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ON THE OSCILLATION OF NONLINEAR TWO-DIMENSIONAL DIFFERENTIAL SYSTEMS

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ABSTRACT. An oscillation criterion is given for a certain form of nonlinear two–dimensional differential systems. This criterion originated in a well-known oscillation result due to Coles (as extended and improved by Wong) concerning second order nonlinear differential equations with alternating coefficients.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The oscillation problem for second order nonlinear differential equations is of particular interest and, therefore, it is the subject of many investigations. It is an interesting problem to extend oscillation criteria for second order nonlinear differential equations to the case of nonlinear two-dimensional differential systems. Such differential systems include in particular the second order nonlinear differential equations. In this paper, a classical oscillation criterion due to Coles [3] (as extended and improved by Wong [11], [12]) for second order nonlinear differential equations is extended to nonlinear two-dimensional differential systems.

Consider the nonlinear two-dimensional differential system

$$x'(t) = b(t)g[y(t)],\ y'(t) = -a(t)f[x(t)], \tag{S}$$

where *a and b are continuous real-valued functions on an interval $[t_0, \infty)$, and f and g are continuous real-valued functions on the real line $\mathbb{R}$ with the sign property*

$$uf(u) > 0 \textit{ and } ug(u) > 0 \textit{ for all } u \in \mathbb{R} - \{0\}.$$

It will be supposed that *b is nonnegative on $[t_0, \infty)$, f is continuously differentiable on $\mathbb{R} - \{0\}$ and satisfies*

$$f'(u) \geq 0 \textit{ for every } u \neq 0,$$

and *g is increasing on $\mathbb{R}$.* Note that *no restriction is imposed on the sign of the coefficient a.*

Throughout the paper, we shall restrict our attention only to the solutions of the differential system (S) which exist on some ray $[T_0, \infty)$, where $T_0 \geq t_0$ may depend on the particular solutions. Note that under quite general conditions there will always exist solutions of (S) which are continuable to an interval $[T_0, \infty), T_0 \geq t_0$, even though there will also exist noncontinuable solutions.

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As usual, a continuous real-valued function defined on an interval $[T_0, \infty)$ is said to be *oscillatory* if it has arbitrarily large zeros, and otherwise it is said to be *nonoscillatory*. A solution (x, y) of the differential system (S) will be called oscillatory if both x and y are oscillatory functions, and otherwise it will be called nonoscillatory. The differential system (S) will be called oscillatory if all its solutions are oscillatory.

It is remarkable that, in the case where the coefficient b is assumed to be not identically zero on any interval of the form $[\tau_0, \infty), \tau_0 \geq t_0$, from the first equation of (S) it follows easily that, for any solution (x, y) of the differential system (S), the oscillation of x implies that y is also oscillatory. So, in this case, if (x, y) is a nonoscillatory solution of (S), then x is always nonoscillatory.

Some oscillation results for the differential system (S) have been presented by Kwong and Wong [5]. More precisely, generalizations of Atkinson's theorem [1] for second order superlinear differential equations and of Belohorec's theorem [2] for second order sublinear equations are given in [5]. Moreover, in [5] an extension of Waltman's classical theorem [9] for second order nonlinear differential equations is established.

The special case where $f(u) = |u|^\lambda \mathrm{sgn} u,\ u \in \mathbb{R}$ and $g(u) = |u|^\mu \mathrm{sgn} u,\ u \in \mathbb{R}$ $(\lambda > 0, \mu > 0)$ is of particular interest. In this case, the differential system (S) becomes

$$(\mathrm{S}_0) \qquad x'(t) = b(t)|y(t)|^\mu \mathrm{sgn} y(t),\ y'(t) = -a(t)|x(t)|^\lambda \mathrm{sgn} x(t),$$

where λ and μ are positive constants. System (S_0) is the prototype of (S). For some oscillation results for the differential system (S_0) we refer to Mirzov [6],[7],[8].

In the particular case where b is positive on $[t_0, \infty)$ and $g(u) = u, u \in \mathbb{R}$, the differential system (S) reduces to the second order nonlinear differential equation

$$[x'(t)/b(t)]' + a(t)f[x(t)] = 0.$$

For $b(t) = 1$ for $t \geq t_0$, the last equation becomes

$$(\mathrm{E}) \qquad x''(t) + a(t)f[x(t)] = 0.$$

The prototype of (E) is the differential equation

$$(\mathrm{E}_0) \qquad x''(t) + a(t)|x(t)|^\lambda \mathrm{sgn} x(t) = 0,$$

where $\lambda > 0$. System (S_0) is reduced to (E_0) when $b(t) = 1$ for $t \geq t_0$ and $\mu = 1$.

Kwong and Wong [5] established that, if the coefficient b is such that

$$(\mathrm{C}_1) \qquad \int_{t_0}^\infty b(t)dt = \infty$$

and g satisfies

$$(\mathrm{C}_2) \qquad \liminf_{u \to \pm\infty} |g(u)| > 0,$$

then the condition

$$(\mathrm{C}_3) \qquad \int_{t_0}^\infty a(s)ds = \infty$$

is sufficient for the oscillation of the differential system (S). [Note that this result holds true without the assumption that g is increasing on $\mathbb{R}$. Clearly, the increasing character of g on $\mathbb{R}$ implies condition (C_2).] This oscillation criterion is an extension of a well-known oscillation theorem due to Waltman [9] (see also Wong [10] and

Kwong and Wong [4]) for second order nonlinear differential equations with alternating coefficients. Our purpose here is to examine the oscillation of the differential system (S) in a case where (C_3) fails. More precisely, our interest is concentrated to the case where

$$(C_4) \qquad \int_{t_0}^{\infty} a(s)ds \quad \textit{exists as a real number.}$$

Throughout the paper, if (C_4) holds, by A we will denote the function defined by

$$A(t) = \int_t^{\infty} a(s)ds, \ t \geq t_0.$$

Coles [3] showed that, if (C_4) holds and $A(t) \geq 0$ for $t \geq t_0$, then the condition

$$(C_5) \qquad \int_{t_0}^{\infty} A(t)dt = \infty$$

suffices for the oscillation of the differential equation (E_0) in the superlinear case $\lambda > 1$. This result has been improved by Wong [11], who proved that the assumption that A is nonnegative on $[t_0, \infty)$ can be removed. In [12], Wong extended Cole's criterion (without the restriction that $A(t) \geq 0$ for $t \geq t_0$) for the more general case of the differential equation (E), where f *is strongly superlinear in the sense that*

$$(C_6) \qquad \int^{\infty} \frac{du}{f(u)} < \infty \textit{ and } \int^{-\infty} \frac{du}{f(u)} < \infty.$$

More precisely, Wong [12] established the following oscillation theorem for the differential equation (E).

Theorem 0. *Let* (E) *be strongly superlinear in the sense that* (C_6) *is satisfied, and suppose that* (C_4) *holds. Then* (C_5) *is a sufficient condition for the oscillation of the differential equation* (E).

Our aim in this paper is to present an extension of Theorem 0 to the differential system (S). For this purpose, we introduce the following assumption:

(C_7) *For all sufficiently small* u *and every* $v > 0$

$$g(u)g(v) \leq g(uv) \leq g(u)[-g(-v)].$$

This assumption holds by itself in the case where $g(u) = |u|^{\mu}\mathrm{sgn}u, u \in \mathbb{R}$ $(\mu > 0)$, and especially when $g(u) = u, u \in \mathbb{R}$. We will consider the case where *the function* $g \circ f$ *is strongly superlinear in the sense that*

$$(C_8) \qquad \int^{\infty} \frac{du}{g[f(u)]} < \infty \textit{ and } \int^{-\infty} \frac{du}{g[f(u)]} < \infty.$$

Note that in the special case where $g(u) = u, u \in \mathbb{R}$, condition ($C_8$) is reduced to ($C_6$).

Our main result is the following theorem.

Theorem. *Let the function* $g \circ f$ *be strongly superlinear in the sense that* (C_8) *is satisfied, and suppose that* (C_1), (C_4) *and* (C_7) *hold. The differential system* (S) *is oscillatory if*

$$(C_9) \qquad \int_{t_0}^{\infty} b(t)g[A(t)]dt = \infty.$$

Our theorem reduces to Theorem 0 in the special case $b(t) = 1$ for $t \geq t_0$ and $g(u) = u, \ u \in \mathbb{R}$.

2. A technical lemma

In order to prove our theorem we make use of the following technical lemma.

Lemma. *Let (x,y) be a solution on an interval $[\tau,\infty)$, $\tau \geq t_0$, of the differential system* (S) *with $x(t) > 0$ for all $t \geq \tau$. Moreover, let $\tau^\star \geq \tau$ and c be a real constant. If*

$$-\frac{y(\tau)}{f[x(\tau)]} + \int_\tau^t a(s)ds + \int_\tau^{\tau^\star} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2}ds \geq c \ for \ every \ t \geq \tau^\star,$$

then

$$y(t) \leq -cf[x(\tau^\star)] \ for \ all \ t \geq \tau^\star.$$

Proof. From the second equation of (S) we obtain for $t \geq \tau^\star$

$$\int_\tau^t a(s)ds = \int_\tau^t \frac{-y'(s)}{f[x(s)]}ds = -\frac{y(t)}{f[x(t)]} + \frac{y(\tau)}{f[x(\tau)]} - \int_\tau^t \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2}ds$$

and so we have

$$\begin{aligned} -\frac{y(t)}{f[x(t)]} &= \left[-\frac{y(\tau)}{f[x(\tau)]} + \int_\tau^t a(s)ds + \int_\tau^{\tau^\star} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2}ds \right] \\ &\quad + \int_{\tau^\star}^t \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2}ds \end{aligned}$$

for every $t \geq \tau^\star$. This, by our hypothesis, gives

$$-\frac{y(t)}{f[x(t)]} \geq c + \int_{\tau^\star}^t \left\{ -\frac{y(s)}{f[x(s)]} \right\} \left\{ -\frac{x'(s)f'[x(s)]}{f[x(s)]} \right\} ds \quad \text{for all} \quad t \geq \tau^\star.$$

Hence, by using a simple result from the theory of integral inequalities (see, for example, Kwong and Wong [4], Lemma 1), we conclude that

$$w(t) \leq -y(t) \text{ for every } t \geq \tau^\star,$$

where w satisfies

$$\frac{w(t)}{f[x(t)]} = c + \int_{\tau^\star}^t \frac{w(s)}{f[x(s)]} \left\{ -\frac{x'(s)f'[x(s)]}{f[x(s)]} \right\} ds \text{ for } t \geq \tau^\star.$$

We can easily see that $w' = 0$ on $[\tau^\star, \infty)$. Moreover, we obviously have $w(\tau^\star) = cf[x(\tau^\star)]$. Thus, $w(t) = cf[x(\tau^\star)]$ for all $t \geq \tau^\star$ and so the proof of the lemma is complete.

3. Proof of the Theorem

Assume that the differential system (S) admits a nonoscillatory solution (x,y) on an interval $[T_0,\infty)$, where $T_0 \geq t_0$. From (C_1) it follows that the coefficient b is not identically zero on any interval of the form $[\tau_0,\infty)$, $\tau_0 \geq t_0$. So, as pointed out in Section 1, x is always nonoscillatory. Without loss of generality, we shall assume that $x(t) \neq 0$ for all $t \geq T_0$. Furthermore, we observe that the substitution $z = -x, w = -y$ transforms (S) into the system

$$z'(t) = b(t)\widehat{g}[w(t)], \ w'(t) = -a(t)\widehat{f}[z(t)],$$

where $\widehat{f}(u) = -f(-u), u \in \mathbb{R}$, and $\widehat{g}(v) = -g(-v), v \in \mathbb{R}$. The functions $\widehat{f}$ and $\widehat{g}$ are subject to the conditions posed on f and g. Thus, we can restrict our discussion only to the case where x is positive on $[T_0,\infty)$. It must be noted that from the first equation of the system (S) it follows that the function yx' is necessarily nonnegative on the interval $[T_0, \infty)$, even though y is oscillatory.

First of all, we will show that

$$(\star) \qquad \int_{T_0}^{\infty} \frac{y(t)x'(t)f'[x(t)]}{\{f[x(t)]\}^2} dt < \infty.$$

To this end, let us assume that $(\star)$ fails. By condition (C_4), there exists a real constant K such that

$$-\frac{y(T_0)}{f[x(T_0)]} + \int_{T_0}^{t} a(s)ds \geq K \text{ for } t \geq T_0.$$

Furthermore, we can choose a point $T_0^\star \geq T_0$ so that

$$\int_{T_0}^{T_0^\star} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds \geq 1 - K.$$

So, we have

$$-\frac{y(T_0)}{f[x(T_0)]} + \int_{T_0}^{t} a(s)ds + \int_{T_0}^{T_0^\star} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds \geq 1 \text{ for every } t \geq T_0^\star$$

and hence, by applying our lemma with $\tau = T_0, \tau^\star = T_0^\star$ and $c = 1$, we obtain

$$y(t) \leq d \text{ for all } t \geq T_0^\star,$$

where $d = -f[x(T_0{}^\star)] < 0$. Next, from the first equation of (S) we derive for $t \geq T_0^\star$

$$x(t) - x(T_0^\star) = \int_{T_0^\star}^{t} b(s)g[y(s)]ds \leq g(d) \int_{T_0^\star}^{t} b(s)ds,$$

which, in view of (C_1), gives $\lim_{t\to\infty} x(t) = -\infty$, a contradiction.

Now, by taking into account (C_4) and the definition of the function A as well as $(\star)$, from the second equation of (S) we get for $t \geq T_0$

$$\begin{aligned} A(T_0) - A(t) &= \int_{T_0}^{t} a(s)ds = \int_{T_0}^{t} \frac{-y'(s)}{f[x(s)]} ds \\ &= -\frac{y(t)}{f[x(t)]} + \frac{y(T_0)}{f[x(T_0)]} - \int_{T_0}^{t} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds \\ &= -\frac{y(t)}{f[x(t)]} + \frac{y(T_0)}{f[x(T_0)]} - \int_{T_0}^{\infty} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds \\ &\quad + \int_{t}^{\infty} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds, \end{aligned}$$

namely

$$(\star\star) \qquad \frac{y(t)}{f[x(t)]} = \theta + A(t) + \int_{t}^{\infty} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds \text{ for every } t \geq T_0,$$

where the real number θ is defined by

$$\theta = \frac{y(T_0)}{f[x(T_0)]} - A(T_0) - \int_{T_0}^{\infty} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds.$$

We claim that the constant θ is nonnegative. Otherwise, from (C_4) and ($\star$) it follows that there exists $T_0^{\star\star} \geq T$ such that

$$\int_{T_0^{\star\star}}^{\infty} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds \leq -\frac{\theta}{4}$$

and

$$\int_t^{\infty} a(s)ds \leq -\frac{\theta}{4} \text{ for all } t \geq T_0^{\star\star}.$$

Thus, by using ($\star\star$), we find for every $t \geq T_0^{\star\star}$

$$\begin{aligned}
&- \frac{y(T_0)}{f[x(T_0)]} + \int_{T_0}^{t} a(s)ds + \int_{T_0}^{T_0^{\star\star}} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds \\
&\quad = -\theta - A(T_0) - \int_{T_0}^{\infty} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds \\
&\qquad + \int_{T_0}^{t} a(s)ds + \int_{T_0}^{T_0^{\star\star}} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds \\
&\quad = -\theta - \int_t^{\infty} a(s)ds - \int_{T_0^{\star\star}}^{\infty} \frac{y(s)x'(s)f'[x(s)]}{\{f[x(s)]\}^2} ds \\
&\quad \geq -\theta + \frac{\theta}{4} + \frac{\theta}{4} = -\frac{\theta}{2}
\end{aligned}$$

and so our lemma ensures that

$$y(t) \leq D \text{ for all } t \geq T_0^{\star\star},$$

where $D = (\theta/2)f[x(T_0^{\star\star})] < 0$. Hence, exactly as in proving ($\star$), we arrive at the contradiction $\lim_{t\to\infty} x(t) = -\infty$, which proves our claim.

Finally, ($\star\star$) guarantees that

$$y(t) \geq A(t)f[x(t)] \text{ for every } t \geq T_0.$$

Hence, by taking into account the fact that $\lim_{t\to\infty} A(t) = 0$ and using condition (C_7), from the first equation of (S) we obtain for $t \geq T_0$

$$x'(t) = b(t)g[y(t)] \geq b(t)g[A(t)f[x(t)]] \geq b(t)g[A(t)]g\{f[x(t)]\}$$

and consequently

$$\int_{x(T_0)}^{x(t)} \frac{du}{g[f(u)]} \geq \int_{T_0}^{t} b(s)g[A(s)]ds \text{ for all } t \geq T_0.$$

So, because of condition (C_8), we have

$$\int_{T_0}^{t} b(s)g[A(s)]ds \leq \int_{x(T_0)}^{\infty} \frac{du}{g[f(u)]} < \infty \text{ for } t \geq T_0,$$

which contradicts condition (C_9). The proof of our theorem is complete.

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