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HERMITE DISTRIBUTIONS ASSOCIATED TO THE GROUP $O(p,q)$

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Abstract. We calculate the tempered $O(p,q)$-invariant eigendistributions of the $O(p,q)$-invariant Hermite operator

$$-\tfrac{1}{2}(\Delta_x - \Delta_y) + \tfrac{1}{2}(|x|^2 - |y|^2) \qquad (x \in \mathbb{R}^p,\ y \in \mathbb{R}^q).$$

They are singular on the cone $|x| = |y|$ and are given elsewhere in terms of confluent hypergeometric functions.

Suppose p and q are positive integers. We consider the quadratic form

$$R_{pq}(x,y) = \sum_1^p x_j^2 - \sum_1^q y_j^2$$

and the differential operator

$$\Delta_{pq} = \sum_1^p \frac{\partial^2}{\partial x_j^2} - \sum_1^q \frac{\partial^2}{\partial y_j^2}$$

on $\mathbb{R}^{p+q}$ with coordinates $x_1, \dots, x_p, y_1, \dots, y_q$. The group of linear transformations of $\mathbb{R}^{p+q}$ that leave R_{pq} (or equivalently Δ_{pq}) invariant is $O(p,q)$.

In what follows, "invariant" will always mean "$O(p,q)$-invariant." We denote by $\mathcal{S}'(\mathbb{R}^{p+q})^{O(p,q)}$ the space of invariant tempered distributions on $\mathbb{R}^{p+q}$.

This paper is concerned with the invariant tempered eigendistributions of the invariant Hermite operator

$$H_{pq} = -\tfrac{1}{2}\Delta_{pq} + \tfrac{1}{2}R_{pq},$$

which we shall call *$O(p,q)$-Hermite distributions.* These distributions play a crucial role in the analysis of the space $\mathcal{S}'(\mathbb{R}^{p+q})^{O(p,q)}$ by Howe and Tan [4], who have proved the following result ([4, p.152]):

Proposition 1. *The set of eigenvalues of $H_{pq}|\mathcal{S}'(\mathbb{R}^{p+q})^{O(p,q)}$ is $\frac{1}{2}(p-q)+2\mathbb{Z}$, and each eigenspace is one-dimensional.*

However, Howe and Tan use the $O(p,q)$-Hermite distributions in a purely formal algebraic way as basis elements for $\mathfrak{sl}(2)$-modules, without investigating their analytic properties at all. Our object here is to calculate them explicitly.

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We begin with some general remarks about elements of $\mathcal{S}'(\mathbb{R}^{p+q})^{O(p,q)}$. The $O(p,q)$-orbits in $\mathbb{R}^{p+q}$ are simply the level sets of R_{pq}, except for the *light cone*

$$C_0 = \big\{(x,y) : R_{pq}(x,y) = 0\big\},$$

which is the union of the two orbits $\{0\}$ and $C_0 \setminus \{0\}$. Therefore, a continuous function f defined on an invariant set in $\mathbb{R}^{p+q}$ is invariant if and only if it depends only on R_{pq}:

$$f(x,y) = g(R_{pq}(x,y)). \tag{1}$$

In this case, if f is C^2, a simple application of the chain rule shows that

$$\Delta_{pq} f = 4R_{pq} g''(R_{pq}) + 2(p+q)g'(R_{pq}). \tag{2}$$

Next, we need the fact that all invariant eigendistributions of H_{pq} are C^∞ functions on the complement of the light cone. One way to see this is to use the results of de Rham [2] or Tengstrand [6] on the structure of invariant distributions, which say essentially that the correspondence $f \leftrightarrow g$ in (1) extends to distributions except for some complications at the origin, and then to invoke the fact that the eigenfunctions of the ordinary differential operator $-2s(d/ds)^2 - (p+q)(d/ds) + \frac{1}{2}s$ corresponding to H_{pq} according to (2) are smooth away from the origin. Another way is to observe that if f is invariant then the wave front set of f is contained in the normal bundles of the $O(p,q)$-orbits, whereas if $H_{pq}f = \lambda f$ then the wave front set of f is contained in the characteristic variety of Δ_{pq}; and these two sets do not intersect except along the light cone.

In short, finding solutions of $H_{pq}f = \lambda f$ away from the light cone, with $\lambda = \frac{1}{2}(p-q) + 2l$ as in Proposition 1, amounts to solving the ordinary differential equation

$$-2sg''(s) - (p+q)g'(s) + \tfrac{1}{2}sg(s) = [\tfrac{1}{2}(p-q) + 2l]g(s) \tag{3}$$

on $\mathbb{R} \setminus \{0\}$ and then setting $f = g \circ R_{pq}$. Moreover, by a simple calculation, the substitution

$$g(s) = e^{-s/2}h(s)$$

turns (3) into

$$sh''(s) + [\tfrac{1}{2}(p+q) - s]h'(s) - [\tfrac{1}{2}q - l]h(s) = 0.$$

This is a special case of the confluent hypergeometric equation

$$sh''(s) + (\gamma - s)h'(s) - \alpha h(s) = 0, \tag{4}$$

so one can find all the necessary information about its solutions in one's favorite book on special functions, such as the Bateman Manuscript Project [1].

The results we shall need are as follows. Here and in the sequel, $\mathbb{N}$ and $\mathbb{N}^+$ denote the nonnegative and positive integers, respectively.

1. ([1, p.256 and p.273]) One solution of (4) is the function $\Psi(\alpha, \gamma; s)$ defined in the half-planes $|\arg s + \phi| < \frac{1}{2}\pi$ with $|\phi| < \pi$ by

$$\Psi(\alpha,\gamma;s) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty e^{i\phi}} e^{-st} t^{\alpha-1}(1+t)^{\gamma-\alpha-1}\, dt$$

for $\operatorname{Re}\alpha > 0$ or by

$$\Psi(\alpha,\gamma;s) = \frac{1}{2\pi i} e^{-\pi i\alpha}\Gamma(1-\alpha)\int_{\infty e^{i\phi}}^{(0+)} e^{-st}t^{\alpha-1}(1+t)^{\gamma-\alpha-1}\,dt$$

for $\alpha \notin \mathbb{N}^+$. In the first integral, the contour is the ray $\{re^{i\phi} : r > 0\}$; in the second, the contour encircles this ray in the positive sense and $\phi - 2\pi < \arg t < \phi$ on the contour.

2. ([1, pp.258–9]) A second independent solution of (4) is given by

$$e^s\Psi(\gamma-\alpha,\ \gamma;\ -s).$$

3. ([1, p.278]) The solution Ψ is distinguished by its behavior at infinity:

$$\Psi(\alpha,\gamma;s) = s^{-\alpha} + O(|s|^{-\alpha-1}) \text{ as } s \to \infty,\ -\tfrac{3}{2}\pi < \arg s < \tfrac{3}{2}\pi. \tag{5}$$

4. ([1, p.257]) If $\gamma \notin \mathbb{Z}$, $\Psi(\alpha,\gamma;s)$ is given in terms of the confluent hypergeometric series

$${}_1F_1(\alpha,\gamma;s) = \sum_0^\infty \frac{\alpha(\alpha+1)\cdots(\alpha+j-1)}{\gamma(\gamma+1)\cdots(\gamma+j-1)}\frac{s^j}{j!}$$

by

(6)
$$\Psi(\alpha,\gamma;s) = \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-\gamma+1)}\,{}_1F_1(\alpha,\gamma;s) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)}s^{1-\gamma}\,{}_1F_1(\alpha-\gamma+1,\ 2-\gamma;\ s).$$

5. ([1, p.268]) If $-\alpha \in \mathbb{N}$, $\Psi(\alpha,\gamma,s)$ is a polynomial of degree $-\alpha$ in s. If $\gamma-\alpha \in \mathbb{N}^+$, $s^{\gamma-1}\Psi(\alpha,\gamma;s)$ is a polynomial of degree $\gamma-\alpha-1$ in s. Up to scalar multiples, these polynomials are the (generalized) Laguerre polynomials $L_{-\alpha}^{\gamma-1}(s)$ and $L_{\gamma-\alpha-1}^{1-\gamma}(s)$.
6. ([1, p.261]) If $\gamma \in \mathbb{N}^+$, $-\alpha \notin \mathbb{N}$, and $\gamma-\alpha \notin \mathbb{N}^+$, $\Psi(\alpha,\gamma;s)$ has the form

$$\Psi(\alpha,\gamma;s) = \sum_0^\infty a_j s^{j-\gamma+1} + \sum_0^\infty b_j s^j \log s, \tag{7}$$

where $a_0 \neq 0$ when $\gamma > 1$ (and for most values of α when $\gamma = 1$).

Returning now to the particular case at hand, we see that the general solution of (3) is

$$c_1 e^{-s/2}\Psi(\tfrac{1}{2}q - l,\ \tfrac{1}{2}(p+q);s) + c_2 e^{s/2}\Psi(\tfrac{1}{2}p + l,\ \tfrac{1}{2}(p+q);\ -s). \tag{8}$$

The general eigendistribution of H_{pq} on $\mathbb{R}^{p+q}$ with eigenvalue $\frac{1}{2}(p-q)+2l$ will agree with a function of $s = R_{pq}$ of the form (8) on the region $R_{pq} > 0$, and with another function of the same form but perhaps with different constants c_1 and c_2 on the region $R_{pq} < 0$. Moreover, we are interested only in the tempered eigendistributions, and for these the function (8) must be of at most polynomial growth as $s \to \pm\infty$. In view of (5), this forces $c_2 = 0$ for $s > 0$ and $c_1 = 0$ for $s < 0$. Hence, in what follows, the arguments of the functions $\Psi(\alpha,\gamma;\cdot)$ will always be positive numbers, and we fix the branch of all powers and logarithms by declaring that $\arg s = 0$ for $s > 0$.

Since functions of the form (8) on $\mathbb{R}^{p+q}$ are generally not integrable near points on the light cone, we need to find a way to interpret them as distributions. Let us define locally integrable functions s_+ and s_- on $\mathbb{R}^{p+q}$ by

$$s_+ = \begin{cases} R_{pq} & \text{if } R_{pq} > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad s_- = \begin{cases} -R_{pq} & \text{if } R_{pq} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $\operatorname{Re}\beta > -1$ the functions $s_\pm^\beta$ and $s_\pm^\beta \log s_\pm$ are locally integrable and tempered. (This is not hard to prove from the fact that R_{pq} vanishes to first order along $C_0 \setminus \{0\}$; cf. [4, pp.153–4].) In view of (6) and (7), we see that the function (8) is of the form

$$\sum_1^{[(p+q)/2]} a_j s^{j-(p+q)/2} + b + c \log s + g(s), \tag{9}$$

where g is continuous and vanishes at $s = 0$.

(The term $c \log s$ is absent if $p+q$ is odd.) Clearly $g(s_\pm)$ is a continuous function on $\mathbb{R}^{p+q}$ which is tempered if (8) is of polynomial growth at infinity. Hence, to define tempered distributions on $\mathbb{R}^{p+q}$ by substituting $s_\pm$ for s in (8) (and in particular substituting $bs_\pm^0$ for the constant function b) we only need to make $s_\pm^\beta$ into a tempered distribution for $-\frac{1}{2}(p+q) < \beta \leq -1$.

The procedure for doing this is well known. By (2), if $\operatorname{Re}\beta > 2$ we have

$$\Delta_{pq}[s_\pm^\beta] = \pm 2\beta(2\beta + p + q - 2)s_\pm^{\beta-1}, \tag{10}$$

$$\Delta_{pq}[s_\pm^\beta \log s_\pm] = \pm 2\beta(2\beta + p + q - 2)s_\pm^{\beta-1} \log s_\pm \pm 2(4\beta + p + q - 2)s_\pm^{\beta-1}. \tag{11}$$

Formulas (10) and (11) remain valid for $\operatorname{Re}\beta > 0$ since the quantities on both sides are analytic distribution-valued functions of β in this range. We shall use them to *define* $s_\pm^\beta$ as a distribution for $-\frac{1}{2}(p+q) < \beta \leq -1$. Namely, the formula (10), rewritten as

$$s_\pm^\beta = \frac{\pm 1}{2(\beta+1)(2\beta+p+q)} \Delta_{pq}[s_\pm^{\beta+1}], \tag{12}$$

defines $s_\pm^\beta$ for $-2 < \operatorname{Re}\beta \leq -1$ except when $\beta = -1$, and hence by recursion for all $\operatorname{Re}\beta < 0$ except when β or $\beta + \frac{1}{2}(p+q)$ is an integer. For $\beta = -1$ (and $p+q > 2$), we use (11),

$$s_\pm^{-1} = \frac{\pm 1}{2(p+q-2)} \Delta_{pq}[\log s_\pm], \tag{13}$$

and then for $\beta = -2, -3, \ldots$ (but $\beta < \frac{1}{2}(p+q)$ if $p+q$ is even) we combine this with (10):

$$s_\pm^{-n} = \frac{(\pm 1)^n}{2(-2)(-4)\cdots(-2n+2)(p+q-2)\cdots(p+q-2n)} \Delta_{pq}^n[\log s_\pm]. \tag{14}$$

(*Warning:* $s_\pm^\beta$, thus defined, is not a continuous function of β at $\beta = -1, -2, \ldots$.)

We can now define tempered distributions $e^{-s_\pm/2}\Psi(\alpha, \gamma; s_\pm)$ on $\mathbb{R}^{p+q}$ as follows: Obtain the series expansion for $e^{-s/2}\Psi(\alpha, \gamma; s)$ by multiplying the Maclaurin series for $e^{-s/2}$ by the series (6) or (7) for $\Psi(\alpha, \gamma; s)$, then substitute $s_\pm$ for s. Alternatively, write $e^{-s/2}\Psi(\alpha, \gamma; s)$ in the form (9), then substitute $s_\pm$ for s. The latter procedure shows that there are no convergence problems and that $e^{-s_\pm/2}\Psi(\alpha, \gamma; s_\pm)$ is always tempered. We have therefore arrived at the following result.

Proposition 2. *Every $O(p,q)$-Hermite distribution with eigenvalue $\frac{1}{2}(p-q)+2l$ agrees on $\mathbb{R}^{p+q} \setminus C_0$ with a distribution of the form*

$$A_+ e^{-s_+/2}\Psi(\tfrac{1}{2}q - l,\ \tfrac{1}{2}(p+q); s_+) + A_- e^{-s_-/2}\Psi(\tfrac{1}{2}p + l,\ \tfrac{1}{2}(p+q);\ s_-), \tag{15}$$

with $A_+, A_- \in \mathbb{C}$.

Next we must determine what happens along the light cone when the Hermite operator $H_{pq} = -\frac{1}{2}\Delta_{pq} + \frac{1}{2}R_{pq}$ is applied to a distribution of the form (15). By (6) and (7), such a distribution can be expanded in a series whose terms are multiples of $s_\pm^\beta$ ($\beta \geq 1-(p+q)/2$) or $s_\pm^\beta \log s_\pm$ ($\beta \geq 0$), so it will suffice to determine the action of Δ_{pq} and R_{pq} on these terms.

By (10–14), applying Δ_{pq} to $s_\pm^\beta$ or $s_\pm^\beta \log s_\pm$ results in more terms of the same form except for $\Delta_{pq}[s_\pm^0]$ and $\Delta_{pq}[s_\pm^{1-(p+q)/2}]$, which vanish off the light cone, and it is these distributions we must examine closely. First, note that s_+^0 and s_-^0 are the characteristic functions of the regions $R_{pq} > 0$ and $R_{pq} < 0$, so $s_+^0 + s_-^0$ is (as a distribution) the constant function 1. Hence,

$$\Delta_{pq}[s_-^0] = -\Delta_{pq}[s_+^0]. \tag{16}$$

The remaining facts we need are summarized in the following proposition.

Proposition 3. *The space $\{f \in \mathcal{S}'(\mathbb{R}^{p+q})^{O(p,q)} : \Delta_{pq} f = 0\}$ is two-dimensional. It is spanned by the constant function 1 and the distribution*

$$\begin{aligned} s_+^{1-(p+q)/2} &\ \textit{if } p \textit{ is even and } q \textit{ is odd},\\ s_-^{1-(p+q)/2} &\ \textit{if } p \textit{ is odd and } q \textit{ is even},\\ \Delta_{pq}^{1-(p+q)/2}[s_+^0] &\ \textit{if } p \textit{ and } q \textit{ are both even},\\ s_+^{1-(p+q)/2} + (-s_-)^{1-(p+q)/2} &\ \textit{if } p \textit{ and } q \textit{ are both odd and } p+q>2,\\ \log s_+ + \log s_- &\ \textit{if } p=q=1. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \Delta_{pq}[s_-^{1-(p+q)/2}] &= \kappa_1\delta_0 \ \textit{if } p \textit{ is even and } q \textit{ is odd},\\ \Delta_{pq}[s_+^{1-(p+q)/2}] &= \kappa_2\delta_0 \ \textit{if } p \textit{ is odd and } q \textit{ is even},\\ \Delta_{pq}[s_+^{1-(p+q)/2} + (-s_-)^{1-(p+q)/2}] &= \kappa_3\delta_0 \ \textit{if } p \textit{ and } q \textit{ are both even},\\ \Delta_{pq}^{(p+q)/2}[s_+^0] &= \kappa_4\delta_0 \ \textit{if } p \textit{ and } q \textit{ are both odd}, \end{aligned}$$

where δ_0 is the point mass at the origin and the constants κ_j are all nonzero.

These results are due to Methée [5] when $p=1$ or $q=1$ and to de Rham [3] in general. All of them except the identity of the second nullvector for Δ_{pq} when p and q are both odd can also be read off from the discussion in Howe and Tan [4, §IV.3]. The first part of the proposition immediately yields:

Corollary. *The distributions $\Delta_{pq}^n[s_\pm^0]$ ($n < \frac{1}{2}(p+q)$) and $\Delta_{pq}[s_\pm^{1-(p+q)/2}]$ ($p+q$ even) are all nonzero.*

Remark. It is not hard to show that $\Delta_{pq}[s_+^0]$ is the invariant measure on the orbit $C_0 \setminus \{0\}$, and if $p+q$ is even, $\Delta_{pq}[s_+^{1-(p+q)/2}]$ is the generator of the "missing piece" of the space of invariant distributions supported on C_0 in Theorem 3.3.1 on p.162 of Howe and Tan [4], as explained on the errata sheet.

Next we study the effect of the operation $f \to R_{pq} f$ on the distributions $s_\pm^\beta$ and $s_\pm^\beta \log s_\pm$.

Proposition 4. *We have*

$$R_{pq} s_\pm^\beta = \pm s_\pm^{\beta+1} \quad \textit{and} \quad R_{pq} s_\pm^\beta \log s_\pm = \pm s_\pm^{\beta+1} \log s_\pm \tag{17}$$

unless β is a negative integer. Moreover,

$$R_{pq} s_\pm^{-n} = \begin{cases} \pm s_\pm^0 & \textit{if } n = 1, \\ \pm s_\pm^{-n+1} + 2(p+q+2-4n)\gamma_\pm^n \Delta_{pq}^{n-1}[s_\pm^0] & \textit{if } n > 1, \end{cases} \tag{18}$$

where $\gamma_\pm^n$ is defined by $s_\pm^{-n} = \gamma_\pm^n \Delta_{pq}^n[\log s_\pm]$.

Proof. (17) obviously holds for $\beta > 0$, and hence by analytic continuation for all β except negative integers. For the latter case we use the commutation relation

$$[R_{pq}, \Delta_{pq}^n] = -2n\Delta_{pq}^{n-1}(2E + p + q - 2n + 2), \tag{19}$$

where E is the Euler degree operator

$$E = \sum_1^p x_j \frac{\partial}{\partial x_j} + \sum_1^q y_j \frac{\partial}{\partial y_j}. \tag{20}$$

(19) is easily verified by induction. A simple calculation shows that $E[\log s_\pm] = 2s_\pm^0$, and $\Delta_{pq}[s_\pm \log s_\pm] = \pm 2(p+q)\log s_\pm \pm 2(p+q+2)s_\pm^0$ by (11). Hence,

$$\begin{aligned} R_{pq} s_\pm^{-n} &= \gamma_\pm^n R_{pq} \Delta_{pq}^n [\log s_\pm] \\ &= \pm \gamma_\pm^n \Delta_{pq}^n [s_\pm \log s_\pm] + \gamma_\pm^n [R_{pq}, \Delta_{pq}^n][\log s_\pm] \\ &= \gamma_\pm^n \Delta_{pq}^{n-1} \Big[\pm \Delta_{pq}[s_\pm \log s_\pm] - 2n(2E + p + q - 2n + 2)[\log s_\pm] \Big] \\ &= (2-2n)(p+q-2n)\gamma_\pm^n \Delta_{pq}^{n-1}[\log s_\pm] + 2(p+q+2-4n)\gamma_\pm^n \Delta_{pq}^{n-1}[s_\pm^0]. \end{aligned}$$

If $n = 1$, the first term vanishes and the second one is $s_\pm^0$, since $\gamma_\pm^1 = \pm 1/2(p+q-2)$ by (13). If $n > 1$, the first term is $s_\pm^{-n+1}$, since $\gamma_\pm^n (2-2n)(p+q-2n) = \pm \gamma_\pm^{n-1}$ by (14). This proves (18). ☐

If we combine Propositions 3 and 4 and use (16), we find:

Proposition 5. *For $l \in \mathbb{Z}$, let F_l be the distribution (15):*

$$F_l = A_+ e^{-s_+/2} \Psi(\tfrac{1}{2}q - l,\ \tfrac{1}{2}(p+q);\ s_+) + A_- e^{-s_-/2} \Psi(\tfrac{1}{2}p + l,\ \tfrac{1}{2}(p+q);\ s_-), \tag{21}$$

and let c_β^+ and c_β^- be the coefficients of s^β in the series expansions derived from (6) or (7) for $e^{-s/2}\Psi(\frac{1}{2}q - l,\ \frac{1}{2}(p+q);\ s)$ and $e^{-s/2}\Psi(\frac{1}{2}p + l,\ \frac{1}{2}(p+q);\ s)$, respectively. Then

$$\begin{aligned} \big[H_{pq} - \tfrac{1}{2}(p-q) - 2l\big] F_l = &- \tfrac{1}{2} A_+ c^+_{1-(p+q)/2} \Delta_{pq}[s_+^{1-(p+q)/2}] \\ &- \tfrac{1}{2} A_- c^-_{1-(p+q)/2} \Delta_{pq}[s_-^{1-(p+q)/2}] \\ &- \tfrac{1}{2}\big(A_+ c_0^+ - A_- c_0^-\big)\Delta_{pq}[s_+^0] \\ &+ \sum_{n=2}^{(p+q)/2-1} (p+q+2-4n)\gamma_\pm^n \big(A_+ c_{-n}^+ - A_- c_{-n}^-\big)\Delta_{pq}^{n-1}[s_0^+], \end{aligned} \tag{22}$$

where $\gamma_\pm^n$ is as in Proposition 4. If $p+q$ is odd or $p+q \le 4$, the last sum is not present; and if $p=q=1$, the first two terms should be replaced by the corresponding logarithmic terms

$$-\tfrac{1}{2}A_+c^+_{\log}\Delta_{pq}[\log s_+] - \tfrac{1}{2}A_-c^-_{\log}\Delta_{pq}[\log s_-].$$

At last we are ready to calculate the $O(p,q)$-Hermite distributions precisely. Proposition 1 assures us that there is exactly one eigendistribution for the eigenvalue $\frac{1}{2}(p-q)+2l$ for each $l \in \mathbb{Z}$, and our job is to find it. By Proposition 2, it must be of the form (21) away from the light cone, so it remains to determine the coefficients $A_\pm$ in (21), or to modify (21) by adding in a distribution supported on the light cone, to make the right side of (22) vanish. The outcome will be different depending on the parities of p and q.

Theorem 1. *If p is even and q is odd, the $O(p,q)$-Hermite distributions are of the form F_l given by (21), with*

$$A_- = 0 \quad \text{if } l > -\tfrac{1}{2}p,$$
$$A_+c_0^+ = A_-c_0^- \quad \text{if } l \le -\tfrac{1}{2}p.$$

Proof. If p is even and q is odd, we have $\Delta_{pq}[s_+^{1-(p+q)/2}] = 0$ and $\Delta_{pq}[s_-^{1-(p+q)/2}] = \kappa_1\delta_0$, so the right side of (22) reduces to

$$-\tfrac{1}{2}A_-c^-_{1-(p+q)/2}\kappa_1\delta_0 - \tfrac{1}{2}\big(A_+c_0^+ - A_-c_0^-\big)\Delta_{pq}[s_+^0]. \tag{23}$$

If $l \le \frac{1}{2}p$, $\Psi(\frac{1}{2}p+l,\, \frac{1}{2}(p+q);\, -s)$ is a polynomial, so $c^-_{1-(p+q)/2} = 0$; hence (23) vanishes if and only if $A_+c_0^+ - A_-c_0^- = 0$. On the other hand, if $l > \frac{1}{2}p$, $\Psi(\frac{1}{2}q-l,\, \frac{1}{2}(p+q);\, s)$ is $s^{1-(p+q)/2}$ times a polynomial, so $c_0^+ = 0$ since $\frac{1}{2}(p+q) \notin \mathbb{Z}$. In this case, (23) vanishes if and only if $A_- = 0$. □

Theorem 2. *If p is odd and q is even, the $O(p,q)$-Hermite distributions are of the form F_l given by (21), with*

$$A_+ = 0 \quad \text{if } l < \tfrac{1}{2}q,$$
$$A_+c_0^+ = A_-c_0^- \quad \text{if } l \ge \tfrac{1}{2}q.$$

The proof is the same as that of Theorem 1.

Theorem 3. *If p and q are both even, the $O(p,q)$-Hermite distributions are of the form*

$$F_l + \sum_{1}^{(p+q)/2-1} a_n(l)\Delta^n_{pq}[s_+^0],$$

where F_l is as in (21) with

$$A_- = 0 \text{ if } l > -\tfrac{1}{2}p \qquad \text{and} \qquad A_+ = 0 \text{ if } l < \tfrac{1}{2}q.$$

(In particular, $F_l = 0$ if $-\frac{1}{2}p < l < \frac{1}{2}q$.)

Proof. Let us start with the case $-\frac{1}{2}p < l < \frac{1}{2}q$. Let $\mathcal{V}$ be the space spanned by $\Delta^n[s_+^0]$, $1 \le n < \frac{1}{2}(p+q)$. It is easily verified by using (19) that

$$R_{pq}\Delta^n_{pq}[s_+^0] = 2(1-n)(p+q-2n)\Delta^{n-1}_{pq}[s_+^0]. \tag{24}$$

In particular, $R_{pq}\Delta_{pq}[s_+^0] = 0$, and $\Delta_{pq}^{(p+q)/2}[s_+^0] = 0$ by Proposition 4, so $\mathcal{V}$ is preserved by the operations Δ_{pq} and multiplication by R_{pq}, and hence by H_{pq}. In fact, Δ_{pq}, R_{pq}, and their commutator $-4E-2(p+q)$ (where E is given by (20)) span a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ under the correspondence

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \longleftrightarrow a\bigl(E+\tfrac{1}{2}(p+q)\bigr)+\tfrac{1}{2}ibR_{pq}+\tfrac{1}{2}ic\Delta_{pq}$$

for which $\mathcal{V}$ is an irreducible module. Another application of (19) together with (24) shows that

$$E\Delta_{pq}^n[s_+^0] = -\bigl(\tfrac{1}{4}[R_{pq},\Delta_{pq}]+\tfrac{1}{2}(p+q)\bigr)\Delta_{pq}^n[s_+^0] = -2n\Delta_{pq}^n[s_+^0],$$

so the eigenvalues of E on $\mathcal{V}$ are $-2,-4,\ldots,2-p-q$. But the map

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \to \begin{pmatrix} \frac{1}{2}(b+c) & -ia+\frac{1}{2}i(b-c) \\ ia+\frac{1}{2}i(b-c) & -\frac{1}{2}(b+c) \end{pmatrix}$$

is an automorphism of $\mathfrak{sl}(2,\mathbb{C})$ that takes $E+\frac{1}{2}(p+q)$ to H_{pq} (a fact used repeatedly in [4]), so the eigenvalues of H_{pq} on $\mathcal{V}$ are $\frac{1}{2}(p+q)-2,\ldots,2-\frac{1}{2}(p+q)$ — in other words, $\frac{1}{2}(p-q)+2l$ where $-\frac{1}{2}p<l<\frac{1}{2}q$. Thus the eigendistributions for these eigenvalues are suitable elements of $\mathcal{V}$, as claimed.

Next, if $l\geq\frac{1}{2}q$, $\Psi(\frac{1}{2}q-l,\ \frac{1}{2}(p+q);\ s)$ is a polynomial, and hence $c_\beta^+=0$ for $\beta<0$. Thus, if we take $A_-=0$ in (21), the right side of (22) reduces to $-\frac{1}{2}A_+c_0^+\Delta_{pq}[s_+^0]$. We can therefore obtain an eigendistribution of the form F_l+v by taking $v\in\mathcal{V}$ to satisfy

$$\bigl[H_{pq}-\tfrac{1}{2}(p-q)-2l\bigr]v = \tfrac{1}{2}A_+c_0^+\Delta_{pq}[s_+^0].$$

This is possible since the results of the preceding paragraph show that $H_{pq}-\frac{1}{2}(p-q)-2l$ is invertible on $\mathcal{V}$ for $l\geq\frac{1}{2}q$. The case $l\leq-\frac{1}{2}p$ is similar. □

To handle the case where p and q are both odd, we need the following description of the invariant distributions supported on the light cone.

Proposition 6. *Let $\mathcal{C}$ be the space of invariant distributions supported on the light cone. If $p+q$ is odd, the distributions $\Delta_{pq}^n\delta_0$ and $\Delta_{pq}^{n+1}[s_+^0]$ ($n\geq 0$) are a basis for $\mathcal{C}$. If $p+q$ is even, $\mathcal{C}$ is spanned by $\Delta_{pq}^n\delta_0$, $\Delta_{pq}^{n+1}[s_+^0]$, and $\Delta_{pq}^{n+1}F$ ($n\geq 0$), where $F=s_+^{1-(p+q)/2}$ if $p+q>2$ and $F=\log s_+$ if $p=q=1$. The only linear relations among these distributions are those implied by Proposition 3.*

This is essentially in Methée [5] when $q=1$, and it can be deduced from the description of invariant distributions in de Rham [2] or Tengstrand [3]. It is also Theorem 3.3.1 on p.162 of Howe and Tan [4], as corrected on the errata sheet. (Howe and Tan identify the third generator of $\mathcal{C}$ when $p+q$ is even only as an element of $\mathcal{C}$ that is independent of δ and annihilated by $(E+p+q)^2$ where E is given by (20); $\Delta_{pq}[s^{1-(p+q)/2}]$ satisfies these conditions.)

Corollary. *Suppose G is an $O(p,q)$-Hermite distribution of eigenvalue $\frac{1}{2}(p-q)+2l$. Then*

$$G = F_l + \sum_{1}^{(p+q)/2-1} a_n\Delta_{pq}^n[s_+^0],$$

where F_l is given by (21); moreover, $a_n=0$ for all n if $p+q$ is odd.

Proof. We know that $G = F_l + T$ where $T \in \mathcal{C}$. T cannot have any terms of the form $\Delta_{pq}^n \delta_0$ or of the form $\Delta_{pq}^{n+1}[s_+^0]$ (if $p+q$ is odd) or $\Delta_{pq}^{n+1}[s^{1-(p+q)/2}]$ (if $p+q$ is even), because application of H_{pq} to such distributions yields even more singular distributions which cannot cancel with each other or with $H_{pq}F_l$. □

We remark that if p and q are not both odd, the arguments in Theorems 1–3 can be combined with this corollary to give a proof of Proposition 1. However, if p and q are odd, we shall need both Proposition 1 and this corollary to get a grasp on the situation. Here is the result.

Theorem 4. *If p and q are both odd, the $O(p,q)$-Hermite distributions are of the form*

$$F_l + \sum_{1}^{(p+q)/2-2} a_n(l)\Delta_{pq}^n[s_+^0] \tag{25}$$

(the sum being omitted if $p+q \le 4$), where F_l is as in (21) with

$$\Gamma(\tfrac{1}{2}p + l)A_+ = (-1)^{(p+q)/2-1}\Gamma(\tfrac{1}{2}q - l)A_-. \tag{26}$$

Proof. Suppose $p+q > 2$. From the explicit formula for $\Psi(\alpha,\gamma;s)$ with $\gamma \in \mathbb{N}^+$ in [1, p.261], the coefficients $c^+_{1-(p+q)/2}$ and $c^-_{1-(p+q)/2}$ are in the same proportion as $1/\Gamma(\frac{1}{2}q - l)$ and $1/\Gamma(\frac{1}{2}p + l)$. Hence if we choose $A_\pm$ to satisfy (26), the first two terms on the right of (22) will cancel by Proposition 3. The remaining terms then have the form

$$\sum_{1}^{(p+q)/2-1} b_n\Delta_{pq}^n[s_+^0].$$

On the other hand, by (24),

$$H_{pq}\Delta_{pq}^n[s_+^0] = -\tfrac{1}{2}\Delta_{pq}^{n+1}[s_+^0] + (n-1)(p+q-2n)\Delta_{pq}^{n-1}[s_+^0],$$

so $H_{pq} - \frac{1}{2}(p-q) - 2l$ will annihilate (25) provided that

$$\begin{aligned}
-\tfrac{1}{2}a_{(p+q)/2-2} + b_{(p+q)/2-1} &= 0,\\
-\tfrac{1}{2}a_{n-1} - [\tfrac{1}{2}(p-q) + 2l]a_n&\\
+n(p+q-2n-2)a_{n+1} + b_n &= 0 \text{ for } \tfrac{1}{2}(p-q) - 2 \ge n \ge 2,\\
-[\tfrac{1}{2}(p-q) + 2l]a_1 + (p+q-4)a_2 + b_1 &= 0,
\end{aligned}$$

where $a_n = a_n(l)$ and $a_{(p+q)/2-1} = 0$. This is an overdetermined system. The first $\frac{1}{2}(p+q) - 2$ equations may be solved successively for $a_{(p+2)/2-2}, \dots, a_1$, but the final equation is left hanging. Of course, this phenomenon is typical for eigenvalue problems — it is what prevents the existence of eigenvectors for arbitrary values of the parameter l — but it does not seem to be easy to see directly that a nontrivial solution exists precisely when l is an integer. For this we must rely on Proposition 1, and Proposition 6 and its corollary ensure that the form of the solution is given by (25) and (26).

We can, however, see explicitly what is going on in the case $p = q = 1$, and this will give an indication of the difficulty in the general case. From the formula for

$\Psi(\alpha, 1; s)$ in [1, p.261], we have

$$\begin{aligned} F_l = & \frac{A_+}{\Gamma(\frac{1}{2}-l)}\big(\log s_+ + [\psi(\tfrac{1}{2}-l) - 2\psi(1)]s_+^0 + \cdots\big) \\ & + \frac{A_-}{\Gamma(\frac{1}{2}+l)}\big(\log s_- + [\psi(\tfrac{1}{2}+l) - 2\psi(1)]s_-^0 + \cdots\big), \end{aligned}$$

where $\psi = \Gamma'/\Gamma$ and the dots denote higher order terms. Hence, if we choose $A_\pm$ to satisfy (26) with $p = q = 1$, $H_{pq} - \frac{1}{2}(p-q) - 2l$ will annihilate F_l if and only if $\psi(\frac{1}{2}-l) = \psi(\frac{1}{2}+l)$. But this happens precisely when l is an integer, for logarithmic differentiation of the functional equation

$$\Gamma(\tfrac{1}{2}+z)\Gamma(\tfrac{1}{2}-z) = \frac{\pi}{\cos \pi z}$$

yields

$$\psi(\tfrac{1}{2}+z) - \psi(\tfrac{1}{2}-z) = \pi \tan \pi z.$$

□

If p and q are both ≥ 2, every distribution invariant under the connected component of the identity in $O(p,q)$ is invariant under $O(p,q)$ itself, as the orbits for these groups are the same. However, if $p = 1$ or $q = 1$, there is more to be said; we consider $p = 1$ to be specific. The region $R_{1q} > 0$ has two connected components, namely its intersection with the half-spaces $x > 0$ and $x < 0$, and one has the group $O^+(1,q)$ consisting of all elements of $O(1,q)$ that map these components into themselves rather than interchanging them. It is of interest to consider the space of all tempered distributions that are $O^+(1,q)$-invariant but odd in x; we denote this space by $\mathcal{S}'(\mathbb{R}^{1+q})^{O^+(1,q)}_{\text{odd}}$. There is an analogue of Proposition 1 in this situation:

Proposition 7. *The set of eigenvalues of* $H_{1q}|\mathcal{S}'(\mathbb{R}^{1+q})^{O^+(1,q)}_{\text{odd}}$ *is* $\frac{1}{2}(3-q) + 2\mathbb{N}$, *and each eigenspace is one-dimensional.*

This result (stated in terms of $O^+(n,1)$ instead of $O^+(1,q)$) is proved in Howe and Tan [4, p.177]. We shall call the eigendistributions of H_{1q} in $\mathcal{S}'(\mathbb{R}^{1+q})^{O^+(1,q)}_{\text{odd}}$ *odd* $O^+(1,q)$*-Hermite distributions.*

The calculation of the odd $O^+(1,q)$-Hermite distributions proceeds much as above. They all vanish in the region $R_{1q} < 0$ since the orbits in that region are invariant under reflection in x, and in the region $R_{1q} > 0$ they agree with C^∞ functions of the form $(\operatorname{sgn} x)e^{-s_+/2}h(s_+)$ where h satisfies the confluent hypergeometric equation

$$sh''(s) + [\tfrac{1}{2}(1+q) - s]h'(s) - [\tfrac{1}{2}(q-1) - l]h(s) = 0 \qquad (l \in \mathbb{N}).$$

By (5), the requirement of temperedeness eliminates all but one of the solutions of this equation, and we arrive at

$$G_l = A(\operatorname{sgn} x)e^{-s_+/2}\Psi(\tfrac{1}{2}(q-1) - l,\ \tfrac{1}{2}(q+1);\ s_+) \qquad (A \in \mathbb{C}). \tag{27}$$

This is to be interpreted as a distribution as before, by using its series expansion, with the following proviso. $(\operatorname{sgn} x)s_+^\beta$ is defined to be $s_{++}^\beta - s_{+-}^\beta$, where $s_{+\pm}^\beta$ is defined in the same way as s_+^β starting from

$$s_{+\pm}(x,y) = \begin{cases} R_{1q}(x,y) & \text{if } \pm x > |y|, \\ 0 & \text{otherwise.} \end{cases}$$

Again, the only terms in G_l that can create a distribution supported on the light cone when H_{1q} acts on them are $(\operatorname{sgn} x)s_+^\beta$ with $-\beta \in \mathbb{N}$ or $\beta = \frac{1}{2}(1-q)$ (and $(\operatorname{sgn} x)\log s_+$, if $q=1$). Concerning these, we have the following variant of Proposition 3:

Proposition 8. *If q is even,*

$$\Delta_{1q}[s_{++}^{(1-q)/2}] = \Delta_{1q}[s_{+-}^{(1-q)/2}] = \kappa_5\delta_0,$$

while if q is odd,

$$\Delta_{1q}^{(q+1)/2}[s_{++}^0] = \Delta_{1q}^{(q+1)/2}[s_{+-}^0] = \kappa_6\delta_0,$$

where κ_5 and κ_6 are nonzero. Consequently,

$$\Delta_{1q}[(\operatorname{sgn} x)s_+^{(1-q)/2}] = 0 \text{ if } q \text{ is even}, \qquad \Delta_{1q}^{(q+1)/2}[(\operatorname{sgn} x)s_+^0] = 0 \text{ if } q \text{ is odd.}$$

See, e.g., Methée [5]. (This result is classic; $\kappa_5^{-1}s_{++}^{(1-q)/2}$ and $\kappa_6^{-1}\Delta_{1q}^{(q-1)/2}[s_{++}^0]$ are the most commonly used fundamental solutions for the wave equation.)

Theorem 5. *If q is even or $q=1$, the odd $O^+(1,q)$-Hermite distributions are of the form G_l given by (27). If q is odd and $q>1$, they are of the form*

$$G_l + \sum_1^{(q-1)/2} a_j(l)\Delta_{1q}^j[(\operatorname{sgn} x)s_+^0],$$

where $A=0$ in (27) if $l \le \frac{1}{2}(q-3)$.

Proof. In each of the following arguments we use Proposition 8.

If q is even, $s^{(q+1)/2}\Psi(\frac{1}{2}(q-1)-l, \frac{1}{2}(q+1); s)$ is a polynomial, so the series expansion of G_l involves only half-integer powers of s_+. Thus s_+^0 does not occur, and $\Delta_{1q}[(\operatorname{sgn} x)s_+^{(1-q)/2}] = 0$, so $H_{1q}G_l$ has no terms supported on the light cone; hence it equals $[\frac{1}{2}(3-q)+2l]G_l$.

If q is odd and $q>1$, the fact that $\Delta_{1q}^{(q+1)/2}[(\operatorname{sgn} x)s_+^0] = 0$ together with the argument in the proof of Theorem 3 shows that the space $\mathcal{W}$ spanned by the distributions $\Delta_{1q}^n[(\operatorname{sgn} x)s_+^0]$, $1 \le n \le \frac{1}{2}(q-1)$, is invariant under H_{1q} and that the eigenvalues of H_{1q} on $\mathcal{W}$ are $-\frac{1}{2}(q-3)+2l$, $0 \le l \le \frac{1}{2}(q-3)$. On the other hand, if $l \ge \frac{1}{2}(q-1)$, $\Psi(\frac{1}{2}(q-1)-l, \frac{1}{2}(q+1); s)$ is a polynomial $\sum_0^{l-(q-1)/2} c_j s^j$, so G_l involves only nonnegative powers of s_+, and we have

$$\big[H_{1q} - \tfrac{1}{2}(3-q) - 2l\big]G_l = -\tfrac{1}{2}Ac_0\Delta_{1q}[(\operatorname{sgn} x)s_+^0].$$

The desired eigendistribution is therefore of the form $G_l + w$ where w is an element of $\mathcal{W}$ chosen to satisfy

$$\big[H_{1q} - \tfrac{1}{2}(3-q) - 2l\big]w = \tfrac{1}{2}Ac_0\Delta_{1q}[(\operatorname{sgn} x)s_+^0].$$

Such a w exists because $H_{1q} - \frac{1}{2}(3-q) - 2l$ is invertible on $\mathcal{W}$.

Finally, if $q = 1$, $\Psi(-l,1;s)$ is again a polynomial, and $\Delta_{11}[(\operatorname{sgn} x)s_+^0] = 0$. Hence $H_{11}G_l$ has no terms supported on the light cone, so it equals $(1+2l)G_l$. □

Concluding Remarks. 1. When $p = q = 1$ there is another family of tempered eigendistributions of H_{11} that are invariant under the connected component of the identity in $O(1,1)$, namely the distributions $h(x,y) = \tilde{h}(y,x)$ where $\tilde{h}$ is an odd $O^+(1,1)$-Hermite distribution. Their eigenvalues are of the form $-1-2l$ with $l \in \mathbb{N}$.

2. From Theorems 1–5 and the facts about confluent hypergeometric functions cited earlier we immediately obtain the following qualitative description of the behavior of Hermite distributions. They are locally integrable functions if and only if $p+q < 4$, and the most singular terms in their series expansions are nonconstant solutions of $\Delta_{pq} f = 0$ as in Proposition 3. If $p+q$ is odd, the $O(p,q)$-Hermite distributions blow up like $|R_{pq}|^{1-(p-q)/2}$ on one side of the light cone (the side $R_{pq} > 0$ if p is even and the side $R_{pq} < 0$ if p is odd) but are bounded, and sometimes zero, on the other side; similarly for odd $O^+(1,q)$-Hermite distributions when q is even. If p and q are both odd, the $O(p,q)$-Hermite distributions blow up like $|R_{pq}|^{1-(p+q)/2}$ (or $\log |R_{pq}|$ if $p = q = 1$) on both sides of the light cone. Each $O(p,q)$-Hermite function for p and q both even, and each odd $O^+(1,q)$-Hermite distribution for q odd, is the sum of a bounded function vanishing on one side of the light cone and a distribution supported on the light cone.

3. For each p and q, the finite linear span of the $O(p,q)$-Hermite distributions is an $\mathfrak{sl}(2,\mathbb{C})$-module as described in the proof of Theorem 3. Moreover, if p (resp. q) is even, the span of the $O(p,q)$-Hermite distributions with eigenvalue $> -\frac{1}{2}(p+q)$ (resp. $< \frac{1}{2}(p+q)$) is a submodule. (See Howe and Tan [4], Proposition 3.2.2 on p.152 and Proposition 1.2.9 on p.63.) *The $O(p,q)$-Hermite distributions that vanish on the region $R_{pq} < 0$ (resp. $R_{pq} > 0$) are precisely those which belong to these submodules.* This fact was established in Howe and Tan [4], p.156, by an entirely different argument. Our calculations (taking [3] rather than [4] as a source for Proposition 3) provide an independent verification of it.

4. It may be of interest to note that whenever a Hermite distribution vanishes on one side of the light cone, on the other side it is the product of $e^{-|R_{pq}|/2}$ and either a polynomial function of R_{pq} or a rational function of $|R_{pq}|^{1/2}$. The other Hermite distributions involve nonelementary functions.

5. A corresponding theory exists for the case $q = 0$ — that is, radial eigenfunctions of the ordinary Hermite operator $H = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2$ on $\mathbb{R}^p$ — but it is much simpler. The set of eigenvalues of H on the space of radial tempered distributions on $\mathbb{R}^p$ is $\frac{1}{2}p + 2\mathbb{N}$, the eigenspaces are one-dimensional, and the eigendistributions are C^∞ functions on all of $\mathbb{R}^p$. In fact, the eigenfunctions for the eigenvalue $\frac{1}{2}p+2l$ are the multiples of $e^{-|x|^2/2}\Psi(-l, \frac{1}{2}p; |x|^2)$, or equivalently of $e^{-|x|^2/2}L_l^{(p/2)-1}(|x|^2)$, where L denotes a Laguerre polynomial.

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