THERE IS A PARACOMPACT Q-SET SPACE IN ZFC

ZOLTAN T. BALOGH

(Communicated by Franklin D. Tall)

Abstract. We construct a paracompact space $QX$ such that every subset of $QX$ is an $F_\sigma$-set, yet $QX$ is not $\sigma$-discrete. We will construct our space not to have a $G_\delta$-diagonal, which answers questions of A.V. Arhangel’ski˘ı and D. Shakhmatov on cleavable spaces.

Introduction

In this paper we will construct a hereditarily paracompact, perfectly normal Q-set space $QX$ without a quasi-$G_\delta$-diagonal. $QX$ answers questions on Q-set spaces, and on cleavable spaces of A.V. Arhangel’ski˘ı.

A topological space $X$ is a Q-set space [B] if every subset of $X$ is a $G_\delta$-set and $X$ is not $\sigma$-discrete. H. Junnila [J] (and Bregman-Shapirovskii-Soštak) asked whether there were any Q-set spaces in ZFC. This problem was answered affirmatively for regular Q-set spaces, and the question was raised whether there are (perfectly) normal Q-set spaces [B]. In this paper we shall combine the technique of the regular examples with a new inductive method to show not only that the answer is yes, but that one can also construct paracompact examples.

A.V. Arhangel’ski˘ı and D.B. Shakhmatov [AS], [A1] raised the question whether every cleavable space has a $G_\delta$-diagonal. Arhangel’ski˘ı [A2] also asked whether spaces cleavable over the rationals had to be $\sigma$-discrete or had to possess $G_\delta$-diagonals. Since normal Q-set spaces are cleavable and also cleavable over the rationals [A2], and our space $QX$ will be constructed not to have a $G_\delta$-diagonal, it settles all of the above questions in the negative. (It should be pointed out here, that a Q-space is defined in [A2] to be a space whose every subset is an $F_\sigma$-set. Thus, Q-set spaces are precisely the non-$\sigma$-discrete Q-spaces).

$QX$ will have cardinality $\mathfrak{c}$, which is necessary only to make it not have a $G_\delta$-diagonal. If we only want to construct a paracompact Q-set space, then it can be done on $\mathfrak{c}$ (Theorem 2.1).

Terminology and notation. We use the standard terminology and notation of set-theoretic topology (see [KV]). $\pi$ will always denote first projection, i.e. $\pi A = \{a: \text{there is } b \text{ with } (a,b) \in A\}$. A sequence of $\langle G_m \rangle_{m \in \omega}$ of families of open subsets of a space $X$ is said to be a quasi-$G_\delta$-diagonal, if for every $x \in X$, $\bigcap \{st(x, G_m): m \in \omega \text{ and } x \in \bigcup G_m \} = \{x\}$.

Received by the editors August 24, 1995.
1991 Mathematics Subject Classification. Primary 54Dxx.
Key words and phrases. Paracompact, Q-set space, $G_\delta$-diagonal, cleavable.
Research supported by NSF Grant DMS-9108476.
1. The space $QX$

**Theorem 1.1.** There is a (hereditarily) paracompact, perfectly normal $Q$-set space $QX$ without a quasi-$G_δ$-diagonal.

**Proof.** The underlying set of $QX$ is $c^+$, the first cardinal bigger than the continuum $c$. The topology of $QX$ will be inductively defined in $\lambda = 2^{c^+}$ steps. For the purposes of making every subset of $QX$ a $G_δ$-set, let $\langle Y_ξ \rangle_{ξ < λ}$ be a one-to-one listing of all subsets of $c^+$. Also, let $\langle U_ξ \rangle_{ξ < λ}$ be a list of all subsets of $c^+ \times c^+$ such that $U_0 = \emptyset$ and each subset is listed $λ$ times. This second list will, in particular, mention codes for all future open covers of $QX$. If such an open cover first occurs at step $ξ$, then we'll add a clopen partition refining that cover to the topology of $QX$. To carry out the program above we shall define, by induction of $ξ < λ$,

(a) a function $g_ξ : c^+ → ω + 1$;
(b) a number $h(ξ) ∈ \{0, 1\}$;
(c) a function $w_ξ : c^+ → c^+ \setminus ω$ if $h(ξ) = 1$.

We will set

(a’) $G_ξ = \{α < c^+ : g_ξ(α) ≥ n\}$ for every $n ∈ ω$;
(b’) $H = \{ξ < λ : h(ξ) = 1\}$;
(c’) $W_ξ = \{α < c^+ : w_ξ(α) = ρ\}$ for every $ξ ∈ H$ and $ρ$ with $ω ≤ ρ < c^+$. □

A subbase for the topology $τ_Q$ of $QX$ will be

$$B = \{G_ξ : ξ < λ, n ∈ ω\} \cup \{W_ξ : ξ ∈ H \text{ and } ρ ∈ c^+ \setminus ω\}.$$ 

Adding the $G_ξ$’s will make every subset of $X$ a $G_δ$-set. $\{W_ξ : ρ ∈ c^+ \setminus ω\}$ will be a clopen partition refining of the open cover coded by rows $ω ≤ ρ < c^+$ of $U_ξ ⊂ c^+ × c^+$ if $ξ ∈ H$.

In order to make sure that $QX$ does not have a quasi-$G_δ$-diagonal we will need the concept of a control pair. We will say that $⟨A, u⟩$ is a control pair if

(C-1) $A ∈ [c^+]^ω$;
(C-2) $u = ⟨u_0, u_1, u_2⟩$, and $u_0$, $u_1$, $u_2$ are functions with domain $A$;
(C-3) for every $α ∈ A$, $u_0(α) ∈ [P(A) × ω]^{<ω}$, $u_1(α) ∈ [P(A × A)]^{<ω}$ and $u_2(α) ∈ [P(A × A) \setminus u_1(α)]^{<ω}$;
(C-4) if $α, α' ∈ A$ and $α ≠ α'$, then $πu_0(α) ∩ πu_0(α') = φ$ and $u_1(α) ∩ u_1(α') = φ$.

(Note that $πu_0(α) = \{B ⊂ A \text{ there is an } n ∈ ω \text{ with } ⟨B, n⟩ ∈ u_0(α)\}$).

Roughly speaking, $⟨A, u⟩$ will code a countable approximation to a neighborhood assignment in $QX$. Let $⟨A_β, u_β⟩_{β < c^+}$ list all control pairs, mentioning each $c^+$ times.

The last ingredient we need is the notion of an initially $ξ$-open set. A subset $E ⊂ c^+$ will be called initially $ξ$-open, if $E$ is an open subset in the topology generated by

$$B_ξ = \{X\} \cup \{G_η : η < ξ, n ∈ ω\} \cup \{W_η : η < ξ, h(η) = 1 \text{ and } ρ ∈ c^+ \setminus ω\}.$$ 

For every $ξ < λ$ and $ρ ∈ c^+ \setminus ω$, let $U_ξ = \{γ < c^+ : ⟨γ, ρ⟩ ∈ U_ξ\}$.

We are going to construct $g_ξ$, $h(ξ)$ and $w_ξ$ (if $h(ξ) = 1$) in such a way that the following hypotheses are satisfied:

(1$ξ$) for every $β < c^+$, $g_ξ(β) = ω$ iff $β ∈ Y_ξ$;
(2$ξ$) if $α < β < c^+$ and $⟨Y_ξ ∩ A_β, n⟩ ∈ u_0(α)$, then $g_ξ(β) ≥ n$;
\((3_\xi)\) \(h(\xi) = 1\) if and only if \(\langle U_{\xi,\rho}^{c^+} \rangle_{\rho \in c^+ \setminus \omega}\) is a cover of \(c^+\) consisting of initially \(\xi\)-open sets and there is no \(\xi' < \xi\) such that \(U_{\xi'} = U_\xi\) and \(\langle U_{\xi',\rho}^{c^+} \rangle_{\rho \in c^+ \setminus \omega}\) is a cover by initially \(\xi'\)-open sets;

\((4_\xi)\) if \(h(\xi) = 1\), then
(a) for every \(\beta < c^+\), \(\beta \in U_{\xi,\omega_1}(\beta)\);
(b) if \(\alpha < \beta < c^+\), \(U_\xi \cap (A_\beta \times A_\beta) \in u_{13}(\alpha)\) and \(\beta \in U_{\xi,\omega_1}(\alpha)\), then \(w_\xi(\beta) = w_\xi(\alpha)\);
(c) if \(\alpha < \beta < c^+\), \(U_\xi \cap (A_\beta \times A_\beta) \in u_{23}(\alpha)\) and \(\beta \in U_{\xi,\omega_1}(\alpha)\), then there is an \(\alpha' \in A_\beta\) such that \(w_\xi(\beta) = w_\xi(\alpha')\).

Let us now pass to the construction. Suppose that \(\xi < \lambda\) and that we are done for \(\eta < \xi\).

We are going to define \(g_\xi(\beta) \in \omega + 1\) by induction on \(\beta < c^+\). Suppose we are done for every \(\alpha < \beta\). We split the definition into two cases.

**Case 1.** Suppose that there is no \(\alpha < \beta\) and \(n \in \omega\) with \(\langle Y_\xi \cap A_\beta, n \rangle \in u_{03}(\alpha)\). Then let \(g_\xi(\beta) = \omega\) if \(\beta \in Y_\xi\), and \(g_\xi(\beta) = 0\) if \(\beta \notin Y_\xi\).

**Case 2.** Suppose now that there is an \(\alpha < \beta\) such that \(\langle Y_\xi \cap A_\beta, n \rangle \in u_{03}(\alpha)\) for some \(n \in \omega\). By (C-4), there is only one such \(\alpha\); furthermore, since \(u_{03}(\alpha)\) is a finite set, there are only finitely many such \(n \in \omega\). Set \(g_\xi(\beta) = \max\{n \in \omega : \langle Y_\xi \cap A_\beta, n \rangle \in u_{03}(\alpha)\}\), if \(\beta \notin Y_\xi\), and \(g_\xi(\beta) = \omega\), if \(\beta \in Y_\xi\).

With these definitions, (1_\xi) and (2_\xi) are clearly satisfied.

Note that (3_\xi) permits exactly one of 0 or 1 to be \(h(\xi)\) and define \(h(\xi)\) according to (3_\xi).

If \(h(\xi) = 0\), then leave \(w_\xi\) undefined.

Suppose now that \(h(\xi) = 1\). We are going to define \(w_\xi(\beta)\) by induction on \(\beta < c^+\). Suppose that we are done for every \(\alpha < \beta\). We consider three cases.

**Case 1.** Suppose that \(U_\xi \cap (A_\beta \times A_\beta) \in u_{13}(\alpha)\) and \(\beta \in U_{\xi,\omega_1}(\alpha)\). Note that by (C-4) there is only one such \(\alpha\) and that \(\alpha \in A_\beta\). Set \(w_\xi(\beta) = w_\xi(\alpha)\).

**Case 2.** Suppose that Case 1 does not hold, but there is an \(\alpha < \beta\) such that \(U_\xi \cap (A_\beta \times A_\beta) \in u_{23}(\alpha)\) and \(\beta \in U_{\xi,\omega_1}(\alpha)\). Note that every such \(\alpha\) belongs to \(A_\beta\). Fix one such \(\alpha\) and set \(w_\xi(\beta) = w_\xi(\alpha)\) for that \(\alpha\).

**Case 3.** Suppose that neither Case 1 nor Case 2 holds. Then pick any \(\rho \in c^+ \setminus \omega\) with \(\beta \in U_{\xi,\rho}\) (since \(\langle U_{\xi,\rho}^{c^+} \rangle_{\rho \in c^+ \setminus \omega}\) is a cover of \(c^+\), there is at least one such \(\rho\)) and set \(w_\xi(\beta) = \rho\).

It is easy to check that (4_\xi) is satisfied in all of the cases above.

To finish our construction, let \(\tau\) denote the topology generated by

\[B = \bigcup_{\xi < \lambda} B_\xi = \{G_{\xi,n} : \xi < \lambda, n \in \omega\} \cup \{W_{\xi,\rho} : \rho \in c^+ \setminus \omega, \xi < \lambda\ and \ h(\xi) = 1\}\]

as a subbase.

Let \(QX = (c^+, \tau)\). The rest of the proof consists of checking that this space possesses the desired properties.

1. To check that every subset of \(QX\) is a \(G_\delta\)-set, let \(Y \subset c^+\). Then there is a \(\xi < \lambda\) such that \(Y = Y_\xi\). By (1_\xi), \(Y = Y_\xi = \bigcap_{n \in \omega} G_{\xi,n}\), i.e. \(Y\) is a \(G_\delta\)-set.

Note that since complements of singletons are \(G_\delta\)-sets (and thus, open sets), every singleton set is closed, i.e. \(X\) is a \(T_1\)-space.
II. In order to show that $QX$ is ultraparacompact it is enough to prove that every open cover of $QX$ has a refinement which is a partition of $c^+$ into pairwise disjoint clopen sets. So let $U$ be an arbitrary open cover of $QX$ and let $\{U_\rho\}_{\rho \in c^+ \setminus \omega}$ be a list of $U \cup \{\phi\}$. Let $U = \bigcup_{\rho \in c^+ \setminus \omega} U_\rho \times \{\rho\}$. Since on the list $\{U_\xi\}_{\xi < \lambda}$ of all subsets of $c^+ \times c^+$, $U$ is listed $\lambda$ times, and because $cf(\lambda) = cf(2^{c^+}) > c^+$, there is a first $\xi < \lambda$ such that $U_\xi = U$ and $U_\rho$ is initially $\xi$-open for every $\rho \in c^+ \setminus \omega$. For this $\xi$, $\{U_\rho\}_{\rho \in c^+ \setminus \omega} = \{U_\rho\}_{\rho \in c^+ \setminus \omega}$ and $h(\xi) = 1$. Therefore $w_\xi : c^+ \to c^+ \setminus \omega$ is defined and $\langle W_\rho \rangle_{\rho \in c^+ \setminus \omega}$ is a refinement of $\{U_\rho\}_{\rho \in c^+ \setminus \omega}$ to a partition of $QX$ into clopen subsets.

III. Perfect normality of $X$ follows from I and II.

IV. The rest of the proof consists of showing that $QX$ does not have a quasi-$G_\delta$-diagonal. From this it automatically follows that $QX$ is not $\sigma$-discrete, because a $\sigma$-discrete space in which every point is a $G_\delta$-set has a quasi-$G_\delta$-diagonal.

First, for every $\langle \xi, \cdot \rangle \in \lambda \times \omega \cup H \times (c^+ \setminus \omega)$, let $Q_\xi = G_\xi n$ if $\langle \xi, \cdot \rangle = (\xi, n)$ for some $n \in \omega$, and let $Q_\xi = W_\xi$ if $\langle \xi, \cdot \rangle = (\xi, \rho)$ for some $\rho \in c^+ \setminus \omega$. Further, let us note that $\{U_\xi\}_{\xi \in H}$ is a one-to-one list.

Now, let us consider an arbitrary sequence $\langle G_m \rangle_{m \in \omega}$ of families of open subsets of $QX$. We are going to show that $\langle G_m \rangle_{m \in \omega}$ does not form a quasi-$G_\delta$-diagonal in $QX$. For this purpose we can assume without loss of generality that each $G_m$ is a non-empty family of non-empty sets and that $G = \bigcup_{m \in \omega} G_m$ covers $QX$.

For each $m \in \omega$ let $q_m : \bigcup G_m \to \{\lambda \times \omega \cup H \times (c^+ \setminus \omega)\}^{< \omega}$ code a refinement of $G_m$ by basic open sets, i.e. for every $\alpha \in \bigcup G_m$ there is a $G \in G_m$ such that

$$\alpha \in q_m(\alpha) = \bigcap \{Q_\xi : (\xi, \cdot) \in q_m(\alpha)\} \subset G.$$

By extending $q_m(\alpha)$, if necessary, we can assume without loss of generality that for every $m \in \omega$ and $\alpha \in \bigcup G_m$,

(F) if $\langle \xi, \rho \rangle \in q_m(\alpha) \cap (H \times (c^+ \setminus \omega))$, then there is a $t_{\xi \rho m}(\alpha) \subset q_m(\alpha)$ such that

(F-a) $\{\eta, \cdot \} \in t_{\xi \rho m}(\alpha)$ implies $\eta < \xi$;

(F-b) $\alpha \in T_{\xi \rho m}(\alpha) = \bigcap \{Q_\eta : (\eta, \cdot) \in t_{\xi \rho m}(\alpha)\} \subset U_{\xi \rho}$. 

This can be done because if $\langle \xi, \rho \rangle \in q_m(\alpha) \cap (H \times (c^+ \setminus \omega))$, then $U_{\xi \rho}$ is an initially $\xi$-open set containing $\alpha$. Also, note that $t_{\xi \rho m}(\alpha)$ does not depend on $\rho$, because $\xi$ and $\alpha$ determine $\rho$ through the condition $\alpha \in W_{\xi \rho}$.

In order to prove that $\langle G_m \rangle_{m \in \omega}$ is not a quasi-$G_\delta$-diagonal it is enough to find two distinct elements $\beta_0, \beta_1$ of $c^+$ such that $\{m \in \omega : \beta_0 \in \bigcup G_m\} = \{m \in \omega : \beta_1 \in \bigcup G_m\}$, and if we denote this subset of $\omega$ by $N$, then

(*) for every $m \in N$ there is an $\alpha \in \bigcup G_m$ such that $\{\beta_0, \beta_1\} \subset q_m(\alpha)$.

In order to find such $\beta_0$ and $\beta_1$, let $M$ be a countable elementary submodel of $H((2^{c^+})^\omega)$ such that $\langle Y_\xi \rangle_{\xi < \lambda}, \langle U_\xi \rangle_{\xi \in H}, \langle g_\xi \rangle_{\xi < \lambda}, \langle w_\xi \rangle_{\xi \in H}, \langle G_m \rangle_{m \in \omega}$ and $\langle g_m \rangle_{m \in \omega}$ are all elements of $M$. Let $A^* = M \cap c^+$ and let $\langle A^*, \_ \rangle$ be a control pair such that if $v : c^+ \to [\lambda \times \omega \cup H \times (c^+ \setminus \omega)]^{< \omega}$ is an infinite partial function in $M$ and $\{\pi v(\alpha) : \alpha \in dom(v)\}$ forms a $\Delta$-system with root $r_v$, then there is an $\alpha \in A^*$ such that

$$u^*_1(\alpha) = \{\langle Y_\xi \cap A^*, n \rangle : (\xi, n) \in v(\alpha) \text{ and } \xi \notin r_v\};$$

$$u^*_2(\alpha) = \{U_\xi \cap (A^* \times A^*) : \xi \in (\pi v(\alpha) \setminus r_v) \cap H\};$$

$$u^*_3(\alpha) = \{\langle \xi, \cdot \rangle \in q_m(\alpha) : m \in \omega \text{ and }\}.$$
To see that such a control pair \( \langle A^*, u^* \rangle \) exists, let \( \langle v_k \rangle_{k \in \omega} \) list all functions above.

By induction on \( k \in \omega \), define a sequence \( \langle \alpha_k \rangle_{k \in \omega} \) of distinct elements of \( A^* = c^+ \cap M \) in such a way that \( \pi v_k(\alpha_k) - r_{v_k}(k \in \omega) \) are pairwise disjoint. Then define \( u^*_k(\alpha_k) \) (\( i = 0, 1, 2 \)) as in (D), writing \( \alpha_k \) and \( v_k \) in place of \( \alpha \) and \( v \), and set \( u^*_k(\alpha) = \emptyset \) for \( \alpha \in A^* \setminus \{ \alpha_k : k \in \omega \} \). Then \( \langle A^*, u^* \rangle \), where \( u^*(\alpha) = \langle u^*_0(\alpha), u^*_1(\alpha), u^*_2(\alpha) \rangle \), is a control pair as desired. (Properties (C-3) and (c-4) of a control pair follow see this, let it follow that \( \langle \beta^* \rangle_{i \in \omega} \) holds for every \( i \in \omega \). Case 1.

Suppose \( \langle \beta^* \rangle_{i \in \omega} \). Subcase 1(a). Case 2.

Suppose \( \beta_0, \beta_1 > \sup A^* \) be such that

(i) \( \langle A^*, u^* \rangle = \langle A_{\beta^i}, \beta_{\beta^i} \rangle \) for \( i = 0, 1 \);

(ii) \( \{ m \in \omega : \beta_0 \in \bigcup \mathcal{G}_m \} = \{ m \in \omega : \beta_1 \in \bigcup \mathcal{G}_m \} = N \);

(iii) for every \( m \in N \)

(iii-a) \( q_m(\beta_0) \cap M = q_m(\beta_1) \cap M \) (denote this set by \( y_m \));

(iii-b) \( \{ \xi \in M \cap H : \langle \xi, w_\xi(\beta_0) \rangle \in q_m(\beta_0) \} = \{ \xi \in M \cap H : \langle \xi, w_\xi(\beta_1) \rangle \in q_m(\beta_1) \} \) (denote this set by \( t_m \)). Note that (iii-a) and (iii-b) together imply

(iii-c) \( \pi q_m(\beta_0) \cap M = \pi q_m(\beta_1) \cap M \). Denote this set by \( S_m \).

Note that \( y_m, t_m, S_m \in M \).

To see that \( \beta_0, \beta_1 \) satisfy (*), fix \( m \in N \). Let \( \varphi(\alpha) \) be the conjunction of the following statements:

(a) \( \alpha \in \bigcup \mathcal{G}_m \);

(b) \( y_m \subset q_m(\alpha) \);

(c) for every \( \langle \xi, n \rangle \in S_m \times \omega \), \( \langle \xi, n \rangle \in q_m(\alpha) \) iff \( \langle \xi, n \rangle \in y_m \);

(d) for every \( \xi \in S_m \cap H, \langle \xi, w_\xi(\alpha) \rangle \in q_m(\alpha) \) iff \( \xi \in t_m \).

Note that all the parameters of \( \varphi(\alpha) \) are from \( M \), and that \( \varphi(\beta_0) \) (as well as \( \varphi(\beta_1) \)) holds. Therefore, by standard reflection, \( \varphi(\alpha) \) is true for infinitely many \( \alpha \in M \), in fact,

\( \psi \colon \) there is an infinite function \( v \) such that \( \text{dom}(v) \subset c^+ \), \( \varphi(\alpha) \) and \( v(\alpha) = q_m(\alpha) \) hold for every \( \alpha \in \text{dom}(v) \), and \( \langle \pi v(\alpha) \rangle_{\alpha \in \text{dom}(v)} \) forms an infinite \( \Delta \)-system with root \( r_v = S_m \).

Since all parameters of \( \psi \) are from \( M \) we can choose a \( v \in M \) as above. Let \( \alpha \in A^* \) be such that (D) holds. We are going to show that \( \{ \beta_0, \beta_1 \} \subset Q_m(\alpha) \). To see this, let \( \xi_0 < \xi_1 < \cdots \xi_{t-1} \ldots \) enumerate \( \pi q_m(\alpha) \). By induction on \( k = 0, \ldots, t-1 \) we are going to prove

\( I_k \)

if \( \langle \xi_k, \cdot \rangle \in q_m(\alpha) \), then \( \{ \beta_0, \beta_1 \} \subset Q_{\xi_k} \).

Let \( 0 \leq k \leq t-1 \) and suppose that \( I_j \) holds for \( j < k \). In order to prove \( I_k \), let \( \langle \xi_k, \cdot \rangle \in q_m(\alpha) = v(\alpha) \).

We are going to split our argument into two cases and consider two subcases in each case.

Case 1. Suppose \( \langle \xi_k, \cdot \rangle = \langle \xi_k, n \rangle \) for some \( n \in \omega \).

Subcase 1(a). Suppose \( \xi_k \in S_m \). Then, since (c) of \( \varphi(\alpha) \) holds and \( \langle \xi_k, n \rangle \in q_m(\alpha) \), it follows that \( \langle \xi_k, n \rangle \in q_m(\beta) \) for \( i = 0, 1 \). Thus \( \beta_0, \beta_1 \subset Q_{\xi_k} \).

Subcase 1(b). Suppose \( \xi_k \notin S_m = r_v \). Then, for \( i = 0, 1 \), it follows that \( \alpha < \beta_i \) and \( \langle Y_{\xi_k} \cap A^*, n \rangle \in u^*_0(\alpha) \) (i.e., \( \langle Y_{\xi_k} \cap A_{\beta}, n \rangle \in u_{0\beta}(\alpha) \)). By (2) it follows that \( g_{\xi_k}(\beta) \geq n \), i.e., \( \beta_i \in G_{\xi_k} = Q_{\xi_k} \).

Case 2. Suppose \( \langle \xi_k, \cdot \rangle = \langle \xi_k, \rho \rangle \) for some \( \rho \in c^+ \).
Subcase 2(a). Suppose \(\xi_k \in S_m\). Then, since \((I_j)\) holds for \(j < k\) and since \(q_m(\alpha)\) satisfies \((F)\), it follows that
\[
\{\beta_0, \beta_1\} \subset T_{\xi_k m}(\alpha) \subset U_{\xi_k \rho} = U_{\xi_k w_{\xi_k}(\alpha)}.
\]
Furthermore, since \(\xi_k \in S_m \cap H = r_m \cap H\), it follows that \(U_{\xi_k} \cap (A^* \times A^*) \in u_{m}^{\alpha}(\alpha)\) (i.e., \(U_{\xi_k} \cap (A_{\beta_i} \times A_{\beta_i}) \in w_{\xi_k}(\alpha)\) for \(i = 0, 1\)). Thus by \((4_{\xi_k} - c)\) there are \(\alpha_0, \alpha_1 A^*\) such that \(w_{\xi_k}(\beta_i) = w_{\xi_k}(\alpha_i)\) for \(i = 0, 1\). Let us denote these common values by \(\rho_i\) (\(i = 0, 1\)). Since \(\xi_k \in M\) and \(\langle w_{\xi_k} \rangle_{\xi_k \in H} \in M\), \(w_{\xi_k} \in M\). Since \(\alpha_i \in M\), \(\rho_i = w_{\xi_k}(\alpha_i) \in M\). Thus \(\langle \xi_k, \rho_i \rangle \in M\) for \(i = 0, 1\). Furthermore, since \(\langle \xi_k, \rho \rangle = \langle \xi_k, w_{\xi_k}(\alpha) \rangle \in q_m(\alpha)\), by part (d) of \(\varphi(\alpha)\), \(\xi_k \in t_m\), so \(\langle \xi_k, \rho_i \rangle \in M \cap q_m(\beta_i) = Y_m\) for \(i = 0, 1\). By part (b) of \(\varphi(\alpha)\), \(Y_m \subset q_m(\alpha)\), so \(\langle \xi_k, \rho_i \rangle \in q_m(\alpha)\). Since \(\langle \xi_k, \rho \rangle \in q_m(\alpha)\), this implies \(\rho = \rho_0 = \rho_1\); hence \(\beta_i \in W_{\xi_k \rho_i} = W_{\xi_k \rho}\) for \(i = 0, 1\).

Subcase 2(b). Suppose \(\xi_k \notin S_m\). Then \(\xi_k \in \pi m(\alpha) \cap H \setminus S_m \subset \pi v(\alpha) \setminus v\), so \(U_{\xi_k} \cap (A^* \times A^*) \subset u_{\xi_k}(\alpha)\) (i.e., \(U_{\xi_k} \cap (A_{\beta_i} \times A_{\beta_i}) \subset u_{\xi_k}(\alpha)\) for \(i = 0, 1\)). Furthermore, \(\alpha_i < \beta_i\), and since \((I_j)\) holds for \(j < k\), and \(q_m(\alpha)\) satisfies \((F)\), it follows that
\[
\{\beta_0, \beta_1\} \subset T_{\xi_k m}(\alpha) \subset U_{\xi_k \rho} = U_{\xi_k w_{\xi_k}(\alpha)}.
\]
Thus by \((4_{\xi_k} - b)\), \(\omega_{\xi}(\beta_i) = w_{\xi_k}(\alpha) = \rho\) holds for \(i = 0, 1\), i.e. \(\{\beta_0, \beta_1\} \subset W_{\xi_k \rho} = Q_{\xi_k \rho}\).

This concludes the proof of Theorem 1.1.

2. Final remarks, open questions

A. As pointed out earlier, \(QX\) is not \(\sigma\)-discrete, because a \(\sigma\)-discrete space in which all points are \(G_{\delta}\)-sets has a quasi-\(G_{\delta}\)-diagonal. (Indeed, if \(Y = \bigcup_{k \in \omega} Y_m\) is such a space (with each \(Y_m\), a discrete subspace), then for each \(y \in Y_m\), let us pick a sequence \(\langle G_{ymk} \rangle_{k \in \omega}\) of open sets such that \(\{y\} = \bigcap_{k \in \omega} G_{ymk}\) and \(G_{ymk} \cap Y_m = \{y\}\) for every \(k \in \omega\). Then \(\langle G_{ymk} \rangle_{m,k \in \omega}\), where \(G_{ymk} = \{G_{ymk} : y \in Y_m\}\), is a quasi-\(G_{\delta}\)-diagonal). Moreover, if we only want to make sure that our \(Q\)-set space is not \(\sigma\)-discrete, then the construction of \(QX\) can be done on \(c\) instead of \(c^+\), with minimal changes.

Theorem 2.1. There is a paracompact perfectly normal \(Q\)-set space of cardinality \(c\).

It is interesting to note that all normal \(Q\)-set spaces of cardinality \(\leq c\) have a \(G_{\delta}\)-diagonal (more generally, all cleavable spaces of cardinality \(\leq c\) have a \(G_{\delta}\)-diagonal \([AS]\)); hence to get “\(QX\) has no (quasi-)\(G_{\delta}\)-diagonal” it was necessary to work on \(c^+\) instead of just \(c\).

B. We can’t hope that a \(Q\)-set space constructed in \(ZFC\) will have properties any closer to metrizability then being paracompact; indeed, under \(V = L\), not only that there are no metrizable \(Q\)-set spaces, but there are no \(Q\)-set spaces with character \(\leq c\) \([R], [H], [BJ]\).

Under \(V = L\), every \(Q\)-set space is \(\sigma\)-left separated \([BJ]\), so a non-\(\sigma\)-left-separated \(Q\)-set space cannot be constructed in \(ZFC\). Of course, under \(MA + \neg CH\), even the real line has \(Q\)-set subspaces (see Miller’s paper in \([KV]\), e.g.).

C. There are several natural questions which are left open.

Problem 1. Is there a connected normal \(Q\)-set space?

For \(Q\)-spaces, this is a question of A.V. Arhangel’skiĭ \([A2]\), who also points out that P. deCaux \([C]\) constructed an infinite, regular, connected, \(\sigma\)-closed-discrete
space. (Note that, of course, a $\sigma$-closed discrete space is a Q-space, but not a Q-set space.)

**Problem 2.** Is there a strong Q-set space in ZFC, i.e. a space $X$ such that all finite powers of $X$ are Q-set spaces? Can such a space be normal or paracompact?

(Note that under MA + $\neg$ CH, the real line has strong Q-set subspaces.)

**Problem 3.** Is there, in ZFC, a Q-set space of size $\omega_1$?

It is interesting to note that the answer is yes both under CH and MA($\omega_1$). Under CH the space in [B] works, and the space of Theorem 2.1 is an example which is even paracompact. Under MA($\omega_1$), any subset of cardinality $\omega_1$ of the real line is an example.

**References**


Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45058  
E-mail address: ZTBalogh@miavx1.muxio.edu