

ENVELOPING ALGEBRAS OF LIE COLOR ALGEBRAS: PRIMENESS VERSUS GRADED-PRIMENESS

JEFFREY BERGEN AND D. S. PASSMAN

(Communicated by Ken Goodearl)

ABSTRACT. Let G be a finite abelian group and let L be a, possibly restricted, G -graded Lie color algebra. Then the enveloping algebra $U(L)$ is also G -graded, and we consider the question of whether $U(L)$ being graded-prime implies that it is prime. The first section of this paper is devoted to the special case of Lie superalgebras over a field K of characteristic $\neq 2$. Specifically, we show that if $i = \sqrt{-1} \in K$ and if $U(L)$ has a unique minimal graded-prime ideal, then this ideal is necessarily prime. As will be apparent, the latter result follows quickly from the existence of an anti-automorphism of $U(L)$ whose square is the automorphism of the enveloping algebra associated with its \mathbb{Z}_2 -grading. The second section, which is independent of the first, studies more general Lie color algebras and shows that if $U(L)$ is graded-prime and if most homogeneous components L_g of L are infinite dimensional over K , then $U(L)$ is prime. Here we use Δ -methods to study the grading on the extended centroid C of $U(L)$. In particular, if G is generated by the infinite support of L , then we prove that $C = C_1$ is homogeneous.

§1. SUPERALGEBRAS

Let $L = L_0 \oplus L_1$ be an ordinary (or restricted) Lie superalgebra over any field K of characteristic not 2, and let $U(L)$ denote its ordinary (or restricted) enveloping algebra. Then $U(L) = U_0 \oplus U_1$ is \mathbb{Z}_2 -graded, and we let σ denote the corresponding automorphism of order 2 defined by $\sigma(u) = u$ if $u \in U_0$ and $\sigma(u) = -u$ if $u \in U_1$. Then it is clear that any subspace of $U(L)$ is graded if and only if it is σ -stable. In particular, an ideal $I \triangleleft U(L)$ is graded-prime if and only if it is σ -prime. The goal of this section is to show that if $i = \sqrt{-1} \in K$, then $U(L)$ is a graded-prime ring if and only if it is a prime ring. Of course, prime always implies graded-prime, so we are essentially concerned with the reverse implication.

Lemma 1.1. *Let $U(L)$ denote the (restricted) enveloping algebra of the (restricted) K -superalgebra L and assume that $i = \sqrt{-1} \in K$. If $\tau: U(L) \rightarrow U(L)$ is given by $\tau(x) = -x$ if $x \in L_0$ and $\tau(x) = ix$ if $x \in L_1$, then τ is a well-defined algebra anti-automorphism of $U(L)$ and $\tau^2 = \sigma$.*

Received by the editors November 22, 1996.

1991 *Mathematics Subject Classification.* Primary 16S30, 16W55, 17B35.

The first author's research was supported by the Faculty Research and Development Fund of the College of Liberal Arts & Sciences at DePaul University. The second author's research was supported in part by NSF Grant DMS-9622566.

Proof. Consider the map $\theta : L \rightarrow U(L)^{\text{op}}$ given by $\theta(x) = -x^{\text{op}}$ if $x \in L_0$ and $\theta(x) = ix^{\text{op}}$ if $x \in L_1$. We show that θ is a Lie superalgebra homomorphism into the superalgebra of the \mathbb{Z}_2 -graded ring $U(L)^{\text{op}}$. To avoid confusion, let us use $[[\ , \]]$ to denote the bracket defined by the latter algebra. Let $x \in L_0$ and let $y \in L_a$ with $a \in \{0, 1\}$. Then $[x, y] \in L_a$, so $\theta(y) = \lambda y^{\text{op}}$ and $\theta([x, y]) = \lambda[x, y]^{\text{op}}$ for the same scalar λ . Thus

$$\begin{aligned} [[\theta(x), \theta(y)]] &= [[-x^{\text{op}}, \lambda y^{\text{op}}]] = -\lambda(x^{\text{op}}y^{\text{op}} - y^{\text{op}}x^{\text{op}}) = -\lambda(yx - xy)^{\text{op}} \\ &= \lambda(xy - yx)^{\text{op}} = \lambda[x, y]^{\text{op}} = \theta([x, y]). \end{aligned}$$

Similarly, if $x, y \in L_1$, then $[x, y] \in L_0$, so

$$\begin{aligned} [[\theta(x), \theta(y)]] &= [[ix^{\text{op}}, iy^{\text{op}}]] = i^2(x^{\text{op}}y^{\text{op}} + y^{\text{op}}x^{\text{op}}) = -(yx + xy)^{\text{op}} \\ &= -(xy + yx)^{\text{op}} = -[x, y]^{\text{op}} = \theta([x, y]). \end{aligned}$$

In particular, if L is an ordinary Lie superalgebra, then the universal property of $U(L)$ implies that θ extends to an algebra homomorphism $\theta : U(L) \rightarrow U(L)^{\text{op}}$. By composing this with the natural algebra anti-isomorphism $U(L)^{\text{op}} \rightarrow U(L)$, we conclude that τ exists and is an algebra anti-homomorphism. Since τ^2 agrees with σ on L , we conclude that $\tau^2 = \sigma$ and hence that τ is an algebra anti-automorphism of order dividing 4.

Finally, if L is a restricted Lie superalgebra in characteristic $p > 2$, then the above argument will apply provided we show that θ also respects the p th power map $^{[p]} : L_0 \rightarrow L_0$. To this end, let $x \in L_0$ and note that $x^p = x^{[p]}$ in the restricted enveloping algebra $U(L)$. Thus

$$\theta(x^{[p]}) = -(x^{[p]})^{\text{op}} = -(x^p)^{\text{op}} = (-x^{\text{op}})^p = \theta(x)^p,$$

and θ extends to an algebra homomorphism $\theta : U(L) \rightarrow U(L)^{\text{op}}$, as required. \square

We remark that any anti-automorphism τ of a ring R permutes its ideals, preserving inclusion, and maps primes to primes. Furthermore, if τ commutes with an automorphism σ , then τ permutes the σ -stable ideals of R and hence it maps σ -primes to σ -primes. With this observation, we have

Theorem 1.2. *Let L be a (restricted) Lie superalgebra over the field K of characteristic $\neq 2$, let $U(L)$ denote its (restricted) enveloping algebra, and assume that $i = \sqrt{-1} \in K$. If $U(L)$ has a unique minimal graded-prime ideal P , then P is the unique minimal prime of $U(L)$.*

Proof. Let σ be the automorphism of $U(L)$ of order 2 associated with the grading and, since $i = \sqrt{-1} \in K$, let τ be defined as in the preceding lemma. Then τ commutes with $\tau^2 = \sigma$, so τ permutes the graded-prime ideals of $U(L)$ and hence τ stabilizes P , the unique minimal graded-prime. In particular, τ gives rise to an anti-automorphism of the σ -prime ring $R = U(L)/P$. Since σ is an automorphism of R of order ≤ 2 , [P, page 133] implies that R has at most two minimal primes, say Q and Q^σ , and that $Q \cap Q^\sigma = 0$. But then the cyclic group $\{1, \tau, \tau^2, \tau^3\}$ permutes these ≤ 2 minimal primes, and hence $\tau^2 = \sigma$ must act trivially as a permutation. Thus $Q = Q^\sigma$ and $0 = Q \cap Q^\sigma = Q$. This means that R is a prime ring and hence that P is a prime ideal of $U(L)$. Finally, if T is any prime ideal of $U(L)$, then $T \cap T^\sigma$ is σ -prime, so $T \cap T^\sigma \supseteq P$ and consequently $T \supseteq P$. Thus P is the unique minimal prime of $U(L)$. \square

Note that [Bl] asks whether the enveloping algebra $U(L)$ of a finite-dimensional Lie superalgebra L must necessarily have a unique minimal prime. In view of the above, we can essentially obtain an affirmative answer here by showing that $U(L)$ has a unique minimal graded-prime ideal. For example, in the following result, we are able to slightly sharpen the conclusion of [L, Theorem I(i)] concerning minimal graded-prime ideals when L is solvable.

Corollary 1.3. *Let L be a finite-dimensional solvable Lie superalgebra over a field K of characteristic 0. Then $U(L)$ has a unique minimal prime ideal.*

Proof. If F is the algebraic closure of K , then [L, Theorem I(i)] and the preceding result imply that $F \otimes U(L) \cong U(F \otimes L)$ has a unique minimal prime, say Q . In particular, since $F \otimes U(L)$ is Noetherian, it follows that Q is nilpotent. Now $U(L)$ is also Noetherian, so there exist minimal primes P_1, P_2, \dots, P_n of $U(L)$, not necessarily distinct, with $P_1 P_2 \cdots P_n = 0$. Lifting these to ideals of $F \otimes U(L)$, we have

$$(F \otimes P_1)(F \otimes P_2) \cdots (F \otimes P_n) = 0 \subseteq Q$$

and therefore $Q \supseteq F \otimes P_k \supseteq P_k$ for some subscript k . But then P_k is also nilpotent, and this easily implies that P_k is the unique minimal prime of $U(L)$. \square

This, of course, extends [KK, Theorem 3.8] and indicates why the following result may be of interest.

Lemma 1.4. *Let $L = L_0 \oplus L_1$ be a finite-dimensional ordinary Lie superalgebra and assume that $U(L)$ has a unique minimal prime ideal. Then every non-nilpotent ideal of $U(L)$ has a nonzero intersection with $U(L_0)$.*

Proof. If T denotes the set of nonzero elements of $U(L_0)$, then clearly T is a multiplicatively closed set of regular elements of $U(L)$. Furthermore, by [Bh, Section 4], T is a right Ore set of $U(L)$ and $U(L)T^{-1}$ is the Artinian right classical quotient ring of the enveloping algebra. Now let N denote the unique minimal prime ideal of $U(L)$, so that N is also the unique largest nilpotent ideal of this Noetherian ring. If I is a non-nilpotent ideal of $U(L)$, then $(I + N)/N$ is a nonzero ideal of the prime Noetherian ring $U(L)/N$, and hence I contains an element c which is regular modulo N . Indeed, [Sm, Theorem 2.12] implies that c is regular in $U(L)$ and hence invertible in $U(L)T^{-1}$. Since the right ideal IT^{-1} contains an invertible element, it follows that $1 \in IT^{-1}$, so $I \cap T \neq \emptyset$ and $I \cap U(L_0) \neq 0$. \square

Another consequence of Theorem 1.2 is a slight sharpening of [BP, Corollary 1.3]. If L is a Lie superalgebra, recall that Δ_L is defined by

$$\Delta_L = \{ \ell \in L \mid \dim_K \ell \cdot \text{ad} U(L) < \infty \}.$$

Then [BP, Lemma 2.1] implies that Δ_L is a characteristic (restricted) Lie superideal of L and that it is the join of all finite-dimensional superideals of L . The following is an immediate consequence of Theorem 1.2 and [BP, Corollary 1.3(ii)].

Corollary 1.5. *Let L be a Lie superalgebra over a field K of characteristic $\neq 2$ and suppose that $i = \sqrt{-1} \in K$. If $\text{char } K = p > 2$, assume further that L is restricted. Then $U(L)$ is prime if and only if $U(\Delta_L)$ is graded L -prime.*

We use this to prove

Corollary 1.6. *Let L be a solvable Lie superalgebra over a field K of characteristic 0. Then $U(L)$ is prime if and only if it is semiprime.*

Proof. Since prime always implies semiprime, we are concerned here with the reverse implication. Thus we suppose that $U(L)$ is semiprime, and we first consider the case where $i = \sqrt{-1} \in K$.

Suppose A and B are nonzero graded L -stable ideals of $U(\Delta_L)$ with $AB = 0$. Since Δ_L is generated by the finite-dimensional superideals of L , there exists such an ideal I of L with $\tilde{A} = A \cap U(I) \neq 0$ and $\tilde{B} = B \cap U(I) \neq 0$. Now I is finite dimensional and solvable, so Corollary 1.3 implies that the Noetherian ring $U(I)$ has a unique minimal prime which is nilpotent. In particular, since $\tilde{A}\tilde{B} = 0$, we see that one of \tilde{A} or \tilde{B} , say \tilde{A} , is nilpotent. But then \tilde{A} is a nonzero graded L -stable nilpotent ideal of $U(I)$, so it is clear that $\tilde{A}U(L)$ is a nonzero nilpotent ideal of $U(L)$, a contradiction. It follows that $U(\Delta_L)$ is graded L -prime, and we conclude from the preceding corollary that $U(L)$ is prime.

Finally, let K be an arbitrary field of characteristic 0 and set $F = K[i]$. Then F/K is a finite Galois extension, so it follows that $F \otimes U(L) = U(F \otimes L)$ is also semiprime. Consequently, by the above, $F \otimes U(L)$ is prime, and hence so is $U(L)$. □

Note that the preceding argument really only requires the fact that each finite-dimensional superideal I of L is solvable. Indeed, we need only assume that, for each such I , $U(I)$ has a unique minimal prime ideal. We remark that if $H \triangleleft L$ and if $U(L)$ is semiprime, then it is not necessarily true that $U(H)$ is also semiprime. This occurs because superderivations need not stabilize minimal primes even if $\text{char } K = 0$. For instance, we have

Example 1.7. If K is any field of characteristic $\neq 2$, then there exists a metabelian (restricted) Lie superalgebra L over K such that $U(L)$ is prime while $U(\Delta_L)$ is not even semiprime.

Proof. Define $L = L_0 \oplus L_1$ so that L_0 has K -basis $\{a, b\}$ and L_1 has K -basis $\{c, d_0, d_1, d_2, \dots\}$. Furthermore, assume that $[c, d_0] = a$, $[b, d_j] = d_{j+1}$ for all $j \geq 0$, and that all other Lie brackets of basis elements are zero. Notice that $A = Ka + Kc$ and $B = Kb + Kd_0 + Kd_1 + Kd_2 + \dots$ are super-subalgebras of L and that $L = A \rtimes B$. Indeed, $B' = Kb + Kd_1 + Kd_2 + \dots$ is a superideal of B and $B \rightarrow B/B' \cong Kd_0 \rightarrow \text{Der}(A)$ describes the action of B on A . With this observation, it is clear that L exists, and then L is metabelian since $[L, L] = Ka + Kd_1 + Kd_2 + \dots$ is abelian.

Now $A \subseteq \Delta_L$ and B has no nonzero finite-dimensional super ideals, so $A = \Delta_L$ and $U(\Delta_L) = K[a, c \mid c^2 = 0]$ is not semiprime. On the other hand, the \mathbb{Z}_2 -grading of this algebra is given by $U(\Delta_L) = K[a] \oplus K[a]c$ and, since $[c, d_0] = a$, it follows that any nonzero graded L -stable ideal of this ring meets $K[a]$ nontrivially. In particular, $U(\Delta_L)$ is graded L -prime, and if we now suppose that $i = \sqrt{-1} \in K$, then Corollary 1.5 implies that $U(L)$ is prime when $\text{char } K = 0$.

If $\text{char } K = p > 2$, let \tilde{L} be the K -subspace of $U(L)$ with K -basis given by $\{a^{p^j}, b^{p^j}, c, d_j \mid j \geq 0\}$. Then we know that \tilde{L} is a restricted Lie superalgebra and that its restricted enveloping algebra $U(\tilde{L})$ is precisely equal to $U(L)$. Furthermore,

$\Delta_{\tilde{L}}$ has K -basis $\{a, a^p, a^{p^2}, \dots, c\}$, so $U(\Delta_{\tilde{L}}) = U(\Delta_L) = K[a, c \mid c^2 = 0]$ is graded \tilde{L} -prime. Thus, Corollary 1.5 again implies that $U(\tilde{L}) = U(L)$ is prime.

Finally, if K is arbitrary, set $F = K[i]$. Then we conclude from the preceding argument applied to $F \otimes L$ that $F \otimes U(L) = U(F \otimes L)$ is prime, and consequently $U(L)$ is also prime. \square

§2. COLOR ALGEBRAS

Let G be a finite abelian group and let $\epsilon : G \times G \rightarrow K^\bullet$ be a bicharacter. If L is a Lie color algebra associated with G and ϵ , then $L = \oplus \sum_{g \in G} L_g$ is a G -graded K -vector space and there is a K -bilinear map $[\cdot, \cdot] : L \times L \rightarrow L$ which satisfies $[L_x, L_y] \subseteq L_{xy}$ for all $x, y \in G$. As usual, the support of L is defined by

$$\text{supp } L = \{g \in G \mid L_g \neq 0\}$$

and we can certainly assume that $G = \langle \text{supp } L \rangle$ since the elements of G outside of this supporting subgroup have no effect on the structure of L or its enveloping algebra. Now each $g \in G$ gives rise to a linear character $\lambda_g \in \widehat{G} = \text{Hom}(G, K^\bullet)$ defined by $\lambda_g(x) = \epsilon(g, x)$, and it is clear that the map $\Lambda : G \rightarrow \widehat{G}$ given by $g \mapsto \lambda_g$ is a group homomorphism. Again, we can assume that Λ is one-to-one, since if $N = \ker \Lambda$ and $\bar{G} = G/N$, then ϵ determines a bicharacter $\bar{\epsilon} : \bar{G} \times \bar{G} \rightarrow K^\bullet$ and L is naturally a Lie color algebra associated with \bar{G} and $\bar{\epsilon}$. For convenience, we say that G is minimal for L if $G = \langle \text{supp } L \rangle$ and $\Lambda : G \rightarrow \widehat{G}$ is one-to-one. In this case, it is clear that $G \cong \widehat{G}$.

Recall that G is the disjoint union of the subsets $G_+ = \{g \in G \mid \epsilon(g, g) = 1\}$ and $G_- = \{g \in G \mid \epsilon(g, g) = -1\}$. Here G_+ is a subgroup of G of index ≤ 2 and G_- is the remaining coset when the index is 2. If L is an ordinary Lie superalgebra we let $U(L)$ denote its universal enveloping algebra, and we define the infinite support of L by

$$\text{supp}_\infty L = \{g \in G_+ \mid L_g \neq 0\} \cup \{g \in G_- \mid \dim_K L_g = \infty\}.$$

Of course, L is always an ordinary algebra when $\text{char } K = 0$.

On the other hand, if $\text{char } K = p > 0$, then L may have the additional structure of a restricted color algebra. This means that there exists a p th power map

$$[\cdot]^{[p]} : L_g \rightarrow L_{g^p} \quad \text{for all } g \in G_+,$$

and here we let $U(L)$ denote the restricted enveloping algebra of L . Furthermore, in this context the infinite support of L is defined by

$$\text{supp}_\infty L = \{g \in G \mid \dim_K L_g = \infty\}.$$

See [BMPZ] or [S] for a more complete description of Lie color algebras and their enveloping algebras.

Now let R be any K -algebra and let G be a finite abelian group with $G \cong \widehat{G} = \text{Hom}(G, K^\bullet)$. If $R = \oplus \sum_{g \in G} R_g$ is G -graded, then the dual group \widehat{G} acts as automorphisms on R by defining

$$\left(\sum_{g \in G} r_g\right)^\lambda = \sum_{g \in G} \lambda(g)r_g$$

for any $\lambda \in \widehat{G}$. Conversely, if \widehat{G} acts on R , then R is G -graded by the eigenspaces

$$R_g = \{r \in R \mid r^\lambda = \lambda(g)r \text{ for all } \lambda \in \widehat{G}\},$$

which are indexed by the elements $g \in G$. Indeed, these are equivalent structures on R , since $G \cong \widehat{G}$ implies that for any two distinct elements $x, y \in G$ there exists $\lambda \in \widehat{G}$ with $\lambda(x) \neq \lambda(y)$.

Suppose, in addition, that R is a semiprime K -algebra, and let $C = \mathcal{C}(R)$ denote its extended centroid (see [A] or [M]). If R is G -graded, then we know that the action of \widehat{G} on R extends naturally to an action on the central closure RC of R . Hence the G -grading on R extends to a grading on RC with C a graded subring. Note that the assumption $G \cong \widehat{G}$ holds when $R = U(L)$ and G is minimal for L . Our goal in this section is to study the grading on the extended centroid of $U(L)$ when this enveloping algebra is semiprime, and following [W, Proposition 2.8] we define

$$\Delta_L = \{ \ell \in L \mid \dim_K \ell \cdot \text{ad}U(L) < \infty \}.$$

Then Δ_L is a characteristic (restricted) Lie color ideal of L which “controls” the linear identities of $U(L)$. We use this fact to prove

Lemma 2.1. *Assume that $R = U(L)$ is semiprime and let c be a nonzero element of the extended centroid $C = \mathcal{C}(R)$. Then there exists a finite-dimensional graded subspace V of R , which is stable under the adjoint action of $U(L)$ on $U(L)$, such that $0 \neq vc \in R$ for some $v \in V$.*

Proof. Let us first assume that either $\text{char } K = 0$, or $\text{char } K = p > 0$ and that L is restricted. If $I = \{ r \in R \mid rc \in R \}$, then the definition of C implies that I is an ideal of R with zero annihilator in R and hence in RC . In particular, $Ic \neq 0$ and we fix $\alpha \in I$ with $0 \neq \beta = \alpha c \in R$. Now let r be any element of I and set $s = rc \in R$. If $x \in R$, then we know that $xc = cx$ and, upon multiplying this equation on the left by α and on the right by r , we obtain

$$\alpha xs = \beta xr \quad \text{for all } x \in R,$$

a linear identity in R . In particular, since $\beta \neq 0$, this linear identity and [W, Theorem 4.8] imply that there exist elements α' and $\beta' \neq 0$ in $U(\Delta_L)$, depending only upon α and β , such that $\alpha' s = \beta' r$. But $s = rc = cr$, so this yields $(\alpha' c - \beta') r = 0$ and, since $r \in I$ is arbitrary, we have $(\alpha' c - \beta') I = 0$ and hence $\alpha' c = \beta'$.

Now, by definition of Δ_L and $U(\Delta_L)$, it follows that there exists a finite-dimensional $\text{ad}U(L)$ -stable graded subspace H of L such that $\alpha' \in (K + H)^n$ for some integer $n \geq 0$. Thus $V = (K + H)^n$ is a finite-dimensional $\text{ad}U(L)$ -stable graded subspace of $U(L)$, and $\alpha' = v \in V$ satisfies $vc = \alpha' c = \beta' \in R \setminus 0$.

Finally, suppose that L is an ordinary Lie color algebra in characteristic $p > 0$ and let \tilde{L} be the restricted Lie color subalgebra of $U(L)$ generated by L . Then it is known that $U(\tilde{L})$, the restricted enveloping algebra of \tilde{L} , is precisely equal to $U(L)$. Thus, the result of the preceding paragraph applied to $U(\tilde{L})$ yields the corresponding result for $U(L)$. □

We remark that if c above is assumed to be homogeneous, then we can find an appropriate finite-dimensional graded subspace V with $0 \neq Vc \subseteq R$. This is essentially contained in the proof of the following result.

Theorem 2.2. *Let L be a (restricted) G -graded Lie color algebra over the field K and assume that G is minimal for L . Suppose that the (restricted) enveloping algebra $U(L)$ is semiprime and let C denote its extended centroid. If G is generated by the infinite support of L , then $C = C_1$ is homogeneous of grade $1 \in G$.*

Proof. Suppose, by way of contradiction, that $C_x \neq 0$ for some $1 \neq x \in G$ and choose $0 \neq c \in C_x$. By the previous lemma, let V be a finite-dimensional graded subspace of $R = U(L)$, stable under the adjoint action of $U(L)$, such that $0 \neq vc \in R$ for some $v \in V$. If $W = \{w \in V \mid wc \in R\}$, then, since R is a graded subring of RC and since c is homogeneous, it follows that W is a graded subspace of V . Furthermore, since $v \in W$, we see that $Wc = \sum_{g \in G} W_g c$ is a nonzero finite-dimensional graded subspace of R . Consequently, we can choose $\alpha \in W$ to be a homogeneous element such that $\deg \alpha c \geq \deg wc$ for all $w \in W$.

Since G is minimal for L and $x \neq 1$, we know that $1 \neq \lambda_x \in \widehat{G}$. In particular, $\ker \lambda_x = \{g \in G \mid \lambda_x(g) = 1\}$ is a proper subgroup of G and hence it cannot contain all members of $\text{supp}_\infty L$, which, by hypothesis, generates G . Thus there exists $y \in \text{supp}_\infty L$ with $1 \neq \lambda_x(y) = \epsilon(x, y)$. Let ℓ be any element of L_y and, for convenience, use ${}^\ell$ to denote the adjoint action of ℓ .

Since α is homogeneous, $c \in C_x$ and $\ell \in L_y$, we have $(\alpha c)^\ell = \alpha^\ell c + k' \alpha c^\ell$ for some nonzero $k' \in K$ depending upon the grades of α and ℓ . Furthermore, since c and ℓ commute, we have

$$c^\ell = \ell c - \epsilon(x, y) c \ell = (1 - \epsilon(x, y)) c \ell = k'' c \ell,$$

where $k'' = 1 - \epsilon(x, y)$ is a nonzero element of K since $\epsilon(x, y) = \lambda_x(y) \neq 1$. Thus, setting $k = -k' k'' \in K \setminus 0$, we have

$$\alpha^\ell c = (\alpha c)^\ell - k' \alpha c^\ell = (\alpha c)^\ell + k(\alpha c) \ell \in R.$$

In particular, since $\alpha^\ell \in V^\ell \subseteq V$, we have $\alpha^\ell \in W$. Now $\deg(\alpha c)^\ell \leq \deg \alpha c$ is always true and $\deg \alpha^\ell c \leq \deg \alpha c$ follows from the maximality of $\deg \alpha c$. Thus the equation $k(\alpha c) \ell = \alpha^\ell c - (\alpha c)^\ell$ implies that $\deg(\alpha c) \ell \leq \deg \alpha c$ for all $\ell \in L_y$.

Since αc is a nonzero element of $U(L)$, there exists a finite-dimensional graded subspace H of L with $\alpha c \in (K + H)^n$ for some $n \geq 0$. In particular, if $\dim_K L_y = \infty$, we can choose $\ell \in L_y \setminus H$, and consequently the PBW theorem implies that $\deg(\alpha c) \ell = 1 + \deg \alpha c > \deg \alpha c$, a contradiction. On the other hand, if $\dim_K L_y$ is finite, then the definition of $\text{supp}_\infty L$ implies that L is an ordinary Lie superalgebra and that $y \in G_+$. In this case, we know that $L_y \neq 0$ and, if $0 \neq \ell \in L_y$, then the PBW theorem again yields $\deg(\alpha c) \ell = 1 + \deg \alpha c > \deg \alpha c$, a contradiction. Thus the assumption that $C_x \neq 0$ for some $1 \neq x \in G$ is false, and the theorem is proved. \square

The preceding argument actually yields information on the allowable support of C when G is not necessarily generated by $\text{supp}_\infty L$. Specifically, it shows that if $x \in G$ satisfies $1 \neq \lambda_x(\text{supp}_\infty L) = \epsilon(x, \text{supp}_\infty L)$, then the homogeneous component C_x must be zero.

Again, let R be a G -graded K -algebra with $G \cong \widehat{G}$. Then any subspace V of R is \widehat{G} -stable if and only if it is G -graded. In particular, R is \widehat{G} -prime if and only if it is graded-prime. Furthermore, when this occurs, then R is \widehat{G} -semiprime and hence also semiprime since \widehat{G} is finite. The following observation is well-known.

Lemma 2.3. *Let R be a G -graded K -algebra with $G \cong \widehat{G}$, and assume that R is graded-prime. If I is a nonzero ideal of R , then there exist a homogeneous element $a \in R$ and an element $c \in C = \mathcal{C}(R)$ with $0 \neq ac \in I$.*

Proof. By assumption, $R = \bigoplus_{g \in G} R_g$ is G -graded, and we can choose $0 \neq \alpha = \sum_{g \in S} \alpha_g \in I$ with support $S \subseteq G$ and with $|S|$ minimal. Fix $h \in S$ and let $x \in R_w$

for some $w \in G$. Then

$$\beta = \alpha_h x \alpha - \alpha x \alpha_h = \sum_{g \in S} (\alpha_h x \alpha_g - \alpha_g x \alpha_h) \in I,$$

and note that $\alpha_h x \alpha_g - \alpha_g x \alpha_h \in R_{hwg}$ since G is commutative. Furthermore, the $g = h$ summand vanishes here, and thus $\beta \in I$ has support contained in $hw(S \setminus \{h\})$, a set of size smaller than $|S|$. Thus $\beta = 0$, and indeed

$$\alpha_h x \alpha_g - \alpha_g x \alpha_h = 0$$

for all $g \in S$ and all $x \in R_w$. Since $R = \sum_{w \in G} R_w$, the above identity clearly holds for all $x \in R$.

Finally, note that the ideals $R\alpha_h R$ and $R\alpha_g R$ are nonzero and graded, and hence they have trivial annihilators in R since R is graded-prime. With this observation, it follows from the linear identity that, for each $g \in S$, there exists $c(g) \in C$ with $\alpha_g = \alpha_h c(g)$. (See, for example, the proof of [M, Theorem 1].) Thus

$$\alpha = \sum_{g \in S} \alpha_g = \alpha_h \sum_{g \in S} c(g) = ac$$

is a nonzero element of I of the required form. \square

As a consequence, we have

Theorem 2.4. *Let L be a (restricted) G -graded Lie color algebra over the field K and assume that G is minimal for L . If the (restricted) enveloping algebra $U(L)$ is graded-prime and $G = \langle \text{supp}_\infty L \rangle$, then $U(L)$ is prime.*

Proof. As we observed, $R = U(L)$ is semiprime since it is graded-prime, and thus we can let $C = \mathcal{C}(R)$ denote its extended centroid. Furthermore, since $G = \langle \text{supp}_\infty L \rangle$, Theorem 2.2 implies that $C = C_1$ is homogeneous. Now let I be any nonzero ideal of R . Then the preceding lemma implies that there exist a homogeneous element $a \in R$ and an element $c \in C$ with $0 \neq ac \in I$. But $C = C_1$, so c is also homogeneous and ac is a nonzero homogeneous element of I . Thus I contains the nonzero graded ideal $RacR$ which has zero annihilator in the graded-prime ring R . It follows that I also has zero annihilator in R , and therefore $R = U(L)$ is a prime ring. \square

We remark that some hypothesis is needed in the preceding theorem, since graded-prime enveloping algebras are not necessarily prime. Indeed, we mention

Example 2.5 ([Pr]). For any field K of characteristic $\neq 2$ there exists a Lie color algebra L , graded by the fours group, such that $U(L)$ is a commutative algebra which is graded-prime but not prime.

Proof. Let $G = \{1, x, y, xy\} \cong \{1, x\} \times \{1, y\}$ be the fours group and let K be a field of characteristic $\neq 2$. If $\epsilon : G \times G \rightarrow K^\bullet$ is the product of the super-bicharacters defined on $\{1, x\}$ and $\{1, y\}$, then $\epsilon(x, x) = \epsilon(y, y) = -1$ and $\epsilon(x, y) = 1$. Now let L be the G -graded Lie color algebra with K -basis $\{a, b, c\}$ such that $L_x = Ka$, $L_y = Kb$, $L_1 = Kc$ and $L_{xy} = 0$. Furthermore, suppose that $[a, a] = [b, b] = 2c$ and that all other Lie products among the basis elements are 0. Then it is easy to check that L is indeed a Lie color algebra. Moreover, $U(L)$ is generated by a, b, c with $a^2 = b^2 = c$ and $ab = ba$, since $[a, b] = 0$ and $\epsilon(x, y) = 1$. Thus the PBW theorem implies that $R = U(L) = K[a, b, c \mid a^2 = b^2 = c]$. Clearly R is a commutative ring, which is not prime since $a \neq \pm b$ but $0 = a^2 - b^2 = (a - b)(a + b)$. On the other hand, the grading of R is given by $R_1 = K[c]$, $R_x = K[c]a$, $R_y = K[c]b$ and

$R_{xy} = K[c]ab$. Consequently, each nonzero homogeneous element of R is regular in R , and hence R is graded-prime.

Note that the above mentioned ring is an ordinary enveloping algebra. However, when $\text{char } K = p > 2$, then the same ring R is also the restricted enveloping algebra of the restricted Lie color algebra \tilde{L} with K -basis $\{a, b, c, c^p, c^{p^2}, \dots\}$. Thus there also exist restricted enveloping algebras which are graded-prime but not prime. \square

Finally, Theorem 2.4 allows us to slightly sharpen [W, Theorem 5.4(ii)] when G is generated by the infinite support of L . Specifically, we have

Corollary 2.6. *Let L be a G -graded Lie color algebra over the field K , and assume that G is minimal for L and that $G = \langle \text{supp}_\infty L \rangle$. Furthermore, let $U(L)$ denote the enveloping algebra of L if $\text{char } K = 0$ or the restricted enveloping algebra of the restricted color algebra L if $\text{char } K = p > 0$. Then $U(L)$ is prime if and only if $U(\Delta_L)$ is graded L -prime.*

REFERENCES

- [A] S. A. Amitsur, *On rings of quotients*, Symposia Math. **8** (1972), 149–164. MR **48**:11180
- [BMPZ] Y. Bahturin, A. Mikhalev, V. Petrogradsky, and M. Zaicev, *Infinite Dimensional Lie Superalgebras*, Walter de Gruyter, Berlin, New York, 1992. MR **94b**:17001
- [BP] J. Bergen and D. S. Passman, *Delta methods in enveloping algebras of Lie superalgebras II*, J. Algebra **166** (1994), 568–610. MR **95j**:17007
- [Bh] E. J. Behr, *Enveloping algebras of Lie superalgebras*, Pacific J. Math. **130** (1987), 9–25. MR **89b**:17023
- [Bl] A. D. Bell, *A criterion for primeness of enveloping algebras of Lie superalgebras*, J. Pure Appl. Algebra **69** (1990), 111–120. MR **92b**:17014
- [KK] E. Kirkman and J. Kuzmanovich, *Minimal prime ideals in enveloping algebras of Lie superalgebras*, Proc. AMS **124** (1996), 1693–1702. MR **96h**:16027
- [L] E. S. Letzter, *Prime and primitive ideals in enveloping algebras of solvable Lie superalgebras*, Contemporary Math. **130** (1992), 237–255. MR **93i**:17003
- [M] W. S. Martindale, III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584. MR **39**:257
- [P] D. S. Passman, *Infinite Crossed Products*, Academic Press, Boston, 1989. MR **90g**:16002
- [Pr] K. L. Price, *Private communication*, October, 1996.
- [S] M. Scheunert, *The Theory of Lie Superalgebras*, Springer Lecture Notes in Mathematics, Vol. 716, Springer-Verlag, Berlin, 1979. MR **80i**:17005
- [Sm] L. W. Small, *Orders in Artinian rings*, J. Algebra **4** (1966), 13–41. MR **34**:199
- [W] M. C. Wilson, *Delta methods in enveloping algebras of Lie colour algebras*, J. Algebra **175** (1995), 661–696. MR **97a**:17013

DEPARTMENT OF MATHEMATICS, DEPAUL UNIVERSITY, CHICAGO, ILLINOIS 60614

E-mail address: jbergen@condor.depaul.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706

E-mail address: passman@math.wisc.edu