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THERE IS A PARACOMPACT Q-SET SPACE IN ZFC

ZOLTAN T. BALOGH

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ABSTRACT. We construct a paracompact space QX such that every subset of QX is an F_σ-set, yet QX is not σ-discrete. We will construct our space not to have a G_δ-diagonal, which answers questions of A.V. Arhangel′skiĭ and D. Shakhmatov on cleavable spaces.

INTRODUCTION

In this paper we will construct a hereditarily paracompact, perfectly normal Q-set space QX without a quasi-G_δ-diagonal. QX answers questions on Q-set spaces, and on cleavable spaces of A.V. Arhangel′skiĭ.

A topological space X is a **Q-set space** [B] if every subset of X is a G_δ-set and X is *not* σ-discrete. H. Junnila [J] (and Bregman-Shapirovskiĭ-Šoštak) asked whether there were any Q-set spaces in ZFC. This problem was answered affirmatively for regular Q-set spaces, and the question was raised whether there are (perfectly) normal Q-set spaces [B]. In this paper we shall combine the technique of the regular examples with a new inductive method to show not only that the answer is yes, but that one can also construct paracompact examples.

A.V. Arhangel′skiĭ and D.B. Shakhmatov [AS], [A1] raised the question whether every cleavable space has a G_δ-diagonal. Arhangel′skiĭ [A2] also asked whether spaces cleavable over the rationals had to be σ-discrete or had to possess G_δ-diagonals. Since normal Q-set spaces are cleavable and also cleavable over the rationals [A2], and our space QX will be constructed not to have a G_δ-diagonal, it settles all of the above questions in the negative. (It should be pointed out here, that a Q-space is defined in [A2] to be a space whose every subset is an F_σ-set. Thus, Q-set spaces are precisely the non-σ-discrete Q-spaces).

QX will have cardinality c^+, which is necesssary only to make it not have a G_δ-diagonal. If we only want to construct a paracompact Q-set space, then it can be done on c (Theorem 2.1).

Terminology and notation. We use the standard terminology and notation of set-theoretic topology (see [KV]). π will always denote *first* projection, i.e. $\pi A = \{a :$ there is a b with $\langle a, b\rangle \in A\}$. A sequence of $\langle \mathcal{G}_m\rangle_{m\in\omega}$ of families of open subsets of a space X is said to be a *quasi-G_δ-diagonal,* if for every $x \in X$, $\bigcap\{st(x,\mathcal{G}_m) :\ m \in \omega$ and $x \in \bigcup \mathcal{G}_m\} = \{x\}$.

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1. The space QX

Theorem 1.1. *There is a (hereditarily) paracompact, perfectly normal Q-set space QX without a quasi-G_δ-diagonal.*

Proof. The underlying set of QX is c^+, the first cardinal bigger than the continum c. The topology of QX will be inductively defined in $\lambda = 2^{c^+}$ steps. For the purposes of making every subset of QX a G_δ-set, let $\langle Y_\xi \rangle_{\xi<\lambda}$ be a one-to-one listing of all subsets of c^+. Also, let $\langle U_\xi \rangle_{\xi<\lambda}$ be a list of all subsets of $c^+ \times c^+$ such that $U_0 = \phi$ and each subset is listed λ times. This second list will, in particular, mention codes for all future open covers of QX. If such an open cover first occurs at step ξ, then we'll add a clopen partition refining that cover to the topology of QX. To carry out the program above we shall define, by induction of $\xi < \lambda$,

(a) a function $g_\xi : c^+ \longrightarrow \omega + 1$;
(b) a number $h(\xi) \in \{0, 1\}$;
(c) a function $w_\xi : c^+ \longrightarrow c^+ \backslash \omega$ if $h(\xi) = 1$.

We will set

(a′) $G_{\xi n} = \{\alpha < c^+ : g_\xi(\alpha) \geq n\}$ for every $n \in \omega$;
(b′) $H = \{\xi < \lambda : h(\xi) = 1\}$;
(c′) $W_{\xi\rho} = \{\alpha < c^+ : w_\xi(\alpha) = \rho\}$ for every $\xi \in H$ and ρ with $\omega \leq \rho < c^+$. □

A subbase for the topology τ_Q of QX will be

$$\mathcal{B} = \{G_{\xi n} : \xi < \lambda,\ n \in \omega\} \cup \{W_{\xi\rho} : \xi \in H \text{ and } \rho \in c^+ \backslash \omega\}.$$

Adding the $G_{\xi n}$'s will make every subset of X a G_δ-set. $\{W_{\xi\rho} : \rho \in c^+ \backslash \omega\}$ will be a clopen partition refinement of the open cover coded by rows $\omega \leq \rho < c^+$ of $U_\xi \subset c^+ \times c^+$ if $\xi \in H$.

In order to make sure that QX does not have a quasi-G_δ-diagonal we will need the concept of a *control pair*. We will say that $\langle A,\ \underline{u} \rangle$ is a control pair if

(C-1) $A \in [c^+]^\omega$;
(C-2) $\underline{u} = \langle u_0,\ u_1,\ u_2 \rangle$, and $u_0,\ u_1,\ u_2$ are functions with domain A;
(C-3) for every $\alpha \in A$, $u_0(\alpha) \in [P(A) \times \omega]^{<\omega}$, $u_1(\alpha) \in [P(A \times A)]^{<\omega}$ and $u_2(\alpha) \in [P(A \times A) \backslash u_1(\alpha)]^{<\omega}$;
(C-4) if $\alpha,\ \alpha' \in A$ and $\alpha \neq \alpha'$, then $\pi u_0(\alpha) \cap \pi u_0(\alpha') = \phi$ and $u_1(\alpha) \cap u_1(\alpha') = \phi$.

(Note that $\pi u_0(\alpha) = \{B \subset A$ there is an $n \in \omega$ with $\langle B,\ n \rangle \in u_0(\alpha)\}$).

Roughly speaking, $\langle A,\ u \rangle$ will code a countable approximation to a neighborhood assignment in QX. Let $\langle A_\beta,\ \underline{u}_\beta \rangle_{\beta<c^+}$ list all control pairs, mentioning each c^+ times.

The last ingredient we need is the notion of an initially ξ-open set. A subset $E \subset c^+$ will be called *initially ξ-open,* if E is an open subset in the topology generated by

$$\mathcal{B}_\xi = \{X\} \cup \{G_{\eta n} : \eta < \xi,\ n \in \omega\} \cup \{W_{\eta\rho} : \eta < \xi,\ h(\eta) = 1 \text{ and } \rho \in c^+ \backslash \omega\}.$$

For every $\xi < \lambda$ and $\rho \in c^+ \backslash \omega$, let $U_{\xi\rho} = \{\gamma < c^+ : \langle \gamma, \rho \rangle \in U_\xi\}$.

We are going to construct g_ξ, $h(\xi)$ and w_ξ (if $h(\xi) = 1$) in such a way that the following hypotheses are satisfied:

(1_ξ) for every $\beta < c^+$, $g_\xi(\beta) = \omega$ iff $\beta \in Y_\xi$;
(2_ξ) if $\alpha < \beta < c^+$ and $\langle Y_\xi \cap A_\beta,\ n \rangle \in u_{0\beta}(\alpha)$, then $g_\xi(\beta) \geq n$;

(3_ξ) $h(\xi) = 1$ if and only if $\langle U_{\xi\rho}\rangle_{\rho\in c^+\setminus\omega}$ is a cover of c^+ consisting of initially ξ-open sets and there is no $\xi' < \xi$ such that $U_{\xi'} = U_\xi$ and $\langle U_{\xi'\rho}\rangle_{\rho\in c^+\setminus\omega}$ is a cover by initially ξ'-open sets;

(4_ξ) if $h(\xi) = 1$, then

(a) for every $\beta < c^+$, $\beta \in U_{\xi w_\xi(\beta)}$;

(b) if $\alpha < \beta < c^+$, $U_\xi \cap (A_\beta \times A_\beta) \in u_{1\beta}(\alpha)$ and $\beta \in U_{\xi w_\xi(\alpha)}$, then $w_\xi(\beta) = w_\xi(\alpha)$;

(c) if $\alpha < \beta < c^+$, $U_\xi \cap (A_\beta \times A_\beta) \in u_{2\beta}(\alpha)$ and $\beta \in U_{\xi w_\xi(\alpha)}$, then there is an $\alpha' \in A_\beta$ such that $w_\xi(\beta) = w_\xi(\alpha')$.

Let us now pass to the construction. Suppose that $\xi < \lambda$ and that we are done for $\eta < \xi$.

We are going to define $g_\xi(\beta) \in \omega + 1$ by induction on $\beta < c^+$. Suppose we are done for every $\alpha < \beta$. We split the definition into two cases.

Case 1. Suppose that there is no $\alpha < \beta$ and $n \in \omega$ with $\langle Y_\xi \cap A_\beta,\ n\rangle \in u_{0\beta}(\alpha)$. Then let $g_\xi(\beta) = \omega$ if $\beta \in Y_\xi$, and $g_\xi(\beta) = 0$ if $\beta \notin Y_\xi$.

Case 2. Suppose now that there is an $\alpha < \beta$ such that $\langle Y_\xi \cap A_\beta,\ n\rangle \in u_{0\beta}(\alpha)$ for some $n \in \omega$. By (C-4), there is only one such α; furthermore, since $u_{0\beta}(\alpha)$ is a finite set, there are only finitely many such $n \in \omega$. Set $g_\xi(\beta) = \max\{n \in \omega : \langle Y_\xi \cap A_\beta, n\rangle \in u_{0\beta}(\alpha)\}$, if $\beta \notin Y_\xi$, and $g_\xi(\beta) = \omega$, if $\beta \in Y_\xi$.

With these definitions, (1_ξ) and (2_ξ) are clearly satisfied.

Note that (3_ξ) permits exactly one of 0 or 1 to be $h(\xi)$ and define $h(\xi)$ according to (3_ξ).

If $h(\xi) = 0$, then leave w_ξ undefined.

Suppose now that $h(\xi) = 1$. We are going to define $w_\xi(\beta)$ by induction on $\beta < c^+$. Suppose that we are done for every $\alpha < \beta$. We consider three cases.

Case 1. Suppose that there is an $\alpha < \beta$ such that $U_\xi \cap (A_\beta \times A_\beta) \in u_{1\beta}(\alpha)$ and $\beta \in U_{\xi w_\xi(\alpha)}$. Note that by (C-4) there is only one such α and that $\alpha \in A_\beta$. Set $w_\xi(\beta) = w_\xi(\alpha)$.

Case 2. Suppose that Case 1 does not hold, but there is an $\alpha < \beta$ such that $U_\xi \cap (A_\beta \times A_\beta) \in u_{2\beta}(\alpha)$ and $\beta \in U_{\xi w_\xi(\alpha)}$. Note that every such α belongs to A_β. Fix one such α and set $w_\xi(\beta) = w_\xi(\alpha)$ for that α.

Case 3. Suppose that neither Case 1 nor Case 2 holds. Then pick any $\rho \in c^+\setminus\omega$ with $\beta \in U_{\xi\rho}$ (since $\langle U_{\xi\rho}\rangle_{\rho\in c^+\setminus\omega}$ is a cover of c^+, there is at least one such ρ) and set $w_\xi(\beta) = \rho$.

It is easy to check that (4_ξ) is satisfied in all of the cases above.

To finish our construction, let τ denote the topology generated by

$$\mathcal{B} = \bigcup_{\xi<\lambda} \mathcal{B}_\xi = \{G_{\xi n} : \xi < \lambda, n \in \omega\} \cup \{W_{\xi\rho} : \rho \in c^+\setminus\omega, \xi < \lambda \text{ and } h(\xi) = 1\}$$

as a subbase.

Let $QX = \langle c^+,\ \tau\rangle$. The rest of the proof consists of checking that this space possesses the desired properties.

I. To check that *every subset* of QX is a G_δ-set, let $Y \subset c^+$. Then there is a $\xi < \lambda$ such that $Y = Y_\xi$. By (1_ξ), $Y = Y_\xi = \bigcap_{n\in\omega} G_{\xi n}$, i.e. Y is a G_δ-set.

Note that since complements of singletons are G_δ-sets (and thus, open sets), every singleton set is closed, i.e. X is a T_1-space.

II. In order to show that QX is **ultraparacompact** it is enough to prove that every open cover of QX has a refinement which is a partition of c^+ into pairwise disjoint clopen sets. So let $\mathcal{U}$ be an arbitrary open cover of QX and let $\langle U_\rho \rangle_{\rho \in c^+ \setminus \omega}$ be a list of $\mathcal{U} \cup \{\phi\}$. Let $U = \bigcup_{\rho \in c^+ \setminus \omega} U_\rho \times \{\rho\}$. Since on the list $\langle U_\xi \rangle_{\xi < \lambda}$ of all subsets of $c^+ \times c^+$, U is listed λ times, and because $cf(\lambda) = cf(2^{c^+}) > c^+$, there is a first $\xi < \lambda$ such that $U_\xi = U$ and U_ρ is initially ξ-open for every $\rho \in c^+ \setminus \omega$. For this ξ, $\langle U_{\xi\rho} \rangle_{\rho \in c^+ \setminus \omega} = \langle U_\rho \rangle_{\rho \in c^+ \setminus \omega}$ and $h(\xi) = 1$. Therefore $w_\xi : c^+ \to c^+ \setminus \omega$ is defined and $\langle W_{\xi\rho} \rangle_{\rho \in c^+ \setminus \omega}$ is a refinement of $\langle U_\rho \rangle_{\rho \in c^+ \setminus \omega}$ to a partition of QX into clopen subsets.

III. **Perfect normality** of X follows from I and II.

IV. The rest of the proof consists of showing that QX **does not have a quasi-G_δ-diagonal.** From this it automatically follows that QX is not σ-discrete, because a σ-discrete space in which every point is a G_δ-set has a quasi-G_δ-diagonal.

First, for every $\langle \xi, \cdot \rangle \in \lambda \times \omega \cup H \times (c^+ \setminus \omega)$, let $Q_{\xi \cdot} = G_{\xi n}$ if $\langle \xi, \cdot \rangle = \langle \xi, n \rangle$ for some $n \in \omega$, and let $Q_{\xi \cdot} = W_{\xi\rho}$ if $\langle \xi \cdot \rangle = \langle \xi, \rho \rangle$ for some $\rho \in c^+ \setminus \omega$. Further, let us note that $\langle U_\xi \rangle_{\xi \in H}$ is a one-to-one list.

Now, let us consider an arbitrary sequence $\langle \mathcal{G}_m \rangle_{m \in \omega}$ of families of open subsets of QX. We are going to show that $\langle \mathcal{G}_m \rangle_{m \in \omega}$ does not form a quasi-G_δ-diagonal in QX. For this purpose we can assume without loss of generality that each $\mathcal{G}_m$ is a non-empty family of non-empty sets and that $\mathcal{G} = \bigcup_{m \in \omega} \mathcal{G}_m$ covers QX.

For each $m \in \omega$ let $q_m : \bigcup \mathcal{G}_m \to [\lambda \times \omega \cup H \times (c^+ \setminus \rho)]^{<\omega}$ code a refinement of $\mathcal{G}_m$ by basic open sets, i.e. for every $\alpha \in \bigcup \mathcal{G}_m$ there is a $G \in \mathcal{G}_m$ such that

$$\alpha \in Q_m(\alpha) = \bigcap \{Q_{\xi \cdot} : \langle \xi, \cdot \rangle \in q_m(\alpha)\} \subset G.$$

By extending $q_m(\alpha)$, if necessary, we can assume without loss of generality that for every $m \in \omega$ and $\alpha \in \bigcup \mathcal{G}_m$,

(F) if $\langle \xi, \rho \rangle \in q_m(\alpha) \cap (H \times (c^+ \setminus \omega))$, then there is a $t_{\xi m}(\alpha) \subset q_m(\alpha)$ such that
(F-a) $\langle \eta, \cdot \rangle \in t_{\xi m}(\alpha)$ implies $\eta < \xi$;
(F-b) $\alpha \in T_{\xi m}(\alpha) = \bigcap \{Q_{\eta \cdot} : \langle \eta, \cdot \rangle \in t_{\xi m}(\alpha)\} \subset U_{\xi\rho}$.

This can be done because if $\langle \xi, \rho \rangle \in q_m(\alpha) \cap (H \times (c^+ \setminus \omega))$, then $U_{\xi\rho}$ is an initially ξ-open set containing α. Also, note that $t_{\xi m}(\alpha)$ does not depend on ρ, because ξ and α determine ρ through the condition $\alpha \in W_{\xi\rho}$.

In order to prove that $\langle \mathcal{G}_m \rangle_{m \in \omega}$ is not a quasi-G_δ-diagonal it is enough to find two distinct elements β_0, β_1 of c^+ such that $\{m \in \omega : \beta_0 \in \bigcup \mathcal{G}_m\} = \{m \in \omega : \beta_1 \in \bigcup \mathcal{G}_m\}$, and if we denote this subset of ω by N, then

(∗) for every $m \in N$ there is an $\alpha \in \bigcup \mathcal{G}_m$ such that $\{\beta_0, \beta_1\} \subset Q_m(\alpha)$.

In order to find such β_0 and β_1, let M be a countable elementary submodel of $H((2^{c^+})^+)$ such that $\langle Y_\xi \rangle_{\xi < \lambda}$, $\langle U_\xi \rangle_{\xi \in H}$, $\langle g_\xi \rangle_{\xi < \lambda}$, $\langle w_\xi \rangle_{\xi \in H}$, $\langle \mathcal{G}_m \rangle_{m \in \omega}$ and $\langle q_m \rangle_{m \in \omega}$ are all elements of M. Let $A^* = M \cap c^+$ and let $\langle A^*, \underline{u}^* \rangle$ be a control pair such that if $v : c^+ \to [\lambda \times \omega \cup H \times (c^+ \setminus \omega)]^{<\omega}$ is an infinite partial function in M and $\{\pi v(\alpha) : \alpha \in dom(v)\}$ forms a Δ-system with root r_v, then there is an $\alpha \in A^*$ such that

$$\text{(D)} \qquad \begin{aligned} u_0^*(\alpha) &= \{\langle Y_\xi \cap A^*, n \rangle : \langle \xi, n \rangle \in v(\alpha) \text{ and } \xi \notin r_v\}; \\ u_1^*(\alpha) &= \{U_\xi \cap (A^* \times A^*) : \xi \in (\pi v(\alpha) \setminus r_v) \cap H\}; \\ u_2^*(\alpha) &= \{U_\xi \cap (A^* \times A^*) : \xi \in r_v \cap H\}. \end{aligned}$$

To see that such a control pair $\langle A^*, u^*\rangle$ exists, let $\langle v_k\rangle_{k\in\omega}$ list all functions above.

By induction on $k \in \omega$, define a sequence $\langle \alpha_k\rangle_{k\in\omega}$ of distinct elements of $A^* = c^+ \cap M$ in such a way that $\pi v_k(\alpha_k) - r_{v_k} (k \in \omega)$ are pairwise disjoint. Then define $u^*_j(\alpha_k)$ $(j = 0, 1, 2)$ as in (D), writing α_k and v_k in place of α and v, and set $u^*_j(\alpha) = \emptyset$ for $\alpha \in A^*\backslash\{\alpha_k : k \in \omega\}$. Then $\langle A^*, \underline{u}^*\rangle$, where $\underline{u}^*(\alpha) = \langle u^*_0(\alpha),\ u^*_1(\alpha),\ u^*_2(\alpha)\rangle$, is a control pair as desired. (Properties (C-3) and (c-4) of a control pair follow from (D), because M is an elementary submodel and the lists $\langle Y_\xi\rangle_{\xi<\lambda},\ \langle U_\xi\rangle_{\xi\in H}$ are one-to-one.)

Now, let $\beta_0, \beta_1 > \sup A^*$ be such that

(i) $\langle A^*, \underline{u}^*\rangle = \langle A_{\beta i}, \underline{u}_{\beta i}\rangle$ for i = 0, 1;
(ii) $\{m \in \omega : \beta_0 \in \bigcup \mathcal{G}_m\} = \{m \in \omega : \beta_1 \in \bigcup \mathcal{G}_m\} = N$;
(iii) for every $m \in N$
(iii-a) $q_m(\beta_0) \cap M = q_m(\beta_1) \cap M$ (denote this set by y_m);
(iii-b) $\{\xi \in M \cap H : \langle \xi, w_\xi(\beta_0)\rangle \in q_m(\beta_0)\} = \{\xi \in M \cap H : \langle \xi, w_\xi(\beta_1)\rangle \in q_m(\beta_1)\}$ (denote this set by t_m). Note that (iii-a) and (iii-b) together imply
(iii-c) $\pi q_m(\beta_0) \cap M = \pi q_m(\beta_1) \cap M$. Denote this set by S_m.

Note that $y_m,\ t_m,\ S_m \in M$.

To see that β_0, β_1 satisfy (*), fix $m \in N$. Let $\varphi(\alpha)$ be the conjunction of the following statements:

(a) $\alpha \in \bigcup \mathcal{G}_m$;
(b) $y_m \subset q_m(\alpha)$;
(c) for every $\langle \xi, n\rangle \in S_m \times \omega,\ \langle \xi, n\rangle \in q_m(\alpha)$ iff $\langle \xi, n\rangle \in y_m$;
(d) for every $\xi \in S_m \cap H, \langle \xi, w_\xi(\alpha)\rangle \in q_m(\alpha)$ iff $\xi \in t_m$.

Note that all the parameters of $\varphi(\alpha)$ are from M, and that $\varphi(\beta_0)$ (as well as $\varphi(\beta_1)$) holds. Therefore, by standard reflection, $\varphi(\alpha)$ is true for infinitely many $\alpha \in M$, in fact,

ψ: there is an infinite function v such that $dom(v) \subset c^+$, $\varphi(\alpha)$ and $v(\alpha) = q_m(\alpha)$ hold for every $\alpha \in dom(v)$, and $\langle \pi v(\alpha)\rangle_{\alpha\in dom(v)}$ forms an infinite Δ-system with root $r_v = S_m$.

Since all parameters of ψ are from M we can choose a $v \in M$ as above. Let $\alpha \in A^*$ be such that (D) holds. We are going to show that $\{\beta_0, \beta_1\} \subset Q_m(\alpha)$. To see this, let $\xi_0 < \xi_1 < \cdots < \xi_{t-1}$ enumerate $\pi q_m(\alpha)$. By induction on $k = 0, ..., t-1$ we are going to prove

$$(I_k) \qquad \text{if } \langle \xi_k, \cdot\rangle \in q_m(\alpha), \text{ then } \{\beta_0, \beta_1\} \subset Q_{\xi_k \cdot}\,.$$

Let $0 \le k \le t-1$ and suppose that (I_j) holds for $j < k$. In order to prove (I_k), let $\langle \xi_k, \cdot\rangle \in q_m(\alpha) = v(\alpha)$.

We are going to split our argument into two cases and consider two subcases in each case.

Case 1. Suppose $\langle \xi_k, \cdot\rangle = \langle \xi_k,\ n\rangle$ for some $n \in \omega$.

Subcase 1(a). Suppose $\xi_k \in S_m$. Then, since (c) of $\varphi(\alpha)$ holds and $\langle \xi_k,\ n\rangle \in q_m(\alpha)$, it follows that $\langle \xi_k,\ n\rangle \in q_m(\beta_i)$ for $i = 0, 1$. Thus $\{\beta_0,\ \beta_1\} \subset Q_{\xi_k n} (= G_{\xi_k n})$.

Subcase 1(b). Suppose $\xi_k \notin S_m = r_v$. Then, for $i = 0, 1$, it follows that $\alpha < \beta_i$ and $\langle Y_{\xi_k} \cap A^*, n\rangle \in u^*_0(\alpha)$ (i.e., $\langle Y_{\xi_k} \cap A_{\beta i}, n\rangle \in u_{0\beta_i}(\alpha)$). By (2_{ξ_k}) it follows that $g_{\xi_k}(\beta_i) \ge n$, i.e. $\beta_i \in G_{\xi_k n} = Q_{\xi_k n}$.

Case 2. Suppose $\langle \xi_k, \cdot\rangle = \langle \xi_k, \rho\rangle$ for some $\rho \in c^+\backslash\omega$.

Subcase 2(a). Suppose $\xi_k \in S_m$. Then, since (I_j) holds for $j < k$ and since $q_m(\alpha)$ satisfies (F), it follows that

$$\{\beta_0, \beta_1\} \subset T_{\xi_k m}(\alpha) \subset U_{\xi_k \rho} = U_{\xi_k w_{\xi_k}(\alpha)}.$$

Furthermore, since $\xi_k \in S_m \cap H = r_v \cap H$, it follows that $U_{\xi_k} \cap (A^* \times A^*) \in u_2^*(\alpha)$ (i.e., $U_{\xi_k} \cap (A_{\beta_i} \times A_{\beta_i}) \in u_{2\beta_i}(\alpha))$ for $i = 0, 1$. Thus by (4_{ξ_k}-c) there are $\alpha_0, \alpha_1 \in A^*$ such that $w_{\xi_k}(\beta_i) = w_{\xi_k}(\alpha_i)$ for $i = 0, 1$. Let us denote these common values by ρ_i ($i = 0, 1$). Since $\xi_k \in M$ and $\langle w_\xi \rangle_{\xi \in H} \in M$, $w_{\xi_k} \in M$. Since $\alpha_i \in M, \rho_i = w_{\xi_k}(\alpha_i) \in M$. Thus $\langle \xi_k, \rho_i \rangle \in M$ for $i = 0, 1$. Furthermore, since $\langle \xi_k, \rho \rangle = \langle \xi_k, w_{\xi_k}(\alpha) \rangle \in q_m(\alpha)$, by part (d) of $\varphi(\alpha)$, $\xi_k \in t_m$, so $\langle \xi_k, \rho_i \rangle \in M \cap q_m(\beta_i) = y_m$ for $i = 0, 1$. By part (b) of $\varphi(\alpha)$, $y_m \subset q_m(\alpha)$, so $\langle \xi_k, \rho_i \rangle \in q_m(\alpha)$. Since $\langle \xi_k, \rho \rangle \in q_m(\alpha)$, this implies $\rho = \rho_0 = \rho_1$; hence $\beta_i \in W_{\xi_k \rho_i} = W_{\xi_k \rho}$ for $i = 0, 1$.

Subcase 2(b). Suppose $\xi_k \notin S_m$. Then $\xi_k \in \pi q_m(\alpha) \cap H \backslash S_m \subset \pi v(\alpha) \backslash r_v$, so $U_{\xi_k} \cap (A^* \times A^*) \in u_1^*(\alpha)$, (i.e. $U_{\xi_k} \cap (A_{\beta_i} \times A_{\beta_i}) \in u_{1\beta_i}(\alpha))$ for $i = 0, 1$. Furthermore, $\alpha < \beta_i$, and since (I_j) holds for $j < k$, and $q_m(\alpha)$ satisfies (F), it follows that

$$\{\beta_0, \beta_1\} \subset T_{\xi_k m}(\alpha) \subset U_{\xi_k \rho} = U_{\xi_k w_{\xi_k}(\alpha)}.$$

Thus by (4_{ξ_k}-b), $\omega_\xi(\beta_i) = w_{\xi_k}(\alpha) = \rho$ holds for $i = 0, 1$, i.e. $\{\beta_0, \beta_1\} \subset W_{\xi_k} \rho = Q_{\xi_k \rho}$.

This concludes the proof of Theorem 1.1.

2. Final remarks, open questions

A. As pointed out earlier, QX is not σ-discrete, because a σ-discrete space in which all points are G_δ-sets has a quasi-G_δ-diagonal. (Indeed, if $Y = \bigcup_{m \in \omega} Y_m$ is such a space (with each Y_m a discrete subspace), then for each $y \in Y_m$, let us pick a sequence $\langle G_{ymk} \rangle_{k \in \omega}$ of open sets such that $\{y\} = \bigcap_{k \in \omega} G_{ymk}$ and $G_{ymk} \cap Y_m = \{y\}$ for every $k \in \omega$. Then $\langle \mathcal{G}_{mk} \rangle_{m, k \in \omega}$, where $\mathcal{G}_{mk} = \{G_{ymk} : y \in Y_m\}$, is a quasi-$G_\delta$-diagonal). Moreover, if we only want to make sure that our Q-set space is not σ-discrete, then the construction of QX can be done on c instead of c^+, with minimal changes.

Theorem 2.1. *There is a paracompact perfectly normal Q-set space of cardinality c.*

It is interesting to note that all normal Q-set spaces of cardinality $\leq c$ have a G_δ-diagonal (more generally, all cleavable spaces of cardinality $\leq c$ have a G_δ-diagonal [AS]); hence to get "QX has no (quasi-)G_δ-diagonal" it was necessary to work on c^+ instead of just c.

B. We can't hope that a Q-set space constructed in ZFC will have properties any closer to metrizability then being paracompact; indeed, under $V = L$, not only that there are no metrizable Q-set spaces, but there are no Q-set spaces with character $\leq c$ ([R], [H], [BJ]).

Under $V = L$, every Q-set space is σ-left separated [BJ], so a non-σ-left-separated Q-set space cannot be constructed in ZFC. Of course, under MA + $\neg$ CH, even the real line has Q-set subspaces (see Miller's paper in [KV], e.g.).

C. There are several natural questions which are left open.

Problem 1. Is there a connected normal Q-set space?

For Q-spaces, this is a question of A.V. Arhangel′skiĭ [A2], who also points out that P. deCaux [C] constructed an infinite, regular, connected, σ-closed-discrete space. (Note that, of course, a σ-closed discrete space is a Q-space, but not a Q-set space.)

Problem 2. Is there a strong Q-set space in ZFC, i.e. a space X such that all finite powers of X are Q-set spaces? Can such a space be normal or paracompact?

(Note that under MA + $\neg$ CH, the real line has strong Q-set subspaces.)

Problem 3. Is there, in ZFC, a Q-set space of size ω_1?

It is interesting to note that the answer is yes both under CH and MA(ω_1). Under CH the space in [B] works, and the space of Theorem 2.1 is an example which is even paracompact. Under MA(ω_1), any subset of cardinality ω_1 of the real line is an example.

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Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45058

E-mail address: `ZTBalogh@miavx1.muohio.edu`