

## SUBAVERAGING ESTIMATE FOR $CR$ FUNCTIONS DEFINED ON A HYPERSURFACE $M$ OF $C^n$

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ABSTRACT. Let  $M$  be a smooth real hypersurface of  $C^n$  and  $N$  a compact submanifold of  $M$ . We generalize a result of A. Boggess and R. Dworkin giving, under some geometric conditions on  $M$  and  $N$ , an estimate of the submeanvalue on  $N$  of any  $CR$  function  $f$  on a neighbourhood of  $N$ , by the  $L^1$  norm of  $f$  on a neighbourhood of  $N$  in  $M$ .

### I. INTRODUCTION

Let  $M$  be a smooth real hypersurface of  $C^n$  and  $N$  a real compact submanifold of  $M$  of dimension  $k$ . In 1988, in the general case of  $M$  a smooth  $CR$  generic submanifold of  $C^n$ , under some conditions of nonvanishing for the Levi form at every point  $p$  of  $M$  and of transversality for  $N$ , A. Boggess and R. Dworkin [BD] estimated the submeanvalue of any  $CR$  function  $f$  on a neighbourhood of  $N$  by the  $L^1(M, d\sigma_M)$  norm of  $f$ , where  $d\sigma_M$  is the measure on  $M$ . Their proof was based on the Henkin formulas and the upper-estimation of integral kernels.

In this article, in the hypersurface case, we generalize this result to a pseudoconvex hypersurface of maximal finite type  $m_0$  (in the sense of Kohn [BG]).

**Theorem 1.** *Let  $M$  be a smooth real hypersurface of  $C^n$ . Let  $N$  be a compact submanifold of  $M$ , not tangent complex of real dimension  $k \geq 1$ , without boundary. Suppose that  $M$  is pseudoconvex, of finite type in a neighbourhood of  $N$ . Then, for any neighbourhood  $\omega$  of  $N$  in  $M$ , there exists a constant  $C = C(\omega)$  such that for all functions  $f \in CR(\omega)$  we have:*

$$(1.1) \quad \left| \int_N f d\sigma_N \right| \leq C(\omega) \int_\omega |f| d\sigma_M.$$

Let us denote by  $B_M(p, \delta)$  the nonisotropic ball of center  $p$ , of radius  $\delta$  in the complex tangent directions and of radius  $\delta^m$  (where  $m$  is the type of the point  $p$ ) in the real tangent direction. Our proof of theorem 1 is based on the following punctual estimation result (1.2) obtained by Boggess, Dworkin and Nagel [BDN] and generalized in [P]:

*There exists  $\delta_0 > 0$  such that, for all  $p \in N$ , for all  $0 < \delta_p \leq \delta_0$  and for every point  $Z$  on the normal line to  $M$  through any point  $q = \Pi(Z)$  of the nonisotropic ball  $B_M(p, \delta_p)$ , at a distance less than  $\tau_0 \delta_Z^m$  (where  $\tau_0 > 0$  is a uniform constant*

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and  $\delta_Z \in ]0, \delta_0]$  denotes the radius of the biggest nonisotropic ball centered in  $q$  and contained in  $B_M(p, \delta)$ , we have the following estimate:

$\exists A, C > 0$  such that if  $|Z' - Z| \leq C \cdot \delta_Z^m$ , we have for every CR function  $u$  in a neighbourhood of  $N$ :

$$(1.2) \quad |u(Z')| \leq \frac{A}{|B_M(\Pi(Z'), \delta)|} \int_{\zeta \in B_M(\Pi(Z'), \delta)} |u(\zeta)| d\sigma_M(\zeta),$$

where  $|B_M(\Pi(Z'), \delta)|$  is the measure of the nonisotropic ball for the measure  $\sigma_M$  of  $M$ , which is roughly  $\delta^{2n-2}\delta^m$ , where  $m$  is the type of  $\Pi(Z')$ .

This property remains true for  $\Pi$  denoting a projection on  $M$  (in the sense of the definition on p.213 of [BDN]) and is based on the fact that the points  $Z$  and  $Z'$  are centers of analytic discs with boundary in  $B_M(p, \delta_p)$ .

If  $\omega$  is a neighbourhood of  $N$  in  $M$ , we denote by  $\mathfrak{D}$  the open set which lies in the pseudoconvex side, whose points  $Z'$  all satisfy (1.2) and to which every CR( $\omega$ ) function  $f$  extends holomorphically to  $F$ .

## II. PROOF OF THE THEOREM

As in the article of Boggess and Dwilewicz [BD], assuming  $N$  not tangent complex allows us to construct locally a manifold  $\tilde{N}$  of dimension  $k+1$  transverse to  $M$  whose boundary contains  $N$ . If  $G$  is a (particular) parametrisation of  $N$  and  $U_j$  the open sets of  $\mathbb{R} \times \mathbb{R}^{k-1}$  associated to this parametrisation,  $\tilde{N}$  is locally written as

$$\tilde{N}_{j,\epsilon'} = \{G(x + iy, u); 0 \leq y \leq \epsilon' \text{ and } (x, u) \in U_j\},$$

where  $G(z, u) = G(x + iy, u)$  is a  $\bar{\partial}$  flat extension of  $G(x, u)$  on  $\mathbb{C} \times \mathbb{R}^{k-1}$  (which means that  $\forall N \in \mathbb{N}$ , there exists a constant  $C(N)$  such that  $|\frac{\partial G}{\partial \bar{z}}(x + iy, u)| \leq C(N)|y|^N$ ).

Now, under the assumption of theorem 1, we choose  $\epsilon' > 0$  such that  $\tilde{N}_{j,\epsilon'} \subset \mathfrak{D}$ .

By compactness of  $N$ , we extract from the  $N_j = G(U_j)$  a finite covering of  $N$ ,  $N = \bigcup_{j=1}^l N_j$ , and if  $(\psi_j)_{j=1\dots l}$  is a partition of unity on  $N$  associated to this covering, we write

$$\int_N f d\sigma_N = \sum_{j=1}^l \int_{N_j} F_j \psi_j d\sigma_N.$$

As in [BD], we then consider the function  $\phi_j$  with compact support in  $U_j$  such that  $\phi_j dx du$  is the pull-back by  $G$  of  $\psi_j d\sigma_N$ , denoted by  $G^*(\psi_j d\sigma_N)$ . We extend  $\phi_j$   $\bar{\partial}$  flat and obtain  $\phi_j(z, u)$ , whose support satisfies  $\phi_j(x + iy, u) = 0$  for  $y \geq \frac{\epsilon'}{2}$ . Thanks to  $G^{-1}$ , we then transport the measure  $\phi_j(z, u) dz du$  on  $\tilde{N}_{j,\epsilon'}$  and get

$$I = \int_{N_j} F_j \psi_j d\sigma_N = \int_{N_j} F_j \Phi_j, \text{ where } \Phi_j = G^{-1*}(\phi_j(z, u) dz du).$$

To simplify, we write  $F$  for  $F_j$ ,  $\phi$  for  $\phi_j$ , etc. Taking into account the support of  $\Phi$ , the Stokes formula gives us  $I = I_1 + I_2$ , where

$$I_1 = \int_{0 < y < \epsilon' z = x + iy} \int_{(x, u) \in U} F(G(z, u)) \left(\frac{\partial \phi}{\partial \bar{z}}(z, u)\right) d\bar{z} \wedge dz \wedge du$$

and

$$I_2 = \sum_{l=1}^{l=n} \int_{\substack{0 < y < \epsilon' \\ z = x + iy}} \int_{(x,u) \in U} \frac{\partial F}{\partial w_l}(G(z, u)) \frac{\partial G_l}{\partial \bar{z}}(z, u) \phi(z, u) d\bar{z} \wedge dz \wedge du.$$

Now, let's denote by  $\delta^*$  the distance between the support of  $\phi$  and the complement of  $N_j$ , and consider  $\delta \leq \inf(\frac{\delta_0}{2}, \frac{\delta^*}{2})$ . From now on, we take

$$\epsilon' \leq \frac{\inf_{p \in N} \tau_0 \delta^{m_p}}{4}$$

and we project each point  $Z = G(z, u) \in \tilde{N}_{j, \epsilon'} \subset \mathfrak{D}$  according to  $\Pi$  on  $M$  along  $\tilde{N}_{j, \epsilon'}$ .

Then, using the estimate (1.2), we find that  $\exists C_3 > 0$  such that

$$I_1 \leq \int_{0 < y < \epsilon'} \int_{(x,u) \in U} \frac{C_3}{\delta_Z^{2(n-1)} \delta_Z^{m_{\Pi(Z)}}} \int_{B_M(\Pi(Z), \delta)} |F(\zeta)| d\sigma_M(\zeta) \frac{\partial \phi}{\partial \bar{z}}(z, u) d\bar{z} dz du.$$

Now, for all compact  $K$  in  $M$ , there exist two constants  $C_1, C_2 > 0$  such that  $\forall p \in K, \forall \delta_p, 0 \leq \delta_p \leq 1$  we have  $C_1 \delta_p^{m_0} \leq \delta_p^{m_p} \leq C_2 \delta_p^2$ .

Taking  $K = \bigcup_{Z \in \tilde{N}_{j, \epsilon'}} B_M(\Pi(Z), \frac{\delta_0}{2})$ , we obtain a upper estimate of the inverse of

$\delta_Z^2$ :

$$I_1 \leq \int_{0 < y < \epsilon'} \int_{(x,u) \in U} \frac{C_3}{(\delta_Z^{m_{\Pi(Z)}})^n} \int_{B_M(\Pi(Z), \delta)} |F(\zeta)| d\sigma_M(\zeta) \frac{\partial \phi}{\partial \bar{z}}(z, u) d\bar{z} dz du.$$

Now, because  $\tilde{N}$  is transverse, there exists a constant  $C$  which doesn't depend on  $x, y, u$  such that for all  $0 \leq y < \epsilon'$  and for all  $(x, u) \in U$  we have

$$|G(z, u) - \Pi(G(z, u))| \geq Cte \left| \frac{\partial G}{\partial y}(x, u) \right| |y| \geq C|y|.$$

We also know that:  $\left| \frac{\partial \phi}{\partial \bar{z}}(z, u) \right| \leq Cte(n)|y|^n$  for all  $n \in \mathbb{N}$ . So, we have

$$I_1 \leq Cte \left[ \int_{0 < y < \epsilon'} \int_{(x,u) \in G^{-1}(\cup_{p \in N} B_M(p, \frac{\delta_0}{2}))} d\bar{z} dz du \right] \int_{\cup_{p \in N} B_M(p, \delta)} |F| d\sigma_M,$$

which gives

$$I_1 \leq Cte \epsilon' \delta_0^{k-1} \delta_0^2 \int_{\cup_{p \in N} B_M(p, \delta)} |F| d\sigma_M \leq Cte \epsilon' \delta_0^{k+1} \int_{\omega} |F| d\sigma_M.$$

Now, to estimate  $I_2$ , we use the harmonicity of  $\frac{\partial F}{\partial w_l}(G(z, u))$  on the  $\mathbb{C}^n$  Euclidian ball  $D(G(z, u), R)$  of center  $G(z, u)$  and of radius  $R$ . Let us take  $R = R_{z,u}$  ranging over the distance of  $G(z, u)$  to  $M$ , i.e. roughly  $|G(z, u) - \Pi(G(z, u))|$ .

The transversality of  $\tilde{N}_j$  allows us to write

$$\left| \frac{\partial F}{\partial w_l}(G(z, u)) \right| \leq \frac{Cte}{|y|^{2n+1}} \int_{D(G(z,u), R)} |F(\zeta)| d\zeta.$$

Now, we know that each point  $\zeta \in D(G(z, u), R)$  is the center of an analytic disc with boundary in  $M$ , and we can again use the subestimate (1.2). Using the reasoning in the estimate for  $I_1$  allows us to estimate  $I_2$  from above:

$$I_2 \leq Cte \epsilon'^{2n+1} \delta_0^{k+1} \int_{\bigcup_{p \in N} B_M(p, \delta)} |F| d\sigma_M.$$

We then obtain an estimate on a neighbourhood of  $N$  in  $M$  which is of the form  $\bigcup_{j=1}^{j=k} B_M(p_j, \delta)$ , and so the expected result follows. □

### III. SOME REMARKS

*Remark 1.* We can't have a comparison between the  $L^1(\omega, d\sigma_M)$  norm and the  $L^1(N, d\sigma_N)$  norm of  $f$ . Indeed, consider the unit sphere  $M = \partial B$  of  $\mathbb{C}^2$  and the transverse circle  $N = M \cap \{(z_1, z_2) \in \mathbb{C}^2 ; z_2 = 0\}$ . If such a comparison holds, we would have, for each function  $F(z_1)$ , depending only on  $z_1$  and holomorphic on a neighbourhood of the ball  $B$ , the following estimate:

$$\int_0^{2\pi} |F(e^{i\theta})| d\theta \leq C \int_D |F(re^{i\theta})| r dr d\theta,$$

where  $C > 0$  is a constant and  $D$  denotes the unit disc of  $\mathbb{C}$ .

This would mean that the Bergman space  $L^1(D) \cap H(D)$  is included in the Hardy space  $H^1(D)$ , which is not the case.

*Remark 2.* Under an assumption of strict pseudoconvexity, we obtain a more precise version of (1.1):

**Proposition 1.** *Let  $M$  be a smooth real hypersurface of  $\mathbb{C}^n$  and  $N$  a compact submanifold of  $M$ , not tangent complex of real dimension  $k$ , without boundary and such that  $M$  is strictly pseudoconvex and of finite type 2 in a neighbourhood of  $N$ .*

*Then, there exists a constant  $C > 0$  such that for all  $\delta, 0 < \delta \leq 1$ , and for all CR functions on  $\bigcup_{p \in N} B_M(p, \delta)$ , we have*

$$\left| \int_N f d\sigma_N \right| \leq \frac{C}{\delta^{2n-k-1}} \int_{\bigcup_{p \in N} B_M(p, \delta)} |f| d\sigma_M.$$

*Proof.* It is sufficient to prove it for  $\delta$  small.

Now the support of  $\phi$  is fixed, and we make  $\delta$  smaller than  $\inf(\frac{\delta_0}{2}, \frac{\delta^*}{2})$ , so  $\phi$  is not zero on  $\{G(x + iy, u); y = \epsilon' \text{ and } (x, u) \in U_j\}$ , and then the Stokes formula involves an other integral, which is

$$J = \int_{\left\{ \begin{array}{l} y = \epsilon' \\ (x, u) \in U \end{array} \right.}} F(G(x + i\epsilon', u)) \phi(x + i\epsilon', u) dx du.$$

By the assumption of strict pseudoconvexity, we know that  $\epsilon' = C\delta^2$ , where  $C$  is a uniform constant. Using (1.2) again, we get the following estimate:

$$J \leq Cte \left[ \int_{(x, u) \in G^{-1}(\bigcup_{p \in N} B_M(p, \delta))} \frac{dx du}{\delta_{G(x+i\epsilon', u)}^{2n}} \right] \int_{\bigcup_{p \in N} B_M(p, \delta)} |F| d\sigma_M.$$

All the points  $G(x + i\epsilon', u)$  lie on  $\tilde{N}_{j,\epsilon'}$  and so satisfy  $\delta^2 = \delta_{G(x+i\epsilon',u)}^2$  for all  $(x, u) \in U$ . As the upper estimates of  $I_1$  and  $I_2$  are still available, we get that the constant  $C(\omega)$  of (1.1) is of the form  $\frac{Cte}{\delta^{2n-k-1}}$  for  $\omega = \bigcup_{p \in N} B_M(p, \delta)$ .  $\square$

*Remark 3.* Boggress and Dwilewicz gave a counterexample to estimate (1.1) if the assumption of transversality of  $N$  was not satisfied. Their example deals with the case where  $M$  is the Heisenberg domain and  $N$  a complex tangent curve to  $M$ . But  $M$  is then strictly pseudoconvex, and so every complex tangent curve in  $M$  is a peak set [DO]. So if we consider the associated  $n$ -powers of the peak function, we immediately get a contradiction of (1.1). So, it could be better to consider the following counterexample:

Let  $M$  be the Heisenberg domain and consider the mixed curve (complex tangent at 0, and transverse to the complex tangent plane at a point different from zero) defined, for  $\lambda$  and  $\mu$  given positive reals, by  $z(1) = z(-1)$  and by

$$z(t) = (z_1(t), z_2(t)) = (te^{it}, t^2) \text{ for } |t| \leq \lambda,$$

$$Re(z_2(t)) \geq \mu \text{ for } \lambda \leq |t| \leq 1.$$

Now consider the function  $f_\epsilon(z) = \frac{1}{z_2 + \epsilon}$ ; we can obtain, by computations analogous to those of [BD], constants  $C_1, C_2, C_3$  and  $C_4$  such that

$$\int_M |f_\epsilon| d\sigma_M \leq -C_1 C_2 \ln(\epsilon) \quad \text{and} \quad \left| \int_N f_\epsilon d\sigma_N \right| \geq \frac{C_4}{\sqrt{\epsilon}} - C_3.$$

These estimates for the function  $f_\epsilon \in CR(M)$  contradict then the estimate (1.1).

*Remark 4.* If  $N$  is a submanifold with boundary, we can't have an estimate of the form (1.1). Indeed, take for instance for  $M$  the unit sphere  $S^3$  of  $\mathbb{C}^2$  and for  $N$  a circle arc with extremities 0 and  $\pi$  contained in the circle  $M \cap \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 = 0\}$ .  $N$  is obviously not tangent complex to  $M$ . Then we consider the functions  $f$  holomorphic in a neighbourhood of the unit ball and depending only on the variable  $z_1$ . These are actually  $CR(S^3)$  functions.

Let's denote by  $\Delta$  the unit disc of  $\mathbb{C}$  and write the inequality (1.1) for the neighbourhood  $\omega = S^3$  and the functions  $f$ . So there exists  $C(N, S^3)$  such that

$$\left| \int_0^\pi f(e^{i\theta}) d\theta \right| \leq C \left| \int_\Delta r dr d\theta \frac{1}{2\pi} \int_{-\pi}^{+\pi} |f(re^{i\theta})| d\phi \right| \leq C \|f\|_{L^1(\Delta)}.$$

Then  $F$  defined on the Bergman space of the disc  $B^2(\Delta)$  by

$$\begin{aligned} F : B^2(\Delta) &\rightarrow \mathbb{C}, \\ f &\mapsto \int_0^\pi f(e^{i\theta}) d\theta, \end{aligned}$$

is a linear continuous bounded operator on  $B^2(\Delta)$ , which involves the existence of a unique function  $g \in B^2(\Delta)$  such that

$$(1.3) \quad \int_0^\pi f(e^{i\theta}) d\theta - \int_\Delta f(z) \overline{g(z)} d\lambda(z) = 0 \quad \forall f \in L^2(\Delta),$$

where  $d\lambda$  denotes the Lebesgue measure on  $\mathbb{C}$ .

In particular,

$$\int_0^\pi e^{in\theta} d\theta = \int_0^1 \int_0^{2\pi} \overline{g(re^{i\theta})} r^{n+1} e^{in\theta} dr d\theta.$$

Now, expressing  $g$  by the mean of its Fourier series  $\sum_{p \in \mathbf{N}} a_p r^p e^{ip\theta}$ , we obtain

$$\frac{2}{n} = \sum_{p \in \mathbf{N}} \overline{a_p} \int_0^1 r^{n+p+1} dr \int_0^{2\pi} e^{i(n-p)\theta} d\theta = \frac{\pi}{n+1} \overline{a_n}.$$

Now  $g \in B^2$  if and only if  $\sum_n \frac{|a_n|^2}{n+1}$  converges. But  $\sum_n (n+1) \left(\frac{2}{n}\right)^2$  diverges, which gives a contradiction.  $\square$

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