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TRANSITIVE AND FULLY TRANSITIVE GROUPS

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Abstract. The notions of transitivity and full transitivity for abelian p-groups were introduced by Kaplansky in the 1950s. Important classes of transitive and fully transitive p-groups were discovered by Hill, among others. Since a 1976 paper by Corner, it has been known that the two properties are independent of one another. We examine how the formation of direct sums of p-groups affects transitivity and full transitivity. In so doing, we uncover a far-reaching class of p-groups for which transitivity and full transitivity are equivalent. This result sheds light on the relationship between the two properties for all p-groups.

1. Introduction

Throughout this note, we will denote the p-height sequence of an element x in a p-local abelian group G by $U_G(x)$ or simply $U(x)$. Recall that G is *transitive* if x can be mapped to y by an automorphism of G whenever $x, y \in G$ satisfy $U(x) = U(y)$; and *fully transitive* if this can be accomplished by an endomorphism of G whenever $U(x) \leq U(y)$ pointwise. Extensive classes of abelian p-groups with both transitivity properties—including separable and totally projective p-groups—are set forth in [Co], [Gr], [Hi] and [Ka]. Examples of p-groups with neither of the properties are given in [Me] and [Hi].

Can an abelian p-group be fully transitive but not transitive, or vice versa? In the earliest account [Ka], Kaplansky proved that transitive p-groups are indeed fully transitive provided $p \neq 2$. More than twenty years later, however, Corner [Co] answered the question in the negative by constructing fully transitive p-groups which fail to be transitive, and a transitive 2-group which is not fully transitive.

Despite the independence of transitivity and full transitivity for abelian p-groups, it has become increasingly clear (see e.g. [CaGo]) that there is, indeed, some basic connection between the two. The most striking of the results in this present note is the fundamental, but apparently unknown

Corollary 3. *A p-group G is fully transitive if and only if its square $G \oplus G$ is transitive.*

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In Theorem 1, we will set forth an extensive class of p-groups for which transitivity and full transitivity are equivalent. Corollary 3 is symptomatic of the fact that this class contains the square of every abelian p-group.

Throughout this note all groups are reduced p-local abelian groups, and we refer to them simply as groups. Notation follows the standard works of Fuchs [Fu] and Kaplansky [Ka] with the exception that maps are written on the right; all undefined terms may be found in these references.

2. Transitivity and full transitivity

It was shown by Megibben [Me, Theorem 2.4] that the direct sum of two fully transitive p-groups need not be fully transitive. In order to obtain closure under direct sums, we first extend the notion of full transitivity in a fairly obvious way.

Definition 1. If G_1 and G_2 are groups, then $\{G_1, G_2\}$ is a *fully transitive pair* if for every $x \in G_i$, $y \in G_j$ $(i, j \in \{1, 2\})$ which satisfy $U_{G_i}(x) \leq U_{G_j}(y)$, there exists $\alpha \in \operatorname{Hom}(G_i, G_j)$ with $x\alpha = y$.

For example, it is an easy exercise to verify that $\{G_1, G_2\}$ is a fully transitive pair whenever G_1 and G_2 are direct summands of a fully transitive group. The next result shows that all fully transitive pairs of p-groups arise in this way.

Proposition 1. *Let $\{G_i\}_{i \in I}$ be a collection of p-groups such that for each $i, j \in I, \{G_i, G_j\}$ is a fully transitive pair. Then the (external) direct sum $\bigoplus_{i \in I} G_i$ is fully transitive.*

Proof. It suffices to consider the case where $I = \{1, ..., n\}$ is finite. Denote $G = G_1 \oplus \cdots \oplus G_n$ and suppose $x, y \in G$ satisfy $U_G(x) \leq U_G(y)$. We will obtain an endomorphism of G mapping x to y by inducting on the order of y. First suppose $py = 0$. Write $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. By relabelling, we may assume that the p-heights satisfy $ht_G(x) = ht_{G_1}(x_1)$. Observe, since $py = 0$, that $U_{G_1}(x_1) \leq U_G(y) \leq U_{G_i}(y_i)$ for all i. By assumption, there exist $\alpha_i \in \operatorname{Hom}(G_1, G_i)$ with $x_1\alpha_i = y_i$ for $1 \leq i \leq n$. Clearly, the $n \times n$ matrix with first row $(\alpha_1, ..., \alpha_n)$ and other rows zero represents an endomorphism of G mapping x to y.

Now assume $o(y) > p$. Note $U_G(px) \leq U_G(py)$. Since $o(py) < o(y)$, induction yields $\theta \in \operatorname{End}(G)$ with $(px)\theta = py$. Set $x' = x\theta$. Then $y - x' \in G[p]$ and $U_G(x) \leq U_G(y - x')$; hence by the first paragraph there exists $\alpha \in \operatorname{End}(G)$ with $x\alpha = y - x'$. Now $\theta + \alpha$ maps x to $x' + y - x' = y$, as desired. □

We will require the following consequence of Proposition 1, indicating how a single fully transitive p-group can be used to produce many more.

Corollary 1. *Let G be a fully transitive p-group and λ any cardinal. Then all direct summands of the power $\bigoplus_\lambda G$ are fully transitive.*

Proof. Because G is fully transitive, $\{G, G\}$ is a fully transitive pair. Proposition 1 implies $\bigoplus_\lambda G$ is fully transitive, and the claim follows since direct summands of fully transitive groups are fully transitive. □

In [Co, Proposition 2.2], Corner proves that if $G = G_1 \oplus G_2$ is a fully transitive p-group such that the Ulm subgroups $p^\omega G_1$, $p^\omega G_2$ are nontrivial and $p^\omega G$ is homocyclic, then G is transitive. His example of a non-transitive, fully transitive p-group G is such that $p^{\omega+1}G = 0$, and he notes the "curious consequence" of Proposition 2.2 that $G \oplus G$ must be transitive (note that it is fully transitive by Corollary 1 above). It was Corner's result that motivated the theorem we shall soon prove. We need two preparatory lemmas.

Lemma 1. *Assume $G = G_1 \oplus G_2$ is a fully transitive group and $x_i, y_i \in G_i$ ($i = 1, 2$). If $U_{G_1}(x_1) \le U_{G_2}(y_2 - x_2)$ and $U_{G_2}(y_2) \le U_{G_1}(y_1 - x_1)$, then there is an automorphism of G mapping (x_1, x_2) to (y_1, y_2).*

Proof. Because $\{G_1, G_2\}$ is a fully transitive pair, there exist $\alpha \in \mathrm{Hom}(G_1, G_2)$ and $\beta \in \mathrm{Hom}(G_2, G_1)$ with $x_1\alpha = y_2 - x_2$ and $y_2\beta = y_1 - x_1$. The matrix $\phi = \begin{pmatrix} 1+\alpha\beta & \alpha \\ \beta & 1 \end{pmatrix}$ represents an automorphism of $G_1 \oplus G_2$, and an easy check verifies that $(x_1, x_2)\phi = (y_1, y_2)$. □

If G is a group and σ an ordinal number, we use $f_G(\sigma)$ to denote the classical Ulm invariant of G at σ (see [Fu] or [Ka]).

Definition 2. If G is a reduced group, the *Ulm support* $\mathrm{supp}(G)$ of G is the set of all ordinal numbers σ less than the p-length of G for which $f_G(\sigma)$ is nonzero.

If G_1 and G_2 are p-groups with $\mathrm{supp}(G_1) \subseteq \mathrm{supp}(G_2)$, it follows that every U-sequence relative to G_1 is also a U-sequence relative to G_2. In particular, we note that for every $x \in G_1$ there is an element $y \in G_2$ such that $U_{G_1}(x) = U_{G_2}(y)$ (see [Ka, Lemma 24]). We employ this fact in the proof of the following crucial lemma.

Lemma 2. *Assume $G = G_1 \oplus G_2$ is a fully transitive p-group and supp$(p^\omega G_1) \subseteq$ supp$(p^\omega G_2)$. If $x \in p^\omega G$, there is an automorphism of G mapping x to an element $(c, d) \in G_1 \oplus G_2$ with $U_G(x) = U_{G_2}(d)$.*

Proof. Write $x = (a, b)$ and assume for the moment that we have shown that there exists an automorphism ϕ of G with $x\phi = (a_1, b_1)$ and $ht_{G_1}(p^i a_1) \neq ht_{G_2}(p^i b_1)$ whenever $p^i a_1 \neq 0$. But then, as noted above, our assumption on the Ulm supports means we can choose $b_2 \in p^\omega G_2$ such that $U_{G_1}(a_1) = U_{G_2}(b_2)$. By full transitivity, $b_2 = a_1\alpha$ for some homomorphism $\alpha : G_1 \to G_2$. The composite automorphism $\phi \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ of $G_1 \oplus G_2$ maps x to $(a_1, b_1 + b_2)$. Since $ht(p^i b_1) \neq ht(p^i b_2)$ when $p^i b_1 \neq 0$, we compute

$$U_{G_2}(b_1 + b_2) = U_{G_2}(b_1) \wedge U_{G_2}(b_2) = U_{G_2}(b_1) \wedge U_{G_1}(a_1) = U_G(x).$$

Choosing $d = (b_1 + b_2)$ gives the desired result. It remains only to establish the existence of the elements a_1, b_1 and the automorphism ϕ as above.

We prove this by induction on the maximum m of the set

$$\mathcal{S}_G(a, b) = \{i < \omega : ht_{G_1}(p^i a) = ht_{G_2}(p^i b) \neq \infty\}.$$

If $\mathcal{S} = \emptyset$, simply take $\phi = 1_G$. If $m = 0$, we proceed as follows. Clearly $ht(pa) > ht(pb)$ or $ht(pa) < ht(pb)$ by definition of $\mathcal{S}_G(a, b)$, say the former. Then $ht_{G_1}(pa) > ht_{G_1}(a) + 1$; hence $pa = pa_1$ for some $a_1 \in G_1$ with $ht(a_1) > ht(a)$. Put $b_1 = b$. Clearly, $ht(p^i a_1) = ht(p^i b_1)$ only if $p^i a_1 = 0$. Since $U_{G_1}(a_1 - a) = (ht(a), \infty, ...) \ge U_{G_2}(b_1)$ and $\{G_1, G_2\}$ is a fully transitive pair, we have $a_1 - a = b_1\alpha$ for some $\alpha \in \mathrm{Hom}(G_2, G_1)$. The automorphism $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ of $G_1 \oplus G_2$ maps (a, b) to (a_1, b_1) as desired. If $ht(pa) < ht(pb)$, we proceed as above to obtain a suitable automorphism of the form $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, finishing the case $m = 0$.

Now assume that $\mathcal{S}_G(a,b)$ is nonempty and has maximum $m > 0$. Note that $\mathcal{S}_G(pa,pb)$ has maximum $< m$. By induction, there exists $\psi \in \mathrm{Aut}(G)$ such that $(pa,pb)\psi = (a_2,b_2)$ and $ht(p^i a_2) \neq ht(p^i b_2)$ whenever $p^i a_2 \neq 0$. Set $x' = (a',b') = x\psi$. Because $px' = (a_2,b_2)$, it follows that $\mathcal{S}_G(a',b')$ is empty or has maximum 0. By the above paragraph, there exists $\phi \in \mathrm{Aut}(G)$ such that $(a',b')\phi = x\psi\phi = (a_1,b_1)$ and $ht(p^i a_1) \neq ht(p^i b_1)$ whenever $p^i a_1 \neq 0$. This establishes our claim. □

Before turning to the main theorem, we observe a consequence of Lemmas 1 and 2 which indicates that a fully transitive p-group G with transitive direct summand H is itself transitive provided $p^\omega G$ and $p^\omega H$ have the same Ulm supports.

Proposition 2. *Assume $G = G_1 \oplus G_2$ is a fully transitive p-group and $supp(p^\omega G_1) \subseteq supp(p^\omega G_2)$. If G_2 is transitive, then G is transitive.*

Proof. By [Co, Lemma 2.1], we need only verify that $\mathrm{Aut}(G)$ acts transitively on $p^\omega G$. Suppose $x,y \in p^\omega G$ have the same Ulm sequences in G. By Lemma 2 there exist $\phi_1,\phi_2 \in \mathrm{Aut}(G)$ such that if $x\phi_1 = (x_1,x_2)$ and $y\phi_2 = (y_1,y_2)$, then $U_{G_2}(x_2) = U_G(x) = U_G(y) = U_{G_2}(y_2)$. Note that $U_{G_2}(x_2) \leq U_{G_1}(y_1 - x_1)$, since $U_G(x_1 - y_1, x_2 - y_2) \geq U_G(x\phi_1)$. Since G_2 is transitive and G is fully transitive, there are $\beta \in \mathrm{Aut}(G_2)$ and $\alpha \in \mathrm{Hom}(G_2,G_1)$ with $x_2\beta = y_2$ and $x_2\alpha = y_1 - x_1$. Put $\psi = \begin{pmatrix} 1 & 0 \\ \alpha & \beta \end{pmatrix}$, an automorphism of G. An easy check verifies that $x\phi_1\psi\phi_2^{-1} = y$, as required. □

The following result is complementary to Corollary 1.

Corollary 2. *Assume the p-group G is transitive and fully transitive. If $\{H_i\}$ is a collection of direct summands of any power of G, then the external direct sum $G \oplus (\bigoplus H_i)$ is transitive and fully transitive.*

Proof. Proposition 1 implies $G \oplus (\bigoplus H_i)$ is fully transitive. Since $\mathrm{supp}(p^\omega G)$ contains $\mathrm{supp}(p^\omega H_i)$, the direct sum is also transitive by Proposition 2. □

We now give a result indicating when transitivity and full transitivity are equivalent.

Theorem 1. *Assume G is a p-group which has a decomposition $G = G_1 \oplus G_2$ such that $p^\omega G_1$ and $p^\omega G_2$ have the same Ulm supports. Then G is fully transitive if and only if G is transitive.*

Proof. Suppose G is fully transitive and that $x,y \in p^\omega G$ satisfy $U_G(x) = U_G(y)$. By Lemma 2, there are $\phi_1,\phi_2 \in \mathrm{Aut}(G)$ such that $x\phi_1 = (x_1,x_2)$, $y\phi_2 = (y_1,y_2) \in G_1 \oplus G_2$ satisfy $U_G(x) = U_{G_1}(x_1)$ and $U_G(y) = U_{G_2}(y_2)$. Because $U_G(x) = U_G(y)$ we have $U_{G_1}(x_1) \leq U_{G_2}(x_2), U_{G_2}(y_2)$; hence $U_{G_1}(x_1) \leq U_{G_2}(y_2 - x_2)$. Similarly, $U_{G_2}(y_2) \leq U_{G_1}(y_1 - x_1)$. The conditions of Lemma 1 are fulfilled, hence there exists $\psi \in \mathrm{Aut}(G)$ with $(x_1,x_2)\psi = (y_1,y_2)$. Now $x\phi_1\psi\phi_2^{-1} = y$, and we see that $\mathrm{Aut}(G)$ acts transitively on $p^\omega G$. By [Co, Lemma 2.1], G is transitive.

Conversely, assume G is transitive. Let B denote the square of the standard basic p-group. Then $H = G \oplus B$ is transitive since B is separable ([CaGo, Proposition 2.6]). The structure of the groups B and $p^\omega H = p^\omega G_1 \oplus p^\omega G_2$ implies that H has no Ulm invariants equal to one. Therefore H is fully transitive by [Ka, Theorem 26(b)], whence G is fully transitive. □

Theorem 1 has many corollaries. Corollary 3 is merely a noteworthy special case of Corollary 4.

Corollary 3. *A p-group G is fully transitive if and only if $G \oplus G$ is transitive.*

Corollary 4. *The following conditions are equivalent for a p-group G.*

(i) *For all cardinals λ, $\bigoplus_\lambda G$ is fully transitive.*
(ii) *For some $\lambda > 0$, $\bigoplus_\lambda G$ is fully transitive.*
(iii) *For all $\lambda > 1$, $\bigoplus_\lambda G$ is transitive.*
(iv) *For some $\lambda > 1$, $\bigoplus_\lambda G$ is transitive.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are trivial. Assume (ii) holds, and $\lambda > 1$ is a fixed cardinal. Note that G is a summand of a fully transitive group, hence is fully transitive. By Corollary 1, $\bigoplus_\lambda G$ is fully transitive. Because $\lambda > 1$ we can obviously decompose $\bigoplus_\lambda G = G_1 \oplus G_2$ in such a way that

$$\operatorname{supp}(p^\omega G_1) = \operatorname{supp}(p^\omega G_2) = \operatorname{supp}(p^\omega G).$$

Hence $\bigoplus_\lambda G$ is transitive by Theorem 1. Therefore (iii) holds.

Finally, assume (iv) holds. Writing $\bigoplus_\lambda G = G_1 \oplus G_2$ as above, it follows from Theorem 1 that $\bigoplus_\lambda G$ is fully transitive since it is transitive. Therefore G is fully transitive, and Corollary 1 yields condition (i). $\square$

Since transitive p-groups are fully transitive if $p \neq 2$, and squares of fully transitive p-groups are necessarily transitive, we obtain

Corollary 5. *For $p \neq 2$, the class of fully transitive p-groups is precisely the class of direct summands of transitive p-groups.*

Corner [Co] has given an example of a 2-group G which is transitive but not fully transitive. It follows from Corollary 4 that for all $\lambda > 1$, the power $\bigoplus_\lambda G$ is neither transitive nor fully transitive. In particular, the square of a transitive 2-group need not be transitive. For $p \neq 2$, Corollary 2 implies that all powers of a transitive p-group are both transitive and fully transitive, simply because the group itself is also fully transitive in this case.

The final corollary extends Theorem 1 and Proposition 1 by exploiting Hill's powerful criteria for transitivity and full transitivity.

Corollary 6. *Let $\{G_i\}_{i\in I}$ be a collection of p-groups. Assume there exists an ordinal σ such that $G_i/p^\sigma G_i$ is totally projective and $\{p^\sigma G_i, p^\sigma G_j\}$ is a fully transitive pair for each $i, j \in I$. Then $\bigoplus_{i\in I} G_i$ is fully transitive. If there exists a partition $I = J \cup K$ such that the groups $\bigoplus_{i\in J} p^{\sigma+\omega} G_i$ and $\bigoplus_{i\in K} p^{\sigma+\omega} G_i$ have equal Ulm supports, then $\bigoplus_{i\in I} G_i$ is also transitive.*

Proof. Let $G = \bigoplus_{i\in I} G_i$. Proposition 1 shows that $p^\sigma G = \bigoplus_{i\in I} p^\sigma G_i$ is fully transitive. Since $\frac{G}{p^\sigma G} \cong \bigoplus_{i\in I} \frac{G_i}{p^\sigma G_i}$ is totally projective, G is fully transitive by [Hi, Theorem 4]. If the second condition in the corollary is also met, then $p^\sigma G$ is transitive by Theorem 1, and it follows again from [Hi] that G is transitive. $\square$

Observe that the total projectivity of the quotients $G_i/p^\sigma G_i$ in Corollary 6 is automatic if σ is a finite ordinal. Megibben's result [Me, Theorem 2.4] and Hill's result [Hi, Theorem 6] demonstrate that Corollary 6 can fail if one of these quotients is not totally projective.

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