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EIGENVALUES OF THE FORM VALUED LAPLACIAN FOR RIEMANNIAN SUBMERSIONS

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Abstract. Let $\pi : Z \to Y$ be a Riemannian submersion of closed manifolds. Let Φ_p be an eigen p-form of the Laplacian on Y with eigenvalue λ which pulls back to an eigen p-form of the Laplacian on Z with eigenvalue μ. We are interested in when the eigenvalue can change. We show that $\lambda \le \mu$, so the eigenvalue can only increase; and we give some examples where $\lambda < \mu$, so the eigenvalue changes. If the horizontal distribution is integrable and if Y is simply connected, then $\lambda = \mu$, so the eigenvalue does not change.

If M is a closed Riemannian manifold, let $E(\lambda, \Delta_M^p)$ be the eigenspace of the Laplacian $\Delta_M^p := d\delta + \delta d$ for the eigenvalue λ on the space of smooth p-forms $C^\infty(\Lambda^p M)$. Let $\pi : Z \to Y$ be a Riemannian submersion of closed manifolds. Pull-back defines a natural map π^* from $C^\infty(\Lambda^p Y)$ to $C^\infty(\Lambda^p Z)$. We are interested in examples where an eigenform on Y pulls back to an eigenform on Z with a different eigenvalue. Let $\mathcal{V}$ and $\mathcal{H}$ be the vertical and horizontal distributions of π. We say that $\mathcal{H}$ is an integrable SL (Special Linear) distribution if $\mathcal{H}$ is integrable and if there exists a measure ν on the fibers of π so that the Lie derivative $\mathcal{L}_H \nu$ vanishes for all horizontal lifts H; this means that the transition functions of the fibration can be chosen to have Jacobian determinant 1.

Theorem 1. *Let $\pi : Z \to Y$ be a Riemannian submersion of closed manifolds. Let $\Phi_p \in E(\lambda, \Delta_Y^p)$ and $\pi^*\Phi_p \in E(\mu, \Delta_Z^p)$. If $\mathcal{H}$ is an integrable SL distribution, $\lambda = \mu$.*

We will show in Lemma 6 that if Y is simply connected and if $\mathcal{H}$ is integrable, then $\mathcal{H}$ is an integrable SL distribution and therefore eigenvalues do not change. This shows the fundamental role that the curvature tensor plays in this subject. We note that the Godbillon-Vey class of the foliation $\mathcal{H}$ vanishes if $\mathcal{H}$ is an integrable SL distribution.

In the general setting, we show that if eigenvalues change, they can only increase.

Theorem 2. *Let $\pi : Z \to Y$ be a Riemannian submersion of closed manifolds. Let $\Phi_p \in E(\lambda, \Delta_Y^p)$ and $\pi^*\Phi_p \in E(\mu, \Delta_Z^p)$. Then $\lambda \le \mu$.*

It is not difficult to use the maximum principle to show that eigenvalues cannot change if $p = 0$. It is not known if eigenvalues can change if $p = 1$. Eigenvalues can change if $p \ge 2$.

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Theorem 3. *Let $0 \leq \lambda < \mu$. If $p \geq 2$, there exists a Riemannian submersion $\pi : Z \to Y$ of closed manifolds so that π has totally geodesic fibers and so that there exists $\Phi_p \in E(\lambda, \Delta_Y^p)$ with $\pi^*\Phi_p \in E(\mu, \Delta_Z^p)$.*

Here is a brief guide to the paper. We first review the spectral geometry of Riemannian submersions. Next we study the geometry of a Riemannian submersion with integrable horizontal distribution and prove Theorem 1. Then we discuss fiber products and give the proof of Theorem 2. The proofs of these results are similar in nature and use the observation that the Laplacian is a non-negative operator. We conclude the paper by using the Hopf fibration to prove Theorem 3; other examples where eigenvalues change may be found in [GLP], [GLPa] and in Muto [Mu], [Mua]. It is a pleasant task to acknowledge helpful conversations with Juha Pohjanpelto.

We adopt the following notational conventions. Let $\pi : Z \to Y$ be a Riemannian submersion of closed manifolds. We shall use capital letters for tensors on Y and lower case letters for tensors on Z. Let $\{e_i\}$, $\{e^i\}$, $\{f_a\}$ and $\{f^a\}$ be local orthonormal frames for $\mathcal{V}$, $\mathcal{V}^*$, $\mathcal{H}$, and $\mathcal{H}^*$. Let $\text{ext}(\xi)$ and $\int(\xi)$ denote exterior multiplication and the dual interior multiplication by the covector ξ. We adopt the Einstein convention and sum over repeated indices. Let

$$\theta := -g_Z(e_i, [e_i, f_a])f^a, \ \omega_{abi} := \tfrac{1}{2}g_Z(e_i, [f_a, f_b]),$$
$$\mathcal{E} := \omega_{abi}\text{ext}(e^i)\text{int}(f^a)\text{int}(f^b).$$

The fibers of π are minimal if and only if the unnormalized mean covector θ is zero. The horizontal distribution $\mathcal{H}$ is integrable if and only if the curvature ω is zero.

Theorem 4. *Let $\pi : Z \to Y$ be a Riemannian submersion of closed manifolds.*

(a) *We have $\Delta_Z^p\pi^* - \pi^*\Delta_Y^p = \{d(\int(\theta) + \mathcal{E}) + (\int(\theta) + \mathcal{E})d\}\pi^*$.*
(b) *If $p = 0$, then the following conditions are equivalent:*
  i) *We have $\Delta_Z^0\pi^* = \pi^*\Delta_Y^0$.*
  ii) *For all $\lambda \in \mathbb{R}$, there exists $\mu(\lambda) \in \mathbb{R}$ so $\pi^*E(\lambda, \Delta_Y^0) \subseteq E(\mu(\lambda), \Delta_Z^0)$.*
  iii) *The fibers of π are minimal.*
(c) *If $1 \leq p \leq \dim(Y)$, then the following conditions are equivalent:*
  i) *We have $\Delta_Z^p\pi^* = \pi^*\Delta_Y^p$.*
  ii) *For all $\lambda \in \mathbb{R}$, there exists $\mu(\lambda) \in \mathbb{R}$ so $\pi^*E(\lambda, \Delta_Y^p) \subseteq E(\mu(\lambda), \Delta_Z^p)$.*
  iii) *The fibers of π are minimal and $\mathcal{H}$ is integrable.*

Remark 5. Assertion (a) of Theorem 4 is the fundamental formula in this subject. If $p = 0$, it follows from the work of Watson [Wa] and if $p > 0$, it follows from the work of Goldberg and Ishihara [GoIs]; these authors used it to prove the equivalence of i) and iii) in assertions (b) and (c). We also refer to [GP] for a slightly different proof of assertion (a) as well as a proof of the equivalence of i) and ii) in assertions (b) and (c). Suppose the pull-back of every eigen p-form on the base is an eigen p-form on the total space. We then have $\pi^*E(\lambda, \Delta_Y^p) \subset E(\mu(\lambda, \Delta_Z^p))$ as the eigenvalue in question must be constant on any given eigenspace. Assertions (b) and (c) then show $\mu(\lambda) = \lambda$; the eigenvalue cannot change.

We begin the proof of Theorem 1 by showing:

Lemma 6. *Let $\pi : Z \to Y$ be a Riemannian submersion of closed manifolds. Assume that $\mathcal{H}$ is integrable.*

(a) *We can find local coordinates $z = (x, y)$ on Z so that $\pi(x, y) = y$ and so that $ds_Z^2 = g_{ij}(x, y)dx^i \circ dx^j + h_{ab}(y)dy^a \circ dy^b$.*

(b) *Let* $g_X = \det(g_{ij})^{1/2}$. *Then* $\theta = -d_Y \ln(g_X)$.
(c) *If* $\pi_1(Y) = 0$, *then we may decompose* $Z = X \times Y$ *such that* $\pi(x, y) = y$. *Furthermore,* $\mathcal{H}$ *is an integrable SL distribution.*
(d) *If we define a new metric on* Z *by replacing the vertical metric* $ds^2_{\mathcal{V}}$ *on the vertical distribution by the rescaled metric* $e^{-2\alpha/\dim(X)}ds^2_{\mathcal{V}}$ *for* $\alpha \in C^\infty(Z)$, *then we have* $\theta_\alpha = \theta + \partial_Y \alpha$.
(e) *If* $\mathcal{H}$ *an integrable SL distribution, then there exists* $\alpha \in C^\infty(Z)$ *so that* $\theta = -d_Y \alpha$.

Proof. We have that Z is a twisted product over Y with fiber X. Choose local coordinates $y = (y^a)$ defined on a neighborhood $\mathcal{O}$ of any point y_0 in Y. Let $\tilde{f}_a$ be the horizontal lift of the coordinate vector fields $\partial/\partial y_a$ from Y to Z over $\pi^{-1}(\mathcal{O})$; this is not an orthonormal frame. Since $\mathcal{H}$ is integrable, we have $[\tilde{f}_a, \tilde{f}_b] \in \mathcal{H}$. We use the equation $\pi_*([\tilde{f}_a, \tilde{f}_b]) = [\partial/\partial y_a, \partial/\partial y_b] = 0$ to see that $[\tilde{f}_a, \tilde{f}_b] = 0$. Choose local coordinates $x = (x^i)$ for the fiber over y_0 near a point $z_0 \in \pi^{-1}(y_0)$. We use the Frobenius theorem to extend x to a system of coordinates $z = (x, w)$ on a neighborhood of z_0 so that $\tilde{f}_a = \partial/\partial w_a$. The projections of the integral curves of the vector fields $\partial/\partial w_a$ are the integral curves of the vector fields $\partial/\partial y_a$. Therefore $y = \pi(x, w) = w$ and π is projection on the second factor. Since $\mathcal{V} = TX = \operatorname{span}\{\partial/\partial x^i\}$ is perpendicular to $\mathcal{H} = TY = \operatorname{span}\{\partial/\partial w^a\}$ and since π_* is an isometry from $\mathcal{H}$ to TY, the metric locally has the form given in assertion (a). If $ds^2_M = g_{rs} du^r \circ du^s$, let $g_M = \det(g_{rs})^{1/2}$. Then $\Delta^0_M = -g_M^{-1}\partial_r g_M g^{rs}\partial_s$. Express the metric on Z locally as in assertion (a). Then

$$\operatorname{int}(\theta) d\pi^*\Phi_0 = (\Delta^0_Z \pi^* - \pi^*\Delta^0_Y)\Phi_0 = -\int (g_X^{-1} d_Y g_X) d\pi^*\Phi_0$$

for any $\Phi_0 \in C^\infty(Y)$, where $g_X = \det(g_{ij}(x, y))^{1/2}$. This shows $\theta = -d_Y \ln(g_X)$. Assertion (b) now follows.

Assume the base Y is simply connected. Parallel translation along a curve defines a diffeomorphism from the fiber of π at the initial point to the fiber of π at the terminal point of the curve. We have that this diffeomorphism is independent of the curve and gives a global splitting $Z = X \times Y$ so that $\pi(x, y) = y$ and $\mathcal{H} = \operatorname{span}\{\partial/\partial y^a\}$, where X is the fiber over the basepoint. If ν is any smooth measure on X, then $\mathcal{L}_H \nu = 0$ for the horizontal lift of any vector field on the base Y, so $\mathcal{H}$ is an integrable SL distribution. Assertions (c) and (d) now follow.

Suppose that $\mathcal{H}$ is an integrable SL distribution. Let ν_X be a measure on the fibers so $\mathcal{L}_H \nu_X = 0$ for all the horizontal lifts H. Let

$$\mathrm{dvol}_Z = e^\alpha \nu_X \mathrm{dvol}_Y \text{ and } \nu_X = e^\beta dx^1 ... dx^{\dim(X)}.$$

Then α is globally defined, β is locally defined, and $g_X = e^{\alpha+\beta}$. Since $\mathcal{L}_{\partial/\partial y}\nu_X = 0$, β is independent of the base. Therefore $\theta = -d_Y \ln(g_X) = -d_Y \alpha$. ☐

Proof of Theorem 1. Let $\mathcal{H}$ be an integrable SL foliation. Let $\Phi_p \in E(\lambda, \Delta^p_Y)$ with $\pi^*\Phi_p \in E(\mu, \Delta^p_Z)$. Since the curvature vanishes, we may apply Theorem 4 to get

$$\begin{aligned}(\mu - \lambda)\pi^*\Phi_p(x, y) &= (\Delta^p_Z \pi^*\Phi_p - \pi^*\Delta^p_Y \Phi_p)(x, y) \\ &= (d_Z \int (\theta)\pi^*\Phi_p + \int (\theta)\pi^* d_Y \Phi_p)(x, y).\end{aligned}$$

Choose $\alpha \in C^\infty(Z)$ so that $\theta = -d_Y \alpha$. We define a new metric $ds^2_Z(\epsilon)$ on Z by replacing the metric $ds^2_{\mathcal{V}}$ on the vertical distribution by $e^{2\epsilon\alpha/\dim(X)}ds^2_{\mathcal{V}}$. We apply

Lemma 6 to obtain $\theta(\epsilon) = (1+\epsilon)\theta$. Thus $\pi^*\Phi_p \in E(\mu(\epsilon), \Delta^p_{Z(\epsilon)})$, where we set $\mu(\epsilon) - \lambda = (1+\epsilon)(\mu - \lambda)$. Since the Laplacian is a non-negative operator, for any $\epsilon \in \mathbb{R}$ we have $\mu(\epsilon) = \lambda + (1+\epsilon)(\mu - \lambda) \geq 0$. This implies $\mu = \lambda$. □

Definition 7 (Fiber products)**.** Let $\pi_U : U \to Y$ and $\pi_V : V \to Y$ be Riemannian submersions of closed manifolds. Denote the horizontal and vertical distributions by $\mathcal{H}_U$, $\mathcal{H}_V$, $\mathcal{V}_U$, and $\mathcal{V}_V$. Let

$$W := \{w = (u,v) \in U \times V : \pi_U(u) = \pi_V(v)\}.$$

We identify $T(U \times V) = T(U) \oplus T(V)$ and embed $T(W)$ in $T(U \times V)$. Define $\pi_W : W \to Y$ by $\pi_W(w) := \pi_U(u) = \pi_V(v)$. Let $\mathcal{V}_W(w) := \mathcal{V}_U(u) \oplus \mathcal{V}_V(v)$ and

$$\mathcal{H}_W(w) := \{(\xi,\eta) \in \mathcal{H}_U(u) \oplus \mathcal{H}_V(v) : (\pi_U)_*\xi = (\pi_V)_*\eta\}.$$

We define a new metric on W by requiring that $\mathcal{H}_W$, $\mathcal{V}_U$, and $\mathcal{V}_V$ are orthogonal, that the metrics on $\mathcal{V}_U$ and $\mathcal{V}_V$ are induced from the metrics on U and on V, and that $(\pi_W)_* : \mathcal{H}_W \to TY$ is an isometry. The metric on $\mathcal{H}_W$ differs from the induced metric by a factor of $2^{-1/2}$. Let $\pi_1(u,v) = u$ and $\pi_2(u,v) = v$. Then $\pi_1 : W \to U$, $\pi_2 : W \to V$, and $\pi_W : W \to Y$ are Riemannian submersions. Let $f_{a,U}$ and $f_{a,V}$ be the horizontal lifts of F_a with respect to the submersions π_U and π_V. Then $f_{a,W} := f_{a,U} + f_{a,V}$ is the horizontal lift of F_a with respect to the submersion π_W; $f_{a,W}$ is also the horizontal lift of $f_{a,U}$ and $f_{a,V}$ to W with respect to the submersions π_1 and π_2. Let $\{e_{i,U}\}$ and $\{e_{\alpha,V}\}$ be local orthonormal frames for the vertical distributions $\mathcal{V}_U$ and $\mathcal{V}_V$ of the submersions π_U and π_V. Then $\{e_{i,U}, e_{\alpha,V}\}$ is a local orthonormal frame for the vertical distribution $\mathcal{V}_W$ of the submersion π_W. Furthermore $\{e_{i,U}\}$ and $\{e_{\alpha,V}\}$ are local orthonormal frames for the vertical distributions $\mathcal{V}_U$ and $\mathcal{V}_V$ of the submersions π_2 and π_1.

Example 8. Suppose that U and V are vector bundles over Y with given fiber metrics and Riemannian connections. This defines Riemannian metrics on U and V so that the projections π_U and π_V are Riemannian submersions; if γ is a curve in Y, then s_γ is the horizontal lift of γ if and only if s_γ is parallel along γ. Then $W = U \oplus V$ is the Whitney sum vector bundle, and the metric on W is defined by the Whitney sum connection and the Whitney sum fiber metric.

Lemma 9. *Adopt the notation of Definition 7.*

(a) *We have* $\theta_W = \pi_1^*\theta_U + \pi_2^*\theta_V$ *and* $\mathcal{E}_W\pi_W^* = \pi_1^*\mathcal{E}_U\pi_U^* + \pi_2^*\mathcal{E}_V\pi_V^*$.
(b) *If* $\Phi_p \in E(\lambda, \Delta^p_Y)$, *if* $\pi_U^*\Phi_p \in E(\lambda + \epsilon_U, \Delta^p_U)$, *and if* $\pi_V^*\Phi_p \in E(\lambda + \epsilon_V, \Delta^p_V)$, *then we have* $\pi_W^*\Phi_p \in E(\lambda + \epsilon_U + \epsilon_V, \Delta^p_W)$.

Proof. We note that pull-back commutes with exterior and interior multiplication. We prove assertion (a) by computing:

$$\begin{aligned}
\theta_W &= -\{g_W(e_{i,U}, [e_{i,U}, f_{a,W}]) + g_W(e_{\alpha,V}, [e_{\alpha,V}, f_{a,W}])\}\pi_W^*(F^a)\\
&= -\{g_U(e_{i,U}, [e_{i,U}, f_{a,U}]) + g_V(e_{\alpha,V}, [e_{\alpha,V}, f_{a,V}])\}\pi_W^*(F^a)\\
&= \pi_1^*\theta_U + \pi_2^*\theta_V,\\
[f_{a,W}, f_{b,W}] &= [f_{a,U}, f_{b,U}] + [f_{a,V}, f_{b,V}],\\
\omega_{Wiab} &= g_W(e_{i,W}, [f_{a,W}, f_{b,W}]) = g_U(e_{i,U}, [f_{a,U}, f_{b,U}]) = \omega_{Uiab},\\
\omega_{W\alpha ab} &= g_W(e_{\alpha,W}, [f_{a,W}, f_{b,W}]) = g_V(e_{\alpha,V}, [f_{a,V}, f_{b,V}]) = \omega_{V\alpha ab}.
\end{aligned}$$

We use assertion (a) and Theorem 4 to prove (b) by computing:

$$\begin{aligned}(\epsilon_U+\epsilon_V)\pi_W^*\Phi_p =&\pi_1^*\{\Delta_U^p\pi_U^*-\pi_U^*\Delta_Y^p\}\Phi_p+\pi_2^*\{\Delta_V^p\pi_V^*-\pi_V^*\Delta_Y^p\}\Phi_p\\ =&(\Delta_W^p\pi_W^*-\pi_W^*\Delta_Y^p)\Phi_p. \quad \square\end{aligned}$$

Proof of Theorem 2. Suppose that $\pi : Z \to Y$ is a Riemannian submersion of closed manifolds. Let $0 \neq \Phi_p \in E(\lambda, \Delta_Y^p)$ and $\pi^*\Phi_p \in E(\lambda+\epsilon, \Delta_Z^p)$. Let $Z_0 = Z$, and inductively let $Z_n = W(Z_{n-1}, Z_{n-1})$ be the fiber product of Z_{n-1} with itself and let $\pi_n : Z_n \to Y$ be the associated projection. We use Lemma 9 (b) and induction to see

$$\pi_n^*\Phi_p \in E(\lambda + 2^n\epsilon, \Delta_{Z_n}^p).$$

Since the Laplacian on Z_n is non-negative, $\lambda + 2^n\epsilon \geq 0$. This implies $\epsilon \geq 0$. $\square$

Proof of Theorem 3. Let $\pi : S^3 \to S^2$ be the Hopf fibration. Let Φ_2 be the volume form on S^2 with respect to the standard metric g. Then $\Phi_2 \in E(0, \Delta_{S^2}^2)$. In [GLP, Theorem 2.3], we constructed a metric $\tilde g(\alpha)$ on S^3 so that π is a Riemannian submersion with totally geodesic fibers from $(S^3, \tilde g(\alpha))$ to (S^2, g) and so that $\pi^*(\Phi_2) \in E(\alpha^2, \Delta_{S^3,\tilde g(\alpha)}^2)$. This completes the proof if $p = 2$ and $\lambda = 0$. If $0 < \lambda < \mu$, let $\lambda = \alpha^2$ and $\mu = \alpha^2 + \beta^2$. Let $W = W(\alpha, \beta)$ be the fiber product of $(S^3, \tilde g(\alpha))$ and $(S^3, \tilde g(\beta))$, where $\lambda = \alpha^2$. Let $\pi_\alpha : W \to S^3$ and let $\pi_W : W \to S^2$. By Lemma 9,

$$\pi^*\Phi_2 \in E(\alpha^2, \Delta_{S^3,\tilde g(\alpha)}^2)$$

and

$$\pi_\alpha^*\pi^*\Phi_2 = \pi_W^*\Phi_2 \in E(\alpha^2+\beta^2, \Delta_W^2).$$

This completes the proof if $p = 2$; the general case now follows by taking suitable Riemannian products. $\square$

This constructs examples where the pull-back of a non-trivial eigenform is an eigenform and where eigenvalues can change. Conversely, in [GLP, Theorem 2.3], we constructed a right invariant metric $\tilde g_3$ on the Lie group S^3 so the Hopf map π is a Riemannian submersion from $(S^3, \tilde g_3)$ to S^2 with the standard metric and so that $\pi^*\Phi_p \in E(\lambda, \Delta_{S^3}^p)$ for $\Phi_p \in C^\infty\Lambda^p S^2$ implies $p = 0$, $\Phi_0 = c$, and $\lambda = 0$. This provides an example where only the constant function on Y pulls back to an eigenform on Z; this is generically the situation for a Riemannian submersion.

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