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THE ENTROPY OF RATIONAL POWERS SHIFTS

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Abstract. The Connes-Størmer entropy of all rational Powers shifts is shown to be $\frac{1}{2}\log 2$.

1. Introduction

In this paper we prove some results about the Connes-Størmer entropy of some automorphisms on R, the hyperfinite II_1 factor, which are related to the Powers shift endomorphisms [Po]. We are particularly interested in those Powers shifts which have finite relative commutant index [Po], [P1]. These are the shifts σ which have the property that $\sigma^k(R)'\cap R$ is nontrivial for some positive integer k. Our main result establishes that if σ is a Powers shift with finite relative commutant index, then the automorphism $\tilde{\sigma}$ which is canonically related to σ has Connes-Størmer entropy $h(\tilde{\sigma}) = \frac{1}{2}\log 2$. This result generalizes a result of M. Choda, which may be stated as follows: if σ above has the property that R is the strong operator closure of the algebra $\bigcup_k \sigma^k(R)'\cap R$, then $h(\tilde{\sigma}) = \frac{1}{2}\log 2$ [C, Example 3] (see also Remark 3.1). Although it is possible to use Choda's result in proving our main result, we present a proof which uses a technique of Narnhofer and Thirring [NT, Proposition 6.16] and which is shorter and more elementary than Choda's original proof of the special case. Our approach also allows us to show that, under a certain type of perturbation of binary shifts with finite commutation index, the entropy remains positive (Proposition 4.2).

Our paper is motivated in part by the recent work of Narnhofer, Størmer, and Thirring, [NST], on the entropy of automorphisms related to Powers shifts (see also [GS], [NT]). In [NST] the authors have constructed automorphisms $\tilde{\sigma}$ related to Powers shifts σ with the property that $h(\tilde{\sigma}) = 0$ whereas $h(\tilde{\sigma}\otimes\tilde{\sigma}) = \log 2$. These automorphisms are the first examples constructed for which the tensor product formula for the Connes-Størmer entropy is known to fail. It also follows from their construction that the trace is the only state on R which is $\tilde{\sigma}$-invariant [NST, Theorem 4.1].

It should be noted that 0 and $\frac{1}{2}\log 2$ are the only values which the Connes-Størmer entropy is known to assume for automorphisms associated with Powers

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shifts. It seems to be a very difficult problem to resolve whether there exist automorphisms of this type whose entropy takes a value strictly between 0 and $\frac{1}{2}\log 2$.

2. Powers shifts and their canonical extensions

In this section we review some of the properties about Powers shifts which will be used in the entropy calculations below. For a more comprehensive description of these shifts the reader may wish to consult [Po], [PP1], [PP2], [P1], [P2], [P3].

Let $\{a_0, a_1, \cdots\}$ be a fixed sequence, or bitstream, consisting of 0's and 1's and satisfying the conditions that $a_0 = 0$, and the doubly-infinite sequence $\{\cdots, a_1, a_0, a_1, \cdots\}$ is not periodic. Let $\{u_0, u_1, \cdots\}$ be a sequence of hermitian unitary elements which satisfy the properties

$$\begin{aligned} u_i u_{i+j} &= -u_{i+j} u_i, \quad \text{if } a_j = 1, \\ u_i u_{i+j} &= +u_{i+j} u_i, \quad \text{if } a_j = 0, \end{aligned} \tag{2.1}$$

for any i, j in $\mathbb{Z}^+$. Then the von Neumann algebra generated by the words in the u_j's is isomorphic to the hyperfinite II_1 factor R [Po], [P3]. The Powers shift σ on R is the *-endomorphism which is completely determined by the definition $\sigma(w) = u_{i_0+1}u_{i_1+1}\cdots u_{i_n+1}$ on words $w = u_{i_0}u_{i_1}\cdots u_{i_n}$ in the unitary generators u_j, $j \in \mathbb{Z}^+$. The endomorphism has the shift property $\bigcap \sigma^k(R) = \mathbb{C}I$. We refer to σ as a shift of index 2 in the sense that the Jones index $[R\colon \sigma(R)]$ of the subfactor $\sigma(R)$ is 2.

It is an easy matter to extend a Powers shift σ in a canonical way to an automorphism $\tilde{\sigma}$ on a hyperfinite II_1 factor $\tilde{R}$ containing R [NST]. To do this define a sequence of hermitian unitary operators $\{u_k\colon\ k \in \mathbb{Z}\}$ which satisfy (2.1) for all $i, j \in \mathbb{Z}$. Then the von Neumann algebra $\tilde{R}$ generated by this sequence is isomorphic to the hyperfinite II_1 factor, and the mapping given on generators by $\tilde{\sigma}(u_k) = u_{k+1}$ can be shown to extend uniquely to an automorphism on $\tilde{R}$.

For $n \in \mathbb{N}$, let $\mathcal{A}_n$ be the algebra generated by the words in the generators $u_0, \cdots, u_{n-1}$ of R. Since the commutation relations (2.1) on the generators show that any word in the generators is either a scalar multiple of the identity or a scalar multiple of a word $w = u_{i_0}u_{i_1}\cdots u_{i_m}$ with the subscripts distinct and in ascending order, it is easy to see that $\mathcal{A}_n$ is an algebra of dimension 2^n. In fact, we have the following characterization of $\mathcal{A}_n$.

Theorem 2.1 ([PP2, Theorem 5.4])**.** *For each positive integer n there is a pair of non-negative integers c_n and d_n such that*

(i) $2d_n + c_n = n$,
(ii) *the center of $\mathcal{A}_n$ is of dimension 2^{c_n}, and*
(iii) *the algebra $\mathcal{A}_n$ is isomorphic to the direct sum of 2^{c_n} copies of the matrix algebra $M_{2^{d_n}}(\mathbb{C})$.*

Definition 2.1. The sequence $\{c_n\colon\ n \in \mathbb{N}\}$ is called the center sequence of the Powers shift σ.

Theorem 2.2 ([PP2, Theorem 5.4], [P1, Corollary 2.10])**.** *Given a Powers shift σ, there is a sequence of positive integers $\{m_i\colon\ i \in \mathbb{N}\}$, not necessarily distinct, such that the center sequence consists of the concatenation of the strings $1, 2, \cdots, m_{i-1}$, $m_i, m_{i-1}, \cdots, 1, 0$.*

Combining the two results gives a characterization of the Bratteli diagram of the AF-algebra $\mathcal{A}$ generated by the ascending union of the algebras $\mathcal{A}_n$. In fact, since the preceding theorem implies that $c_n = 0$ for infinitely many even positive integers n, it follows that for infinitely many n, $\mathcal{A}_n$ is isomorphic to the algebra of $2^{n/2}$ by $2^{n/2}$ matrices over $\mathbb{C}$. Hence we have the following.

Theorem 2.3. *$\mathcal{A}$ is the uniformly hyperfinite algebra of type 2^∞.*

The following definition appears in [Po].

Definition 2.2. A Powers shift is said to be rational if its commutator bitstream $\{a_0, a_1, a_2, \dots\}$ is eventually periodic.

For the remainder of this section, as well as the next, we specialize to the case where σ is a rational shift. The following theorem gives a characterization of such shifts.

Theorem 2.4 ([P1, Theorem 5.8])**.** *A Powers shift σ is rational if and only if it has finite relative commutant index. If $k \in \mathbb{N}$ is this index, then there is a non-trivial word $w = u_{i_0} u_{i_1} \cdots u_{i_m}$, $i_0 < i_1 < \cdots < i_m$, $i_0 = 0$, which generates $\sigma^k(R)' \cap R$. For $j \in \mathbb{Z}^+$ the algebra $\mathcal{B}_j = \sigma^{k+j}(R)' \cap R$ is the algebra generated by the words $\{w, \sigma(w), \dots, \sigma^j(w)\}$. The von Neumann alegbra generated by $\bigcup \mathcal{B}_j$ is a subfactor $\mathfrak{B}$ of R of finite Jones index in $R = \{u_i\colon\ i \in \mathbb{Z}^+\}$. The rational shift σ restricts to a rational shift on $\mathfrak{B}$ with corresponding commutator bitstream of the form $\{b_0, \dots, b_{k-1}, 0, 0, \dots\}$.*

Theorem 2.5 ([PP2, Theorem 6.8])**.** *If σ is a rational shift, then its center sequence is eventually periodic. (Hence there is a c such that $c_n \le c$ for all $n \in \mathbb{N}$.)*

3. Entropy of rational shifts

Throughout this section we refer frequently to the entropy axioms and results of [CS]. Below we show that if $\tilde{\sigma}$ is the canonical extension of a rational Powers shift, then its Connes-Størmer entropy is $\frac{1}{2}\log 2$. This result is proved by combining a technique in [NT, Proposition 6.16] along with the previous theorem.

In what follows we shall use the notation $\mathcal{A}_{mn}$, for integers $m < n$, to denote the finite-dimensional C^*-subalgebra of $\tilde{R}$ generated by $u_m, \dots, u_{n-1}$. Then $\dim \mathcal{A}_{mn} = 2^{n-m}$, the number of distinct ordered words in these generators.

Lemma 3.1. *The algebras $\mathcal{A}_{mn}$ and $\mathcal{A}_{n-m}$ are isomorphic.*

Proof. This result follows immediately from the shift invariance of the commutation relations (2.1), i.e., u_i and u_j commute if and only if u_{i+k}, u_{j+k} do, for any $i, j, k \in \mathbb{Z}$. ☐

Proposition 3.2. *If σ is any Powers shift, then $h(\tilde{\sigma})$, the Connes-Størmer entropy of $\tilde{\sigma}$, does not exceed $\frac{1}{2}\log 2$.*

Proof. It is clear from the definition of a Powers shift that the union of the algebras $\mathcal{A}_{-n,n}$ is strongly dense in $\tilde{R}$. From the Connes-Størmer version of the Kolmogorov-Sinai Theorem, [CS, Theorem 1], $h(\tilde{\sigma}) = \sup_{n\in\mathbb{N}} h(\mathcal{A}_{-n,n}, \tilde{\sigma})$, so that it suffices to show that $h(\mathcal{A}_{-n,n}, \tilde{\sigma}) \le \frac{1}{2}\log 2$ for all $n \in \mathbb{N}$. Moreover, from the definition of Connes-Størmer entropy it follows (see also [CNT, Prop. III.6]) that $h(N, \tilde{\sigma}) = h(\tilde{\sigma}^k(N), \tilde{\sigma})$, for any $k \in \mathbb{Z}$ and any von Neumann subalgebra N of $\tilde{R}$, so it suffices to show that $h(\mathcal{A}_n, \sigma) \le \frac{1}{2}\log 2$ (cf. [CS, Definition 2.1]).

From [CS, Axiom C]

$$h(\mathcal{A}_n, \tilde{\sigma}(\mathcal{A}_n), \ldots, \tilde{\sigma}^{k-1}(\mathcal{A}_n)) \leq h(\{\mathcal{A}_n, \tilde{\sigma}(\mathcal{A}_n), \ldots, \tilde{\sigma}^{k-1}(\mathcal{A}_n)\}''),$$

for all $k \in \mathbb{N}$. But $\{\mathcal{A}_n, \tilde{\sigma}(\mathcal{A}_n), \ldots, \tilde{\sigma}^{k-1}(\mathcal{A}_n)\}'' = \{u_0, \ldots, u_{n+k-1}\}'' = \mathcal{A}_{n+k-1}$. From Theorem 2.2 above there exists a sequence $\{k_m\colon\ r \in \mathbb{N}\}$ of positive integers such that $\mathcal{A}_{n+k_m-1}$ is isomorphic to a matrix algebra. Since each minimal projection $e \in \mathcal{A}_{n+k_m-1}$ has trace $\tau(e) = 2^{-1/2(n+k_m-1)}$, we have by Axiom D of [CS],

$$\begin{aligned} H(\mathcal{A}_{n+k+m-1}) &= 2^{1/2(n+k_m-1)} \cdot \eta(\tau(e)) \\ &= 2^{1/2(n+k_m-1)}(-\tau(e)\log(\tau(e))) \\ &= \frac{1}{2}(n+k_m-1)\log 2, \end{aligned}$$

and therefore,

$$\begin{aligned} h(\mathcal{A}_n, \tilde{\sigma}) &= \lim_{m\to\infty} (1/k_m) H(\mathcal{A}_n, \sigma(\mathcal{A}_n), \ldots, \sigma^{k_m-1}(\mathcal{A}_n)) \\ &\leq \lim_{m\to\infty} (1/k_m) \cdot H(\mathcal{A}_{n+k_m-1}) \\ &= \frac{1}{2}\log 2. \qquad \square \end{aligned}$$

The proof of the following proposition combines a technique used in [NT, Proposition 6.16] with the results above on the structure of a rational Powers shift.

Theorem 3.3. *If $\tilde{\sigma}$ is the extension of a rational Powers shift σ, then $h(\tilde{\sigma}) = \frac{1}{2}\log 2$.*

Proof. As above, let R be the hyperfinite II_1 factor generated by $u_0, u_1, \ldots$. By Theorem 2.4 there is a positive integer k and a word w in these generators such that $\{w, \sigma(w), \ldots, \sigma^{j-1}(w)\}'' = \sigma^{k+j}(R)' \cap R$, for all $j \in \mathbb{Z}^+$. Let N be the subfactor of R generated by $w, \sigma(w), \ldots$. Then (again by Theorem 2.4) σ restricts to a Powers shift on N, which is rational since the corresponding bitstream is finitely nonzero. By Theorem 2.2 there are infinitely many positive integers m for which $B_m = \{w, \ldots, \sigma^{m-1}(w)\}''$ is a matrix algebra. Choosing such an m and setting $L = m+k$, then since $w, \sigma(w), \ldots, \sigma^{m-1}(w)$ all commute with $u_{m+k}, u_{m+k+1}, \ldots$, it follows that B_m and $\sigma^L(B_m)$ are mutually commuting. Using the shift invariance of the commutation relations (2.1), it follows that the algebras $\tilde{\sigma}^{rL}(B_m)$ and $\tilde{\sigma}^{sL}(B_m)$ mutually commute for all $r, s \in \mathbb{Z}$. Therefore $H(B_m) = H(\tilde{\sigma}_{|B})$, by [CS, Theorem 3], where B is the factor generated by the algebras $\tilde{\sigma}^{rL}(B_m)$, $r \in \mathbb{Z}$. Then by [CS, Remark 6], $h(\tilde{\sigma}) = L^{-1}h(\tilde{\sigma}^L) \geq L^{-1}h(\tilde{\sigma}^L_{|B}) = L^{-1}H(B_m) = L^{-1}(\frac{1}{2}m\log 2) = \frac{1}{2}(m/m+k)\log 2$. Letting $m \to \infty$ (and running over those positive integers for which B_m is a matrix algebra) shows that $H(\tilde{\sigma}) \geq \frac{1}{2}\log 2$. Combining this inequality with the previous proposition yields the result. $\square$

Remark 3.1. In [C] M. Choda showed that the rational shifts whose corresponding bitstreams are finitely nonzero have entropy $\frac{1}{2}\log 2$. It follows from Choda's result that $H(\tilde{\sigma}_{|B}) = \frac{1}{2}\log 2$. Although this result may be used to prove the proposition, the calculation above lends itself to the proof of the result of the next section.

4. Entropy of perturbed rational shifts

In this section we show that if an eventually periodic bitstream is altered on a certain suitably sparse subset of its entries, the new bitstream is associated with a Powers shift having positive entropy. More precisely, suppose $q \geq 2$ is a fixed integer. Let $Q = \{q^n \colon n \in \mathbb{N}\}$. Let $\{a_0, a_1, \dots\}$ be an eventually periodic bitstream associated with a rational Powers shift σ, and let $\{b_0, b_1, \dots\}$ be a bitstream which satisfies $b_j = a_j$ for all $j \notin Q$. Then we show below that the automorphism $\tilde{\rho}$ induced from the Powers shift ρ associated with the latter bitstream has positive entropy. We suspect that $\tilde{\rho}$ actually has entropy $\frac{1}{2}\log 2$ but are unable to prove this. We need the following lemma, the proof of which was communicated to us by Courtney Moen.

Lemma 4.1. *Let $q \geq 2$ be a fixed positive integer. Then for fixed $m \in \mathbb{N}$ there exists a positive integer d such that* $\min\{|q^n - kd| \colon k, n \in \mathbb{N}\} > m$.

Proof. If $q > 2$ let r be chosen such that $q^{r+1} - 2q^r > m$. Let $d = q^{r+1} - q^r$. For $1 \leq j \leq r$ and for any $k \in \mathbb{N}$ note that $|q^j - kd| \geq d - q^r = q^{r+1} - 2q^r > m$. For $j \geq r+1$ it is easy to show that q^j is congruent $\operatorname{mod} d$ to q^r, so that $\min\{|q^j - kd| \colon k \in \mathbb{N}\} \geq \min\{|q^r - kd| \colon k \in \mathbb{N}\} = d - q^r$. If $q = 2$, let r be sufficiently large so that $2^{r-1} > m$, and let $d = 2^r + 2^{r-1}$. If $1 \leq j \leq r-1$, then for any $k \in \mathbb{N}$, $|2^j - kd| \geq d - 2^{r-1} = 2^r$. For $j = r$, $|2^r - kd| \geq d - 2^r = 2^{r-1}$ and for $j = r+1$, $|2^{r+1} - kd| \geq d - 2^r = 2^{r-1}$. For $s \geq r+1$ it is easy to show that 2^s is congruent to $2^{s-2} \mod 2$, so that $\min\{|2^s - kd| \colon k \in \mathbb{N}\}$ coincides either with $\min\{|2^r - kd| \colon k \in \mathbb{N}\}$ or with $\min\{|2^{r-1} - kd| \colon k \in \mathbb{N}\}$, both of which exceed m. □

Proposition 4.2. *Let ρ be as above. Then ρ has positive Connes-Størmer entropy.*

Proof. Let σ be as above. By Theorem 2.4, σ has finite commutant index, say k, and there exists a word w in the generators $u_0, u_1, \cdots$ of the factor R such that

$$\{w, \sigma(w), \cdots, \sigma^{j-1}(w)\}'' = \sigma^{k+j}(R)' \cap R, \qquad j \in \mathbb{N}.$$

By Theorem 2.4, σ restricts to a rational shift, with finitely nonzero bitstream, on the subfactor generated by w and all of its shifts. Suppose $w = u_0^{c_0} \cdots u_s^{c_s}$. Set $m = k + s$. Note that for any $d \geq m$ the words $w, \sigma^d(w), \sigma^{2d}(w), \cdots$ all commute. By the shift invariance, then $\tilde{\sigma}^{rd}(w)$ commutes with $\tilde{\sigma}^{sd}(w)$ for all $r, s \in \mathbb{Z}$.

For a fixed integer $q \geq 2$, let ρ and its extension $\tilde{\rho}$ be defined as above, with bitstream $\{b_0, b_1, \cdots\}$. Let $\{v_j \colon j \in \mathbb{Z}\}$ be the hermitian unitary elements which generate $\tilde{R}$ and on which $\tilde{\rho}$ satisfies $\tilde{\rho}(v_j) = v_{j+1}$ for all $j \in \mathbb{Z}$. Let $z = v_0^{c_0} v_1^{c_1} \cdots v_s^{c_s}$. Let d be chosen as in the lemma. Applying (2.1) it follows for $p \in \mathbb{N}$ that whether z and $\rho^{pd}(z) = v_p^{c_0} v_{pd+1}^{c_1} \cdots c_{pd+s}^{c_s}$ commute depends only on the values $b_{pd-s}, b_{pd-s+1}, \cdots, b_{pd+s}$ of the bitstream. Note that by the choice of d these values coincide with a_{pd-s}, $a_{pd-s+1}, \cdots, a_{pd+s}$. Hence since w and $\sigma^{pd}(w)$ commute, so do $\rho(z)$ and $\rho^{pd}(z)$. Hence $\mathfrak{Z} = \{\tilde{\rho}^{pd}(z) \colon p \in \mathbb{Z}\}''$ is a commutative von Neumann subalgebra of R which is invariant under $\tilde{\rho}^d$. Calculating as in Theorem 3.3, $h(\tilde{\rho}) = d^{-1} h(\tilde{\rho}^d) \geq d^{-1} h(\tilde{\rho}_{|\mathfrak{Z}}) = d^{-1} H(\{z\}'')$, and by [CS, Axiom D], $H(\{z\}'') = H(M_1(\mathbb{C}) \oplus M_1(\mathbb{C})) = \log 2$. Hence $h(\tilde{\rho}) > 0$. □

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