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A GENERALIZATION OF CARLEMAN'S UNIQUENESS THEOREM AND A DISCRETE PHRAGMÉN-LINDELÖF THEOREM

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ABSTRACT. Let $d\mu \geq 0$ be a Borel measure on $[0,\infty)$ and $A_n = \int\limits_0^\infty t^n d\mu(t) < \infty$ $(n = 0, 1, 2, ...)$ be its moments. T. Carleman found sharp conditions on the magnitude of $\{A_n\}_0^\infty$ for $d\mu$ to be uniquely determined by its moments. We show that the same conditions ensure a stronger property: if $A'_n = \int\limits_0^\infty t^n d\mu_1(t)$ are the moments of another measure, $d\mu_1 \geq 0$, with $\limsup\limits_{n\to\infty} |A_n - A'_n|^{\frac{1}{n}} = \rho < \infty$, then the measure $d\mu - d\mu_1$ is supported on the interval $[0, \rho]$. This result generalizes both the Carleman theorem and a theorem of J. Mikusiński. We also present an application of this result by establishing a discrete version of a Phragmén-Lindelöf theorem.

§1. PRELIMINARIES

Definition 1.1. A sequence $\{A_n\}_0^\infty$ of positive numbers is called *logarithmically convex* if $A_n \leq \sqrt{A_{n-1}A_{n+1}}$ $(n = 1, 2, ...)$.

Clearly, the moments $A_n = \int\limits_0^\infty t^n d\mu(t)$ of any Borel measure $d\mu \geq 0$, with $A_n < \infty$, form a logarithmically convex sequence.

Definition 1.2. If $\{A_n\}_0^\infty$ is an arbitrary sequence of positive numbers, then the *convex regularization of* $\{A_n\}_0^\infty$ *by means of logarithms*, denoted $\{A_n^c\}_0^\infty$, is formed by setting

$$A_n^c = \sup\{B_n : \{B_\nu\}_0^\infty \text{ is logarithmically convex, and } B_\nu \leq A_\nu \ (\nu = 0, 1, ...)\}.$$

Proposition 1.3 ([Ma]). *Given a sequence* $\{A_n\}_0^\infty$, $A_n > 0$, *the following three conditions are equivalent:*

Carleman's condition:

$$\text{(C)} \qquad \sum_{n=0}^\infty \frac{1}{\beta_n} = \infty, \qquad \textit{where} \qquad \beta_n = \inf_{k\geq n} A_k^{\frac{1}{k}};$$

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Ostrowski's condition:

$$\text{(O)} \qquad \int_1^\infty \frac{\log T(r)dr}{r^2} = \infty, \qquad \textit{where} \qquad T(r) = \sup_{n\geq 0} \frac{r^n}{A_n};$$

Mandelbrojt's condition:

$$\text{(M)} \qquad \textit{either} \qquad \liminf_{n\to\infty} A_n^{\frac{1}{n}} < \infty \qquad \textit{or} \qquad \sum_{n=0}^\infty \frac{A_n^c}{A_{n+1}^c} = \infty.$$

We shall refer to the above as the (COM) condition.

Definition 1.4. We say a measure, $d\mu$, on $[0,\infty)$ is *supported on* $[0,\rho]$*,* if the total variation of $d\mu$ on (ρ,∞) is 0, and ρ is the smallest number having this property.

Theorem 1.5 ([C] Carleman's Uniqueness Theorem)**.** *If* $A_n = \int_0^\infty t^n d\mu(t),\ d\mu \geq 0,$ *and* $\{\sqrt{A_n}\}_0^\infty$ *satisfies the (COM) condition, then no other measure* $d\mu_1 \geq 0$ *has the same moments,* $A_n,$ *for* $n = 0, 1, ...$

Proposition 1.6 ([K] Analytic Quasianalyticity)**.** *Let* $\{A_n\}_0^\infty$ *be a sequence of positive numbers. Let* $C\{A_n\}$ *be the class of functions,* $f(z)$*, analytic in* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ *and infinitely differentiable on* $\overline{\mathbb{D}}$ *such that* $\max_{z\in\mathbb{D}} |f^{(n)}(z)| \leq C_f A_n$*. Then the following are equivalent:*

$$\text{(1)} \qquad \textit{The only } f \in C\{A_n\} \textit{ vanishing with all of its derivatives at a point } \zeta_0 \in \partial\mathbb{D} \textit{ is } f(z) \equiv 0$$

$$\text{(2)} \qquad \{\sqrt{A_n}\}_0^\infty \textit{satisfies the (COM) condition.}$$

§2. The main theorem

Theorem 2.1. *Let* $d\mu \geq 0$ *and* $d\mu_1 \geq 0$ *be two Borel measures on* $[0,\infty)$ *with moments*

$$A_n = \int_0^\infty t^n d\mu(t) \qquad \textit{and} \qquad A'_n = \int_0^\infty t^n d\mu_1(t) \quad (n = 0, 1, ...).$$

If $\{\sqrt{A_n}\}_0^\infty$ *satisfies the (COM) condition, and*

$$\text{(3)} \qquad \limsup_{n\to\infty} |A_n - A'_n|^{\frac{1}{n}} = \rho < \infty,$$

then the measure $d\mu - d\mu_1$ *is supported on* $[0,\rho]$*. Conversely, if* $\{A_n\}_0^\infty$ *is a sequence of positive numbers such that* $\{\sqrt{A_n}\}_0^\infty$ *does not satisfy the (COM) condition, then there are distinct measures* $d\mu \geq 0$ *and* $d\mu_1 \geq 0$ *with*

$$\int_0^\infty t^n d\mu(t) = \int_0^\infty t^n d\mu_1(t) \leq A_n \qquad (n = 0, 1, ...).$$

Remark 2.2. The uniqueness case of the above theorem corresponds to $\rho = 0$:

$$\text{(4)} \qquad \lim_{n\to\infty} |A_n - A'_n|^{\frac{1}{n}} = 0.$$

The theorem then asserts that if $\{\sqrt{A_n}\}_0^\infty$ and $\{\sqrt{A'_n}\}_0^\infty$ satisfy the (COM) condition, and (4) holds as well, then $A_n = A'_n$ for $n = 1, 2, ...$ ($A_0 \neq A'_0$ is possible).

Remark 2.3. If we replace the hypothesis that $\{\sqrt{A_n}\}_0^\infty$ satisfies (COM) by the much stronger requirement

$$\limsup_{n\to\infty} A_n^{\frac{1}{n}} < \infty,$$

then we obtain a theorem of Mikusiński [Mi].

Corollary 2.4. *Let $d\mu$ be a signed Borel measure on $[0,\infty)$ with*

$$B_n = \int_0^\infty t^n |d\mu(t)| < \infty \quad (n = 0, 1, \dots)$$

($|d\mu|$ is the variation measure of μ). If $\{\sqrt{B_n}\}_0^\infty$ satisfies the (COM) condition, then

$$\rho = \limsup_{n\to\infty} \left|\int_0^\infty t^n d\mu(t)\right|^{\frac{1}{n}} = \limsup_{n\to\infty} B_n^{\frac{1}{n}}$$

and $d\mu$ is supported on $[0,\rho]$ ($\rho = +\infty$ is not exluded).

§3. Proof of Theorem 2.1

Define the signed measure, $d\sigma$, by $d\sigma = d\mu - d\mu_1$. We then define

$$M_n = A_n - A'_n = \int_0^\infty t^n d\sigma(t).$$

Also, let $B_n = \int_0^\infty t^n |d\sigma(t)| \le A_n + A'_n$. Define the function

$$\Phi(z) = \int_0^\infty e^{tz} d\sigma(t) \qquad \text{for } z \in \overline{\mathbb{C}_-} = \{z : Re(z) \le 0\}.$$

Φ is easily seen to be analytic in $\mathbb{C}_-$, and differentiable in $\overline{\mathbb{C}_-}$, with

$$\Phi^{(n)}(z) = \int_0^\infty t^n e^{tz} d\sigma(t).$$

We see that $\Phi^{(n)}(0) = M_n$ and $|\Phi^{(n)}(z)| \le B_n$, for all $n = 0, 1, 2, \ldots$ and for all $z \in \overline{\mathbb{C}_-}$.

Define another function, $F(z)$, via

$$F(z) = \sum_{n=0}^\infty \frac{M_n}{n!} z^n.$$

F is an entire function, and is of exponential type ρ (i.e. $|F(z)| \le C_\epsilon e^{(\rho+\epsilon)|z|}$ for all $\epsilon > 0$).

Now, we observe that $F^{(n)}(0) = \Phi^{(n)}(0) = M_n$ $(n = 0, 1, 2, \ldots)$. We know that $|\Phi^{(n)}(z)| \le B_n$. Also,

$$F^{(n)}(z) = \sum_{k=0}^\infty \frac{M_{n+k}}{k!} z^k,$$

and therefore

$$|F^{(n)}(z)| \le C_R R^n \ (n = 0, 1, \ldots) \quad \text{for all } R > \rho \text{ and } |z| \le 2.$$

Hence, we can conclude that on the closed disk $\{z : |z+1| \leq 1\}$,

$$|\Phi^{(n)}(z) - F^{(n)}(z)| \leq B_n + C_R R^n \leq C'_R B_n R^n.$$

By Proposition 1.6, $\Phi - F \equiv 0$ on $\{z : |z+1| \leq 1\}$; that is, F is an entire extension of Φ.

We next let $F_\delta(z) = e^{-(\rho+\delta)z}F(z)$. This function is bounded on the imaginary axis (by B_0), bounded on the non-negative real axis (by a constant dependent on δ) and of exponential type ρ. Hence, applying the Phragmén-Lindelöf theorem [Mar, vol. 2, p. 214], we can conclude that $|F_\delta(z)|$ is bounded on all of $\overline{\mathbb{C}_+} = \{z : Re(z) \geq 0\}$ by B_0. Taking limits (as $\delta \to 0^+$), we see that

$$|F(z)| \leq B_0 e^{\rho Re(z)} \qquad \text{on } \overline{\mathbb{C}_+}.$$

Define the function $G(z)$ by setting

$$G(z) = \frac{F(z)e^{-\rho z}}{1+z}.$$

Clearly, G is analytic on $\mathbb{C}_+$ and is square summable on the imaginary axis. Thus, we can apply the Paley-Wiener theorem [PW, p. 8, Theorem V] to conclude that

$$G(z) = \int_{-\infty}^{0} \Psi(t)e^{tz}dt \quad \text{for some } \Psi \in L^2((-\infty,0)).$$

We will assume that Ψ is defined for all real numbers (by setting $\Psi(t) = 0$ for all $t > 0$), and we will also extend our signed measure, $d\sigma$, to the entire real line (by requiring $|d\sigma((-\infty,0))| = 0$).

On the imaginary axis, we have two representations for G:

$$\int_{-\infty}^{\infty} \Psi(t)e^{iyt}dt = G(iy) = \frac{e^{-i\rho y}F(iy)}{1+iy}.$$

So, we can conclude that

$$\int_{-\infty}^{\infty} \Psi(t-\rho)e^{iyt}dt = \frac{1}{1+iy}\int_{-\infty}^{\infty} e^{iyt}d\sigma(t).$$

Using the notation $\tilde{f}$ for the Fourier transform of f (i.e. $\tilde{f}(x) = \int_{-\infty}^{\infty} f(y)e^{iyx}dy$), and letting $\Psi_\rho(t) = \Psi(t-\rho)$, we arrive at

$$\widetilde{\Psi_\rho}(y) = \widetilde{\gamma}(y)\widetilde{d\sigma}(y),$$

where γ is the function

$$\gamma(x) = \begin{cases} e^x & \text{if } x \leq 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Hence,

$$\Psi_\rho(y) = (\gamma * d\sigma)(y),$$

so

$$\Psi(t-\rho) = \int_t^\infty e^{t-x} d\sigma(x).$$

Thus, for all $t > \rho$, we have $\int_t^\infty e^{-x} d\sigma(x) = 0$, which implies that the total variation of $d\sigma$ on (ρ, ∞) is 0; i.e., $d\sigma = d\mu - d\mu_1$ is supported on $[0, \rho]$.

Conversely, if we are given a logarithmically convex sequence of positive numbers, $\{A_n\}_0^\infty$, with $\sum_{n=0}^\infty \sqrt{\frac{A_n}{A_{n+1}}} < \infty$, we set $A_{-1} = A_0$. A version of Proposition 1.6 for the half-plane [K] shows that there is some function, $f \not\equiv 0$, analytic in $\mathbb{C}_+$ and continuous on $\overline{\mathbb{C}_+}$, with

$$\sup_{x \geq 0} \int_{-\infty}^\infty |f^{(n)}(x+iy)|^2 dy \leq A_{n-1}^2 \qquad (n = 0, 1, 2, ...)$$

and $f^{(n)}(0) = 0$ $(n = 0, 1, 2, ...)$. Applying the same Paley-Wiener theorem, we obtain

$$f(z) = \int_0^\infty \phi(t) e^{-tz} dt \qquad \text{for some } \phi \in L^2((0, \infty)),$$

as well as

$$f^{(n)}(z) = \int_0^\infty \phi(t)(-t)^n e^{-tz} dt.$$

Applying Plancherel's theorem, we find

$$\int_{-\infty}^\infty |f^{(n)}(iy)|^2 dy = 2\pi \int_{-\infty}^\infty t^{2n} |\phi(t)|^2 dt.$$

So, $\int_0^\infty t^{2n} |\phi(t)|^2 \leq A_{n-1}^2$. Hence, by the Cauchy-Schwarz inequality,

$$\int_0^\infty t^n |\phi(t)| dt = \int_0^1 t^n |\phi(t)| dt + \int_1^\infty t^{n+1} |\phi(t)| \frac{dt}{t} \leq A_{n-1} + A_n \leq K A_n$$

for some constant K.

Letting $d\sigma = \frac{1}{K}\phi(t) dt$, and then defining $d\mu = d\sigma^+$ and $d\mu_1 = d\sigma^-$, we get the desired measures. □

§4. Some applications

In this section we examine some consequences of Theorem 2.1. The first result is a discrete Phragmén-Lindelöf type theorem.

Theorem 4.1. *Let $f(z)$ be analytic in $\mathbb{C}_+ = \{z : Re(z) > 0\}$ and continuous in $\overline{\mathbb{C}_+}$. Define*

$$A_n = \sup_{0 \leq x \leq n} |f(x+iy)| \qquad \textit{for } n=1,2,....$$

Assume $A_n < \infty$ for all n.

If $\{\sqrt{A_n}\}_0^\infty$ *satisfies the (COM) condition, and* $\sup_{n\in\mathbb{N}} |f(n)| < \infty$, *then* $f(z)$ *is bounded on* $\overline{\mathbb{C}_+}$.

Proof. Consider the function

$$g(z) = f(z)\left(\frac{1-e^{-z}}{z}\right)$$

and let

$$B_n = \sup_{0\le x\le n}\left(\int_{-\infty}^{\infty} |g(x+iy)|^2 dy\right)^{\frac{1}{2}}.$$

We find that $B_n \le 4A_n$. Using the Paley-Wiener theorem for the strip [PW, p. 7, Theorem IV], we have

$$g(x+iy) = \int_{-\infty}^{\infty} \phi(\sigma)e^{\sigma(x+iy)}d\sigma,$$

where this integral converges for all $x > 0$. Applying Plancherel's theorem yields

$$\int_{-\infty}^{\infty} |g(n+iy)|^2 dy = 2\pi\int_{-\infty}^{\infty} e^{2n\sigma}|\phi(\sigma)|^2 d\sigma \le B_n^2 \le 16A_n^2.$$

Setting $\Psi(t) = \phi(\log t)$ we get

$$g(1+z) = \int_0^\infty t^z\Psi(t)dt \quad (Re\, z > -1).$$

Also $(n \ge 0)$,

$$\begin{aligned}
C_n &= \int_0^\infty t^n|\Psi(t)|dt = \int_{-\infty}^{\infty} e^{(n+1)t}|\phi(t)|dt \\
&= \int_{-\infty}^{0} e^{nt}|\phi(t)|e^t dt + \int_0^\infty e^{(n+2)t}|\phi(t)|e^{-t}dt \\
&\le \left(\frac{1}{2}\int_{-\infty}^{0} e^{2nt}|\phi(t)|^2 dt\right)^{\frac{1}{2}} + \left(\frac{1}{2}\int_0^\infty e^{2(n+2)t}|\phi(t)|^2 dt\right)^{\frac{1}{2}} \\
&\le \frac{B_n + B_{n+2}}{2\sqrt{\pi}} \le \frac{4}{\sqrt{\pi}}A_{n+2}.
\end{aligned}$$

Thus $\{\sqrt{C_n}\}_0^\infty$ also satisfies the (COM) condition. Since $\sup_{n\in\mathbb{N}} |g(n)| \le \sup_{n\in\mathbb{N}} |f(n)|$, Corollary 2.4 yields $\Psi(t) = 0$ $(t > 1)$ and $\phi(t) = 0$ $(t > 0)$. Therefore

$$f(z) = \frac{z}{1-e^{-z}}\int_{-\infty}^{0} \phi(\sigma)e^{\sigma z},$$

which is bounded on every strip $\{0 < Re\, z < a\}$, satisfies $|f(z)| = O(|z|)$ $(|z| \to \infty)$. Applying the Phragmén-Lindelöf theorem [Mar, vol. 2, p. 214], we see that f is bounded on $\overline{\mathbb{C}_+}$. □

Theorem 4.2. *Let* $f(z)$ *be analytic in* $\mathbb{C}_+$ *and continuous on* $\overline{\mathbb{C}_+}$. *Assume*

$$|f(iy)| \le M \text{ for all real } y \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} |f(n)| < \infty.$$

If $|f(z)| \leq Ce^{p|z|log|z|}$ *for some* $p < 2$ *and some constant* C*, then* $|f(z)| \leq M$ *for all* $z \in \overline{\mathbb{C}_+}$.

Proof. Consider the function

$$h_1(z) = f(z)e^{i\frac{p\pi}{2}z}z^{-pz}$$

in the first quadrant: $Q_1 = \{z : 0 \leq Arg(z) \leq \frac{\pi}{2}\}$. This function is bounded on ∂Q_1, and in Q_1

$$|h_1(z)| = O(e^{|z|^{1+\epsilon}}) \qquad \text{for } |z| \to \infty, \epsilon > 0.$$

Applying the same Phragmén-Lindelöf theorem [Mar] gives that $h_1(z)$ is bounded in Q_1. Using similar estimates on $h_2(z) = f(z)e^{-i\frac{p\pi}{2}z}z^{-pz}$ in the fourth quadrant $Q_4 = \{z : -\frac{\pi}{2} \leq Arg(z) \leq 0\}$ gives that $h_2(z)$ is bounded in Q_4. These estimates yield

$$|f(x+iy)| \leq C\ \exp\{pxlog|x+iy| - (py)tan^{-1}(\frac{y}{x}) + \frac{p}{2}\pi|y|\}$$

in $\mathbb{C}_+$. Now, consider the function

$$g(z) = f(z)e^{-\delta z^{\frac{p}{2}}} \qquad \text{for any } \delta > 0.$$

If

$$D_n = \sup_{0 \leq x \leq n} |g(z)|,$$

then by a straightforward computation it is not hard to see that

$$D_n \leq C_\delta^n e^{\pi n} n^{2n}.$$

So, in fact, the sequence $\{\sqrt{D_n}\}_0^\infty$ satisfies the (COM) condition. This implies that $|g(z)|$ is bounded in $\overline{\mathbb{C}_+}$ by M. Letting $\delta \to 0^+$, we get

$$|f(z)| \leq M \text{ on } \overline{\mathbb{C}_+}. \quad \square$$

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Added in proof

After the paper had gone to print, the authors learned that a result related to Theorem 2.1 is due to V. P. Havin and V. G. Maz′ya, Trudy Moskov. Mat. Obšč. **30** (1974), 61–114. MR **52:**6000. The authors are grateful to Michael Sodin for this reference.

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