon/2}[\tilde{\gamma}_{n_k}]$, there is $z \in \tilde{\gamma}_{n_k}$ such that $|z_k - z| < (\epsilon/2)|z|$; in particular $|z_k| < (1 + \epsilon/2)|z|$, so by (4.1)

$$B(z_k, c|z_k|) \subset B(z, \epsilon|z|) \subset \lambda^{n_k} G$$

where $c = (\epsilon/2)/(1+\epsilon/2)$. Now, multiply each z_k by λ^{m_k}, where the power $m_k \in \mathbb{N}$ is chosen so that for all $k \in \mathbb{N}$

$$\tilde{z}_k = \lambda^{m_k} z_k \in Q = \{z \in \mathbb{C} : \; 1 \leq |z| \leq |\lambda|^{-1}\}.$$

Notice that $m_k \geq 0$, because $z_k \notin \mathbb{D}$. Then, for infinitely many k we have:

$$(a) \; B(\tilde{z}_k, c|\tilde{z}_k|) \subset \lambda^{n_k} G \qquad \text{and} \qquad (b) \; \tilde{z}_k \in \partial(\lambda^{m_k} V_0).$$

Again assume, by passing to a subsequence, that $\tilde{z}_k$ converges to a point $\xi \in Q$. By the same argument as above, using (a), $B(\xi, c|\xi|)$ is contained in the invariant set V. Let V_1 be the connected component of V containing $B(\xi, c|\xi|)$. Choose $\tilde{z}_k \in B(\xi, c|\xi|)$. Then, by (b), $V_1 \cap \lambda^{m_k} V_0 \neq \emptyset$. Hence $V_1 = \lambda^{m_k} V_0$ and $\tilde{z}_k \in \lambda^{m_k} V_0$, which contradicts (b). Therefore, there is $k_0 \in \mathbb{N}$ such that, for $k \geq k_0$,

$$(S_{\epsilon/2}[\tilde{\gamma}_{n_k}] \setminus \mathbb{D}) \subset V_0.$$

□

From the proof of Claim 4.3, we could already deduce that V_0 must be a periodic component of V. But, we also want to show that its period must be 1. That's the content of our next claim

Claim 4.4. *The component V_0 is periodic of period 1, i.e. $\lambda V_0 = V_0 = \lambda^{-1} V_0$.*

Proof. Let $\tilde{\gamma} = \tilde{\gamma}_N$ be given as in Claim 4.3.

Definition 4.5. Let Ω be a simply connected region with hyperbolic metric ρ_Ω, and let $F \subset \Omega$. We say that U is a **hyperbolic neighborhood** of F in Ω, if

$$U = \bigcup_{z \in F} \{w \in \Omega : \; \rho_\Omega(w, z) < C\}$$

for some constant $0 < C < \infty$.

Then, $S_\epsilon[\tilde{\gamma}]$ contains a hyperbolic neighborhood of $\tilde{\gamma}$ in G. In fact, if $w \in G$ satisfies $\rho_G(w, z) < C$, for some $z \in \tilde{\gamma}$, then

$$\frac{1}{2} \log\left(1 + \frac{|w - z|}{d(z, \partial G)}\right) < C.$$

See [10], Lemma 3.2. But, since $G \neq \mathbb{C}$, there is a constant $C_1 > 0$ such that $d(z, \partial G) \leq C_1|z|$. Hence, if C is chosen small enough,

$$|w - z| \leq C_1(e^{2C} - 1)|z| \leq (\epsilon/2)|z|.$$

On the other hand, notice that $\Sigma^{-1}(\tilde{\gamma}) = \Phi_N(\{yi : \ y \geq y_N\})$. The fact that Φ has an angular derivative at infinity implies that Φ_N also has an angular derivative at infinity. So $\Sigma^{-1}(\tilde{\gamma})$ is asymptotic in the Riemann sphere to the imaginary axis at infinity, i.e. given any hyperbolic neighborhood of $\Sigma^{-1}(\tilde{\gamma})$ in $\mathbb{H}$, the upper half imaginary axis is eventually contained in it. So, for $R > 0$ large enough, $\Sigma(\{yi : \ y \geq R\})$ is contained in $S_{\epsilon/2}[\tilde{\gamma}]$. By the same argument, we also have $R' > R > 0$, such that $\Sigma \circ \Phi(\{yi : \ y \geq R'\})$ is contained in $S_{\epsilon/2}[\tilde{\gamma}]$. That is to say, $\Sigma(\{yi : \ y \geq R'\})$ and $\lambda\Sigma(\{yi : \ y \geq R'\})$ are both contained in $S_{\epsilon/2}[\tilde{\gamma}]$. But, since $S_{\epsilon/2}[\tilde{\gamma}] \setminus \mathbb{D} \subset V_0$, we must have $\lambda V_0 = V_0$. □

Let Γ_0 be the axis of V_0. The preimage $\Sigma^{-1}(\Gamma_0)$ is a curve in $\mathbb{H}$ starting at i and converging to a unique point of $\partial\mathbb{H}$ in the Riemann sphere. The next claim shows that such a point must be infinity, and hence proves the converse, completing the proof of part (*ii*) of Theorem 3.1.

Claim 4.6. *The curve* $\Sigma^{-1}(\Gamma_0)$ *is contained in a hyperbolic neighborhood of* $\{yi : y \geq 1\}$ *in* $\mathbb{H}$.

Proof. Fix $0 < \delta < \pi/2$, and consider

$$U_\delta = \{z \in \mathbb{H} : \ |arg(z) - \pi/2| < \pi/2 - \delta\}.$$

Let $K_\delta = \mathbb{H} \setminus U_\delta$. We claim that we can choose δ small enough so that for every $z \in \psi_0(K_\delta)$, $d(z, \partial V_0) < (\epsilon/4)|z|$. Suppose not. Then we can pick a sequence $z_n \in \psi_0(K_{\delta_n})$, with δ_n tending to zero as n tends to infinity, such that for all $n \in \mathbb{N}$

$$d(z_n, \partial V_0) \geq \frac{\epsilon}{4}|z_n|. \tag{4.2}$$

Let $\tilde{z}_n = \lambda^{m_n} z_n$, with the appropriate power m_n so that

$$1 \leq |\tilde{z}_n| \leq |\lambda|^{-1}$$

Then (4.2) still holds with z_n replaced by $\tilde{z}_n$. Now, let ξ be the limit of a subsequence $\tilde{z}_{n_j}$. For some $j_0 \in \mathbb{N}$, and for all $j \geq j_0$

$$B(\xi, (\epsilon/8)|\xi|) \subset B(\tilde{z}_{n_j}, (\epsilon/4)|\tilde{z}_{n_j}|) \subset V_0.$$

Let Δ be a small open neighborhood of $\psi_0^{-1}(\xi)$ in $\mathbb{H}$. Since δ_n tends to zero, eventually

$$\psi_0^{-1}(\tilde{z}_{n_j}) = t_0^{m_{n_j}} \psi_0^{-1}(z_{n_j}) \notin \Delta,$$

which is a contradiction, because eventually $\tilde{z}_{n_j} \in \psi_0(\Delta)$. Therefore, by Claim 4.3, if $\tilde{\gamma} = \tilde{\gamma}_N$, we can choose $\delta_0 > 0$ such that $\tilde{\gamma} \subset \psi_0(U_{\delta_0})$, i.e. $\psi_0^{-1}(\tilde{\gamma}) \subset U_{\delta_0}$. Notice that $\psi_0^{-1}(\tilde{\gamma})$ is a curve which tends to infinity. Thus, $\{yi : \ y \geq R\}$ is contained in a hyperbolic neighborhood of $\psi_0^{-1}(\tilde{\gamma})$ in $\mathbb{H}$ for some $R > 0$. That is to say, applying the map $\Sigma^{-1} \circ \psi_0$, $\Sigma^{-1}(\Gamma_0)$ is contained in a hyperbolic neighborhood of $\Sigma^{-1}(\tilde{\gamma}) = \Phi_N(\{yi : \ y \geq y_N\})$ in $\Sigma^{-1}(V_0)$. But, since the hyperbolic metric of $\mathbb{H}$ is bounded above by the hyperbolic metric of $\Sigma^{-1}(V_0)$, and since $\Phi_N(\{yi : \ y \geq y_N\})$ is asymptotic in the Riemann sphere to the imaginary axis at infinity, we have that $\Sigma^{-1}(\Gamma_0)$ is contained in a hyperbolic neighborhood of $\{yi : \ y \geq R'\}$, for some large R'. The claim follows from this. □

Now, we want to show part (*iii*) of Theorem 3.1. For all $n \in \mathbb{N}$, let $z_n = \psi_0(t_0^{-n} i) \in \Gamma_0$, and $w_n = \Sigma^{-1}(z_n)$. We need the following geometric property of $\Sigma^{-1}(\Gamma_0)$. For all $n \in \mathbb{N}$, consider the curve $\tilde{\Gamma}_n = \Sigma^{-1} \circ \psi_0(\{yi : \ y \geq t_0^{-n}\})$, which is the part of the curve $\Sigma^{-1}(\Gamma_0)$ which starts at w_n and, by Claim 4.6, tends to infinity in a hyperbolic neighborhood $U_\theta = \{z \in \mathbb{H} : \ |arg(z) - \pi/2| < \theta\}$, $\theta \in (0, \pi/2)$, of the imaginary axis. Let $\tilde{w}_n$ be a point of $\tilde{\Gamma}_n$ such that

$$\operatorname{Im}(\tilde{w}_n) = \min\{\operatorname{Im}(w) : \ w \in \tilde{\Gamma}_n\}.$$

Claim 4.7. *There is a constant $C_0 > 0$ and some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

$$\operatorname{Im}(\tilde{w}_n) \geq C_0 \operatorname{Im}(w_n)$$

Proof. Since Φ has a finite angular derivative $A > 1$ at infinity, there is a constant $R > 0$ such that, for all $z \in U_\theta \setminus \{|z| \leq R\}$,

$$\operatorname{Im}(z) \geq \operatorname{Im}(\phi(z)). \tag{4.3}$$

Choose n_0 so that $|w_n| > R$ for all $n \geq n_0$. For $m \in \mathbb{N}$, $m \geq 1$, let v_m be a point such that

$$\operatorname{Im}(v_m) = \min\{\operatorname{Im}(w) : \ \psi_0^{-1} \circ \Sigma(w) \in \{yi : \ t_0^{-(n+m)} \leq y \leq t_0^{-(n+m+1)}\}\}.$$

Then

$$\rho_{\mathbb{H}}(v_m, w_{n+m+1}) \leq \rho_{\mathbb{H}}(\psi_0^{-1} \circ \Sigma(v_m), t_0^{-(n+m+1)} i) \leq \log(1/t_0).$$

On the other hand,

$$\rho_{\mathbb{H}}(v_m, w_{n+m+1}) \geq \frac{1}{2} \log\left(1 + \frac{|w_{n+m+1} - v_m|}{\operatorname{Im}(v_m)}\right).$$

So, for some constant $C_0 > 0$ depending only on t_0, and by (4.3), $\operatorname{Im}(v_m) \geq C_0 \operatorname{Im}(w_{n+m+1}) \geq C_0 \operatorname{Im}(w_n)$. The claim follows because m is arbitrary. $\square$

Now, consider the following sequence of maps:

$$\Psi_n(w) = \lambda^n \Sigma(\operatorname{Re}(w_n) + w \operatorname{Im}(w_n)).$$

For all $n \in \mathbb{N}$, Ψ_n maps $\mathbb{H}$ conformally onto $\lambda^n G$, and $\Psi_n(i) = \lambda^n z_n = z_0$. In particular, $\{\Psi_n\}$ is a normal family. Let Ψ be the normal limit of a subsequence Ψ_{n_k}. Then, since $\Psi_{n_k}(\mathbb{H}) = \lambda^{n_k} G$, $\Psi(\mathbb{H}) \subset V$, and since $\Psi_{n_k}(i) = z_0$, $\Psi(\mathbb{H}) \subset V_0$. On the other hand, $V_0 \subset \Psi_{n_k}(\mathbb{H})$, for all $k \in \mathbb{N}$. So $\Psi(\mathbb{H}) = V_0$. Moreover, by the Hurwitz theorem, Ψ is one-to-one. Thus $\Psi^{-1} \circ \psi_0$ is an automorphism of $\mathbb{H}$ that fixes i.

Claim 4.8. *$\Psi^{-1} \circ \psi_0$ is the identity map on $\mathbb{H}$.*

Proof. Note that $\Psi_{n_k}^{-1}$ converges uniformly on compact subsets of V_0 to Ψ^{-1}. So, fix $m \geq 1$, and consider the compact set $K_m = \psi_0(\{yi : \ 1 \leq y \leq t_0^{-m}\})$ in V_0. Then

$$\Psi_{n_k}^{-1}(K_m) = \frac{\Sigma^{-1}(\lambda^{-n_k} K_m) - \operatorname{Re}(w_{n_k})}{\operatorname{Im}(w_{n_k})}. \tag{4.4}$$

Since $\Sigma^{-1}(\lambda^{-n_k} K_m) \subset \Sigma^{-1}(\Gamma_0)$, by Claim 4.6 there is an absolute constant $0 < \theta < \pi/2$ such that $\Sigma^{-1}(\lambda^{-n_k} K_m)$ is contained in a sector U_θ. In particular, there is a constant $C_1 > 0$ such that $|\operatorname{Re}(w_{n_k})|/\operatorname{Im}(w_{n_k}) \leq C_1$, for all k. Therefore, by (4.4), $\Psi_{n_k}^{-1}(K_m)$ is contained in $\bigcup_{-C_1 < x < C_1} (x + U_\theta)$. Moreover, by Claim 4.7,

$$\min\{\operatorname{Im}(w) : \ w \in \Psi_{n_k}^{-1}(K_m)\} \geq C_0 > 0.$$

So, we can find a hyperbolic neighborhood U of $\{yi : \ y \geq 1\}$ in $\mathbb{H}$ such that $\Psi_{n_k}^{-1}(K_m) \subset U$ for all $k \in \mathbb{N}$ and all $m \geq 1$. Letting k tend to infinity, $\Psi_{n_k}^{-1}$ converges uniformly to $\Psi^{-1} \circ \psi_0$ on $\{yi : 1 \leq y \leq t_0^{-m}\}$. But, $\Psi^{-1} \circ \psi_0(\{yi : 1 \leq y \leq t_0^{-m}\})$ is an arc of a geodesic in $\mathbb{H}$, since $\Psi^{-1} \circ \psi_0$ is an automorphism. Letting m become large, we see that this geodesic must tend to $\partial\mathbb{H}$ while staying in U, and therefore it can only be $\{yi : y \geq 1\}$. Hence $\Psi^{-1} \circ \psi_0$ fixes the upper half of the imaginary axis and fixes i. So, it must be the identity. □

We conclude that the only possible limit for any subsequence of $\{\Psi_n\}$ is ψ_0. Therefore, the whole sequence $\{\Psi_n\}$ converges to ψ_0.

We can now compute the angular derivative of Φ at infinity in terms of V_0. Let $\pi_n(w) = \mathrm{Re}(w_n) + w\,\mathrm{Im}(w_n)$. Then, $\Psi_n = \lambda^n \Sigma \circ \pi_n$. So,

$$\pi_n \circ (\Psi_n^{-1} \circ \lambda \Psi_n) = \Phi \circ \pi_n.$$

Therefore

$$\Phi' \circ \pi_n = (\Psi_n^{-1} \circ \lambda \Psi_n)' \to (\psi_0^{-1} \circ \lambda \psi_0)' = t_0$$

as n tends to infinity. In particular, $\Phi' \circ \pi_n(i) = \Phi'(w_n)$ tends to t_0 as n tends to infinity. So, by Claim 4.6, the angular derivative of Φ at infinity is $1/t_0$, completing the proof of (iii).

Finally, we show part (iv) of Theorem 3.1. Note that $\sigma^{-1}(\Gamma_0)$ is perpendicular to $\partial\mathbb{D}$ at $\zeta(V_0)$ if and only if $\sigma^{-1}(\Gamma_0)$ is eventually contained in any hyperbolic neighborhood of the ray $[0, \zeta(V_0)]$ in $\mathbb{D}$. So, consider $I_1 = \{yi : 1 \leq y \leq t_0^{-1}\}$. Then, $\Psi_n^{-1} \circ \psi_0$ converges to the identity map on I_1. So, $\Psi_n^{-1} \circ \psi_0(I_1)$ is eventually contained in any hyperbolic neighborhood of I_1. Notice that

$$\Sigma^{-1}(\Gamma_0) = \bigcup_{k=-\infty}^{+\infty} \pi_k \circ \Psi_k^{-1} \circ \psi_0(I_1).$$

Therefore, if $P = \bigcup_{k=-\infty}^{+\infty} \pi_k(I_1)$, then $\Sigma^{-1}(\Gamma_0)$ is eventually contained in any hyperbolic neighborhood of P. Note that $\pi_k(I_1) = [w_n, w_n + i(t_0^{-1} - 1)\,\mathrm{Im}(w_n)]$. So, to show that $\Sigma^{-1}(\Gamma_0)$ is eventually contained in any hyperbolic neighborhood of the imaginary axis, it is enough to show this for the sequence $\{w_n\}_{n=0}^{\infty}$. Notice that

$$\frac{\mathrm{Re}(w_{n+1}) - \mathrm{Re}(w_n)}{\mathrm{Im}(w_n)} + i\frac{\mathrm{Im}(w_{n+1})}{\mathrm{Im}(w_n)} = \Psi_n^{-1} \circ \psi_0(t_0^{-1} i) \longrightarrow t_0^{-1} i \tag{4.5}$$

as n tends to infinity. So,

$$\frac{|\,\mathrm{Re}(w_{n+1}) - \mathrm{Re}(w_n)|}{\mathrm{Im}(w_{n+1}) - \mathrm{Im}(w_n)} = \frac{|\,\mathrm{Re}(w_{n+1}) - \mathrm{Re}(w_n)|}{\mathrm{Im}(w_n)} \frac{1}{\frac{\mathrm{Im}(w_{n+1})}{\mathrm{Im}(w_n)} - 1} \longrightarrow 0$$

as n tends to infinity, by (4.5).

To show that $\sigma^{-1} \circ \psi_0(\mathbb{H})$ has an inner tangent at $\zeta(V_0)$ we proceed similarly, by considering the fact that $\Psi_n^{-1} \circ \psi_0$ converges to the identity map on every fixed hyperbolic neighborhood of I_1. So Theorem 3.1 is proved.

5. Remark on the degree of contact of periodic components

Does the map $\Sigma^{-1} \circ \psi_0$ have an angular derivative at infinity? By Claim 4.8,

$$\pi_n^{-1} \circ \Sigma^{-1} \circ \psi_0 \circ t^{-n} = \pi_n^{-1} \circ \Sigma^{-1} \circ \lambda^{-n} \circ \psi_0 = \Psi_n^{-1} \circ \psi_0 \to \mathrm{Id}_{\mathbb{H}}$$

as n tends to infinity. Taking the derivatives, we get

$$\frac{t_0^{-n}}{\operatorname{Im}(w_n)}(\Sigma^{-1}\circ\psi_0)'\circ t_0^{-n}\to 1$$

uniformly on compact subsets of $\mathbb{H}$. By taking the arguments, one sees that $\Sigma^{-1}\circ\psi_0$ is semi-conformal at infinity; see [5], definition 2. To prove that $\Sigma^{-1}\circ\psi_0$ has an angular derivative at infinity, it would be enough to show that $t_0^n\operatorname{Im}(w_n)$ converges as n tends to infinity. Notice that $\Sigma^{-1}\circ\Psi_n=\Phi_n\circ\pi_n$. Thus, $(\Phi_n\circ\pi_n)'$ converges to $(\Sigma^{-1}\circ\psi_0)'$ uniformly on compact subsets of $\mathbb{H}$. Evaluating at i and multiplying above and below by t_0^n, we obtain

$$t_0^n\operatorname{Im}(w_n)\prod_{j=1}^{n}\frac{\Phi'(w_j)}{t_0}\longrightarrow(\Sigma^{-1}\circ\psi_0)'(i).$$

So $t_0^n\operatorname{Im}(w_n)$ converges if and only if the infinite product converges, i.e. if and only if $\sum_{j=1}^{\infty}(t_0-\Phi'(w_j))$ converges.

However, consider the following example. Let $\mathbb{S}=\{x+iy:\ |y|<\pi/2\}$ be the "standard strip". Let

$$\mathcal{G}=\{x+iy:\ x<1,\ |y|<\pi\}\cup\{x+iy:\ x>1,\ |y|<\frac{\pi}{2}+\frac{1}{x}\}.$$

Then, $\mathcal{G}-1\subset\mathcal{G}$. By exponentiating, $\mathcal{G}$ is transformed into a region $G\subset\mathbb{C}\setminus(-\infty,0]$ such that $e^{-1}G\subset G$ and $\mathbb{D}\setminus(-1,0]\subset G$. Hence, by adding the slit $(-1,0]$ to such a region G, we obtain a geometric model. Iterating translation by -1 on $\mathcal{G}$, we are left with $\mathbb{S}$. Therefore, the invariant set in this case has only one component, which is periodic of period 1. Let Σ denote the Kœnigs map from $\mathbb{H}$ to $G\cup\mathbb{D}$. Consider $\tilde{\Sigma}(z)=\Sigma(ie^z)$ defined on $\mathbb{S}$. There is a curve $\gamma\subset\mathbb{S}$ starting at 0, with $\sup\{\operatorname{Re}(z):\ z\in\gamma\}<+\infty$, such that $\tilde{\Sigma}$ maps γ onto $(-1,0]$. So, $F=\log\tilde{\Sigma}$ maps $\mathbb{S}\setminus\gamma$ conformally onto $\mathcal{G}$. Moreover, F does not have an angular derivative at $+\infty$; see Definition 7. of [5], because $\operatorname{Area}(\{z\in\mathcal{G}\setminus\mathbb{S}:\ \operatorname{Re}(z)>s\})=\infty$, for all $s>0$. This can be shown using Ahlfors's distortion theorem; see equation (15) of [5]. In particular, F^{-1} restricted to $\mathbb{S}$ also doesn't have an angular derivative at $+\infty$.

Now, the map ψ_0 of $\mathbb{H}$ onto the periodic component $\{z\in\mathbb{C}:\ \operatorname{Re}(z)>0\}$ can be chosen to be $\psi_0(z)=-iz$. Thus, the map

$$\Sigma^{-1}\circ\psi_0=i\exp((F^{-1})_{|\mathbb{S}}(\log(\psi_0)))$$

does not have an angular derivative at infinity.

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