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CURVES IN GRASSMANNIANS

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Abstract. This paper considers curves in Grassmannians which are themselves immersed in projective space by the Plücker map. It is shown that for a generic vector bundle of high enough degree, the image curve lies in a proper linear subvariety of this projective space and satisfies good conditions on syzygies as a curve in this subspace. For very small degree and generic vector bundle, the curve is non-degenerate.

Introduction

Given a curve C and a line bundle L with sections, one can consider the rational map

$$C \to \mathbb{P}(H^0(L))$$

from the curve to projective space associated to the complete linear series of L. A great deal is known about the image curve when the degree of the line bundle is high enough. For example, the curve is projectively normal if $degL \geq 2g+1$ (cf. [C], [M]). Saint Donat [SD] proved that the ideal of the curve is generated by quadrics and cubics and, if $degL \geq 2g+2$, it is generated by quadrics alone. In the same vein, one can define condition N_p for a curve. Roughly, it means that the curve is projectively normal and in a minimal resolution of the ideal generators have the smallest possible degree up to the p^{th} term. Green [G] showed that if $degL \geq 2g+p+1$, then the curve satisfies property N_p.

Consider now a vector bundle E of degree d and rank $n \geq 2$ on the curve. Assume that it has enough sections. We obtain a map

$$C \to \mathbb{G}(n, H^0(E))$$

where $\mathbb{G}$ denotes the Grassmannian. In contrast with the line bundle case, nothing seeems to be known about the geometry of the image curve in such a Grassmanian (say under good conditions for E). In this paper, we try to clarify the situation somehow.

Our main results are in Propositions (1.1) and (1.2). We consider the composition f of the following two maps:

$$C \to \mathbb{G}(n, H^0(E)) \to \mathbb{P}(\wedge^n H^0(E))$$

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where the last map is the Plücker immersion of the Grassmannian in projective space. In (1.1), we show that for very small degree the image of f is a non-degenerate curve in projective space. In (1.2) we show that for sufficiently high degree the image of f lies on a proper linear subvariety and satisfies property N_p as a curve in this variety.

Setup of the problem and proof of the first results

We want to look at a curve from an external point of view. We first need to decide on the setup. Most interesting vector bundles are stable or at least semistable. In contrast with line bundles, even stable vector bundles of high degree don't behave all in the same way. We shall restrict our attention to properties that hold for the generic vector bundle. As we want to define a map to the Grassmannian, we need $h^0(E) \geqq n+1$. If the vector bundle is generic, this implies $d \geqq ng+1$.

The map f is given by means of a (not necessarily complete) linear series. The line bundle that gives rise to this linear series is the determinant of the vector bundle E. The subspace of the space of sections is the image of the map

$$\psi : \wedge^n H^0(E) \to H^0(\wedge^n E).$$

The first question that we want to ask is whether the curve is degenerate in projective space (i.e. lies on some hyperplane). The answer is yes for sufficiently high degree. In fact the curve is degenerate in projective space when the map ψ above is not injective. For a generic vector bundle of degree $d \geq n(g-1)$, $h^1(E) = 0$. Therefore, $h^0(E) = d + n(1-g)$ and $dim \wedge^n H^0(E) = \binom{d+n(1-g)}{n}$. On the other hand, $\wedge^n E$ is a line bundle and $h^0(\wedge^n E) = d+1-g$. Hence, when d is large, $dim \wedge^n H^0(E) \geqq h^0(\wedge^n E)$ and the map cannot be injective. For example, in rank two, injectivity implies $d \leqq \frac{4g-1+\sqrt{8g+1}}{2}$.

We do not know whether injectivity holds in the whole allowable range (for generic vector bundle). We prove in the next two sections that for very small degree, the map is indeed injective. On the other hand, one can show that the map is surjective for generic E if the degree is high enough. The next two propositions show our present state of knowledge. One could ask in general whether ψ always has maximal rank for a generic E.

(1.1) Proposition. *Let E be a generic stable vector bundle of degree $ng+1$ or $ng+2$. Then, the map ψ is injective and therefore the image by f does not lie on any proper linear subvariety.*

(1.2) Proposition. *Let E be a generic stable vector bundle of degree $d \geqq 2ng+1$. Then, the map ψ is surjective. Hence the set of hyperplanes in the ambient space of the Grassmannian cut a complete linear system on the curve. If moreover $d \geqq 2g+p+1$, then the image curve satisfies property N_p in the sublinear variety that it spans.*

Proposition (1.1) will be proved in the next two sections.

Proof of (1.2). In order to prove (1.2), it is enough to exhibit a vector bundle of degree d for which the map is surjective. We shall do this by induction on n. For rank one, there is nothing to prove. Assume the result is true for vector bundles of rank $n-1$. Consider a generic extension

$$0 \to L \to E \to F \to 0,$$

where $degL = 2g$. Then F is a generic vector bundle of rank $n-1$ and degree $degF = degE - 2g \geqq 2(n-1)g + 1$. Hence, the induction assumption applies to F and the map

$$\wedge^{n-1}H^0(F) \to H^0(\wedge^{n-1}F)$$

is surjective. We need to use the following theorem of Mumford [M].

(1.3) Proposition (Mumford). *If L_1, L_2 are line bundles with $degL_1 \geq 2g$ and $degL_2 \geqq 2g+1$, then the map*

$$H^0(L_1) \otimes H^0(L_2) \to H^0(L_1 \otimes L_2)$$

is surjective.

By (1.3), the map

$$H^0(L) \otimes H^0(\wedge^{n-1}F) \to H^0(L \otimes \wedge^{n-1}F)$$

is surjective. Moreover, as $degL = 2g$, $h^1(L) = 0$ and $H^0(E) = H^0(L) \oplus H^0(F)$. Also $\wedge^n E = L \otimes \wedge^{n-1}F$. The surjectivity of ψ then follows.

The second statement follows directly from Green and Lazarsfeld's theorem. □

(1.4) Remark. In characteristic zero, the hypothesis of genericity in (1.2) is unnecessary: It was proved by Butler [B] that if E, F are semistable vector bundles satisfying $degE = d > 2g, \mu(F) \geqq 2g$, then the map $H^0(E) \otimes H^0(F) \to H^0(E \otimes F)$ is surjective. Hence, by induction on n, the map $\otimes^n H^0(E) \to H^0(\wedge^n E)$ is surjective. Moreover the tensor product of semistable vector bundles is semistable. As $\otimes^n E$ and $\wedge^n E$ have the same slope, the symmetric product $S^n(E)$ is also semistable of the same slope. Hence, $h^1(S^n(E)) = 0$ and the map $H^0(\otimes^n(E)) \to H^0(\wedge^n(E))$ is surjective.

$N+1$ SECTIONS

In this section, we prove (1.1) for $d = ng+1$. Our method of proof is to degenerate the generic E to a direct sum of a generic line bundle L of degree g and a generic vector bundle F of rank $n-1$ and degree $(n-1)g+1$. The map is not injective in this case, but the elements in the kernel cannot be deformed to a generic infinitessimal deformation of E. The techniques used are inspired by [W], Prop. (1.2).

With our assumptions, $H^0(E) = H^0(L) \oplus H^0(F)$ and $h^0(L) = 1$, $\wedge^n H^0(E) = (H^0(L) \otimes \wedge^{n-1}H^0(F)) \oplus \wedge^n H^0(F)$. As F is generic, it has n sections and by induction on n, we can assume that the map

$$\wedge^{n-1}H^0(F) \to H^0(\wedge^{n-1}F).$$

is injective. Then, as $h^0(L) = 1$, the map

$$H^0(L) \otimes (\wedge^{n-1}H^0(F)) \to H^0(L \otimes \wedge^{n-1}F) = H^0(E)$$

is also injective. On the other hand, as F has rank $n-1$, $\wedge^n H^0(F)$ is contained in the kernel of ψ. Therefore, the kernel of ψ in this case is $\wedge^n H^0(F)$.

Denote by $s^k, k = 1, \ldots, n$, a basis for $H^0(F)$. We want to see that $s^1 \wedge \ldots \wedge s^n$ does not extend to an element in the kernel of ψ_ϵ for a generic infinitessimal deformation E_ϵ of E. Recall that an infinitessimal deformation E_ϵ of E is given by an element

$e \in H^1(E^* \otimes E)$. Up to the choice of a suitable covering $C = \bigcup U_i$, we can represent e by a cocicle

$$\varphi_{ij} \in H^0(U_i \cap U_j, E^* \otimes E).$$

Then, E_ϵ can be represented locally as $E_{|U_i} \oplus \epsilon E_{\ |U_i}$ with gluings on $U_i \cap U_j$ given by the matrix

$$\begin{pmatrix} Id & 0 \\ \varphi_{ij} & Id \end{pmatrix}$$

In the case $E = L \oplus F$,

$$\varphi_{ij} = \begin{pmatrix} f_{ij}^{11} & f_{ij}^{21} \\ f_{ij}^{12} & f_{ij}^{22} \end{pmatrix}$$

where $f_{ij}^{kl} \in H^0(U_i \cap U_j, Hom(L_k, L_l)), L_t = L$ (resp. F) if $t = 1$ (resp.2).

Denote by

$$s_\epsilon^k = (0, s^k) + \epsilon(s_{1i}^k, s_{2i}^k)$$

an extension to E_ϵ of the section s^k. Then, as $\epsilon^2 = 0$ and $\wedge^n H^0(F) \subset Ker\psi$,

$$\psi_\epsilon(s_\epsilon^1 \wedge ... \wedge s_\epsilon^n) = \epsilon \sum_{k=1}^{n} (-1)^k s_{1i}^k \wedge (\wedge_{l \neq k} s^l)$$

As F is a vector bundle of rank $n-1$, it is locally a direct sum of $n-1$ copies of the trivial bundle. When we take such a local representation, the condition

$$\psi_\epsilon(s_\epsilon^1 \wedge ... \wedge s_\epsilon^n) = 0$$

can be written as

$$det \begin{pmatrix} s_{1i}^1 & \dots & s^1 \\ \dots & \dots & \dots \\ s_{1i}^n & \dots & s^n \end{pmatrix} = 0,$$

where each s^i stands for a row of length $n-1$. This means that the first column in the matrix is a linear combination of the remaining $n-1$. Equivalently, there exists a local section $\phi_i : F \to L$ on U_i such that $s_{1i}^k = \phi_i(s^k)$. The gluing conditions can be read as $s_{2j}^k = s_{2i}^k + f_{ij}^{22} s^k$, $s_{1j}^k = s_{1i}^k + f_{ij}^{12} s^k$. The substitution $s_{1i}^k = \phi_i(s^k)$ gives $\phi_i + f_{ij}^{21} = \phi_j$.

We now consider the set

$$\{(\varphi_{ij}, \phi_i, s_{2i}^1 ... s_{2i}^n) \in C^1(E^* \otimes E) \oplus C^0(F^* \otimes L) \oplus (C^0(F))^n$$

$$|\phi_j = \phi_i + f_{ij}^{21}, s_{2j}^k = s_{2i}^k + f_{ij}^{22} s^k\}.$$

We can realize this set as the first hypercohomology group $\mathbb{H}$ of the double complex of sheaves

$$\begin{array}{ccccc} C^0(E^* \otimes E) & \to & C^1(E^* \otimes E) & \to & C^2(E^* \otimes E) \\ \downarrow & & \downarrow & & \downarrow \\ C^0(L \otimes F^*) \oplus (C^0(F))^n & \to & C^1(L \otimes F^*) \oplus (C^1(F))^n & \to & C^2(L \otimes F^*) \oplus (C^2(F))^n \end{array}$$

The vertical maps in this diagram are induced by $(\pi^{21}, \pi^{22}(s^k))$.

We then have an exact sequence

$$0 \to H^0(E^* \otimes E) \to H^0(L \otimes F^*) \oplus (H^0(F))^n \to \mathbb{H}$$
$$\to H^1(E^* \otimes E) \to H^1(L \otimes F^*) \oplus (H^1(F))^n.$$

Note how this last map is defined by

$$\alpha : H^1(E^* \otimes E) \to H^1(L \otimes F^*) \oplus (H^1(F))^n$$
$$(\varphi_{ij}) \quad \to \quad (f_{ij}^{21}, f_{ij}^{22}(s^k)).$$

As $degF = (n-1)g+1, h^1(F) = 0$. As $degL = g, h^1(L \otimes F^*) = (n-1)(g-1)+1 \neq 0$, $Ker\alpha = \{\varphi_{ij} | f_{ij}^{21} = 0\}$. Therefore, if the deformation of E does not preserve F as a subbundle, $s_\epsilon^1 \wedge ... \wedge s_\epsilon^n \notin Ker\psi_\epsilon$. This completes the proof of the statement.

$N + 2$ SECTIONS

We now prove the analogous result for a generic vector bundle of rank n with $n+2$ sections.

We first degenerate the vector bundle E to a direct sum $E = L_1^1 \oplus F_2 \oplus L_1^4 \oplus ... \oplus L_1^n$ where subindices denote ranks $degL_1^1 = g+1, degF_2 = 2g+1, degL_1^4 = ... = degL_1^n = g$. We next degenerate $F_2 = L_1^2 \oplus L_1^3, degL_1^2 = g+1$. We then obtain

$$E = L_1^1 \oplus L_1^2 \oplus ... \oplus L_1^n, degL_1^1 = degL_1^2 = g+1, degL_1^3 = ... = degL_1^n = g.$$

We assume all vector bundles appearing in the decompositions above to be generic. Hence, $h^0(L_1^1) = h^0(L_1^2) = 2, h^0(L_1^3) = ... = h^0(L_1^n) = 1$. Then

$$\wedge^n H^0(E) = \wedge^2 H^0(L_1^1) \otimes \wedge^2 H^0(L_1^2) \otimes (\oplus_{3 \leq i_5 < .. < i_n} H^0(L_1^{i_5}) \otimes ... \otimes H^0(L_1^{i_n}))$$

$$\oplus \wedge^2 H^0(L_1^1) \otimes (\oplus_{i_k \neq 1} H^0(L_1^{i_3}) \otimes ... \otimes H^0(L_1^{i_n}))$$

$$\oplus \wedge^2 H^0(L_1^2) \otimes (\oplus_{i_k \neq 2} H^0(L_1^{i_3}) \otimes ... \otimes H^0(L_1^{i_n}))$$

$$\oplus H^0(L_1^1) \otimes ... \otimes H^0(L_1^n)$$

As L_1^1 is generic and $h^0(L_1^1) = 2$, the kernel of the map $H^0(L_1^1) \otimes H^0(L_1^2) \to H^0(L_1^1 \otimes L_1^2)$ is $H^0(L_1^2 \otimes (L_1^1)^{-1}) = 0$. Moreover, as $H^0(L_1^i) = 1, i = 3, \ldots, n$, the composition map

$$H^0(L_1^1) \otimes ... \otimes H^0(L_1^n) \to H^0(L_1^1 \otimes L_1^2) \otimes H^0(L_1^3) \otimes \cdots \otimes H^0(L_1^n)$$
$$\to H^0(L_1^1 \otimes ... \otimes L_1^n)$$

is injective. So, the kernel of the map ψ_ϵ is given in this case by the other three summands in $\wedge^n H^0(E)$.

We check first that the elements in

$$\wedge^2 H^0(L_1^1) \otimes \wedge^2 H^0(L_1^2) \otimes (\oplus_{3 \leq i_5 \leq .. \leq i_n} H^0(L_1^{i_5}) \otimes ... \otimes H^0(L_1^{i_n}))$$

can be deformed to elements in the kernel of ψ_ϵ for any infinitessimal deformation E_ϵ of E.

We choose bases $s^1, s'^1 \in H^0(L_1^1), s^2, s'^2 \in H^0(L_1^2), s^i \in H^0(L_1^i)$. We consider deformations of these sections to a generic E_ϵ. With the local description $E_\epsilon =$

$E \oplus \epsilon E$ and taking into account that $E = L_1^1 \oplus L_1^2 \oplus ... \oplus L_1^n$, the sections can be written as

$$s_\epsilon^1 = s^1 + \epsilon \overline{s}_i^1 = (s^1, 0, .., 0) + \epsilon(s_{1i}^1, ..., s_{ni}^1)$$
$$...$$
$$s_\epsilon^n = s^n + \epsilon \overline{s}_i^n = (0, .., 0, s^n) + \epsilon(s_{1i}^n, ..., s_{ni}^n).$$

Independently of the choice of the $\overline{s}_i$, we get

$$\psi_\epsilon(s_\epsilon^1 \wedge s'^1_\epsilon \wedge s_\epsilon^2 \wedge s'^2_\epsilon \wedge s_\epsilon^{i_5} \wedge ... \wedge s_\epsilon^{i_n}) = 0.$$

We want to check now that these are the only elements that deform in an arbitrary direction as elements of the kernel. Assume the opposite. Then we can find an element in

$$\wedge^2 H^0(L_1^2) \otimes (\oplus_{i_k \neq 2} H^0(L_1^{i_3}) \otimes ... \otimes H^0(L_1^{i_n}))$$

$$(*) \qquad \oplus \wedge^2 H^0(L_1^1) \otimes (\oplus_{i_k \neq 1} H^0(L_1^{i_3}) \otimes ... \otimes H^0(L_1^{i_n}))$$

in the kernel of ψ_ϵ that deforms in an arbitrary direction. We first consider an element of the form $s^1 \wedge s'^1 \wedge s^{i_3} \wedge ... \wedge s^{i_n}, \{i_3, .., i_n\} \cup \{k\} = \{2, .., n\}$. We want to show that these elements deform in the infinitessimal directions $\varphi_{ij} = (f_{ij}^{pq})$ such that $f_{ij}^{pq} = 0$ if $p = 1, q = k$. Note that $\psi_\epsilon(s^1 \wedge s'^1 \wedge s^{i_3} \wedge ... \wedge s^{i_n}) = \epsilon(s^1 s'^1_{ki} - s_{ki}^1 s'^1) s^{i_3} ... s^{i_n}$. This expression will be zero if and only if $s^1 s^{1\prime}_{ki} - s_{ki}^1 s^{1\prime} = 0$. As $s^1, s^{1\prime}$ have no common zeroes, $s_{ki}^1 = s^1 t_i, s^{1\prime}_{ki} = s^{1\prime} t_i$. Define

$$A = (L_1^1)^n \oplus ... \oplus (L_1^k (L_1^1)^{-1}) \oplus (L_1^k)^{n-2} \oplus ... \oplus (L_1^n)^n$$

We now consider the set

$$\{(\varphi_{ij}, s_{1i}^1, s^{1\prime}_{1i} .. s_{1i}^n, ..., t_i, s_{ki}^3 .. s_{ki}^n ... s_{ni}^n) \in C^1(E^* \otimes E) \oplus C^0(A)$$

$$| t_j = t_i + f_{ij}^{1k}, s_{mj}^l = s_{mi}^l + f_{ij}^{lm} s^l\}.$$

We can realize this set as the first hypercohomology group $\mathbb{H}$ of the double complex of sheaves

$$\begin{array}{ccccc} C^0(E^* \otimes E) & \to & C^1(E^* \otimes E) & \to & C^2(E^* \otimes E) \\ \downarrow & & \downarrow & & \downarrow \\ C^0(A) & \to & C^1(A) & \to & C^2(A) \end{array}$$

We then have an exact sequence

$$0 \to H^0(E \otimes E^*) \to H^0(A) \to \mathbb{H} \to H^1(E \otimes E^*) \to H^1(A).$$

We are interested in the kernel of this last map. Note that $h^1(L_1^i) = 0$ and $h^1(L_1^k \otimes (L_1^1)^{-1}) = g$. Hence, the kernel consists of those φ_{ij} such that $f_{ij}^{1k} = 0$.

Assume now that an element in (*) deforms in a generic direction Write this element as

$$\sum a_{i_3 ... i_n} s^1 \wedge s'^1 \wedge s^{i_3} \wedge ... \wedge s^{i_n} + \sum b_{i_3 ... i_n} s^2 \wedge s'^2 \wedge s^{i_3} \wedge ... \wedge s^{i_n}.$$

Then, it deforms also in the directions that have

$$f_{ij}^{1k} = 0, k = 2, ..., n, f_{ij}^{2k} = 0, k \neq 2, l, f_{ij}^{2l} \neq 0.$$

Then, the coefficient $b_{i_3,\dots,i_n} = 0$. Reasoning in this way for all l and replacing 2 by 1, we obtain that all coefficients are 0. In particular, this completes the proof for $n = 2, 3$.

We now go back to the degeneration $E = L_1^1 \oplus F_2 \oplus L_1^4 \oplus ... \oplus L_1^n$. We need only show that the elements in

$$\oplus_{\{i_5,\dots,i_n\}\subset\{4,\dots,n\}} \wedge^2 H^0(L_1^1) \otimes \wedge^2 H^0(L_1^1) \otimes \wedge^2 H^0(F_2) \otimes H^0(L_1^{i_5}) \otimes \cdots \otimes H^0(L_1^{i_n})$$
$$\oplus \oplus_{\{i_6,\dots,i_n\}\subset\{4,\dots,n\}} \wedge^2 H^0(L_1^1) \otimes \wedge^3 H^0(F_2) \otimes H^0(L_1^{i_6}) \otimes \cdots \otimes H^0(L_1^{i_n})$$

are not in the kernel. Reasoning as before, we see that the the only sections that deform to sections in the kernel are those in

$$\oplus_{\{i_6,\dots,i_n\}\subset\{4,\dots,n\}} \wedge^2 H^0(L_1^1) \otimes \wedge^3 H^0(F_2) \otimes H^0(L_1^{i_6}) \otimes ... \otimes H^0(L_1^{i_n}).$$

Therefore, the only sections that could originally be deformed in all directions are those in

$$\oplus_{\{i_6,\dots,i_n\}\subset\{4,\dots,n\}} \wedge^2 H^0(L_1^1) \otimes \wedge^2 H^0(L_1^2) \otimes H^0(L_1^3) \otimes H^0(L_1^{i_6}) \otimes \cdots \otimes H^0(L_1^{i_n}).$$

By symmetry in the $L^i, i = 3, ..., n$, this concludes the proof.

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