A GENERALIZATION
OF CARLEMAN’S UNIQUENESS THEOREM
AND A DISCRETE PHRAGMÉN-LINDELÖF THEOREM

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Abstract. Let \( d\mu \geq 0 \) be a Borel measure on \([0, \infty)\) and \( A_n = \int_0^\infty t^n d\mu(t) < \infty \) \((n = 0, 1, 2, \ldots)\) be its moments. T. Carleman found sharp conditions on the magnitude of \( \{A_n\}_{n=0}^\infty \) for \( d\mu \) to be uniquely determined by its moments. We show that the same conditions ensure a stronger property: if \( A'_n = \int_0^\infty t^n d\mu_1(t) \) are the moments of another measure, \( d\mu_1 \geq 0 \), with \( \limsup_{n \to \infty} \frac{|A_n - A'_n|}{n} = \rho < \infty \), then the measure \( d\mu - d\mu_1 \) is supported on the interval \([0, \rho]\). This result generalizes both the Carleman theorem and a theorem of J. Mikusiński. We also present an application of this result by establishing a discrete version of a Phragmén-Lindelöf theorem.

\[ \sum_{n=0}^{\infty} \frac{1}{\beta_n} = \infty, \quad \text{where} \quad \beta_n = \inf_{k \geq n} A_k^{\frac{1}{k}}; \]

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Ostrowski’s condition:

\[ \int_1^\infty \log T(r) \frac{dr}{r^2} = \infty, \quad \text{where} \quad T(r) = \sup_{n \geq 0} \frac{r^n}{A_n}; \]

Mandelbrojt’s condition:

\[ \text{(M)} \quad \text{either} \quad \liminf_{n \to \infty} \frac{A_n^{1/n}}{A_{n+1}} < \infty \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{A_n^{c_n}}{A_{n+1}} = \infty. \]

We shall refer to the above as the (COM) condition.

**Definition 1.4.** We say a measure, \( d\mu \), on \([0, \infty)\) is supported on \([0, \rho]\), if the total variation of \( d\mu \) on \((\rho, \infty)\) is 0, and \( \rho \) is the smallest number having this property.

**Theorem 1.5** ([C] Carleman’s Uniqueness Theorem). If \( A_n = \int_0^\infty t^n d\mu(t) \), \( d\mu \geq 0 \), and \( \{\sqrt{A_n}\}_{n}^{\infty} \) satisfies the (COM) condition, then no other measure \( d\mu_1 \geq 0 \) has the same moments, \( A_n \), for \( n = 0, 1, \ldots \).

**Proposition 1.6** ([K] Analytic Quasianalyticity). Let \( \{A_n\}_{n=0}^{\infty} \) be a sequence of positive numbers. Let \( C\{A_n\} \) be the class of functions, \( f(z) \), analytic in \( D = \{z \in \mathbb{C} : |z| < 1\} \) and infinitely differentiable on \( \overline{D} \) such that \( \max_{z \in \overline{D}} |f^{(n)}(z)| \leq C_f A_n \). Then the following are equivalent:

1. The only \( f \in C\{A_n\} \) vanishing with all of its derivatives at a point \( \zeta_0 \in \partial D \) is \( f(z) \equiv 0 \).

2. \( \{\sqrt{A_n}\}_{n=0}^{\infty} \) satisfies the (COM) condition.

**§2. The main theorem**

**Theorem 2.1.** Let \( d\mu \geq 0 \) and \( d\mu_1 \geq 0 \) be two Borel measures on \([0, \infty)\) with moments

\[ A_n = \int_0^\infty t^n d\mu(t) \quad \text{and} \quad A'_n = \int_0^\infty t^n d\mu_1(t) \quad (n = 0, 1, \ldots). \]

If \( \{\sqrt{A_n}\}_{n=0}^{\infty} \) satisfies the (COM) condition, and

\[ \limsup_{n \to \infty} \left| \frac{A_n - A'_n}{A_{n+1}^{1/2}} \right| = \rho < \infty, \]

then the measure \( d\mu - d\mu_1 \) is supported on \([0, \rho]\). Conversely, if \( \{A_n\}_{n=0}^{\infty} \) is a sequence of positive numbers such that \( \{\sqrt{A_n}\}_{n=0}^{\infty} \) does not satisfy the (COM) condition, then there are distinct measures \( d\mu \geq 0 \) and \( d\mu_1 \geq 0 \) with

\[ \int_0^\infty t^n d\mu(t) = \int_0^\infty t^n d\mu_1(t) \leq A_n \quad (n = 0, 1, \ldots). \]

**Remark 2.2.** The uniqueness case of the above theorem corresponds to \( \rho = 0 \):

\[ \lim_{n \to \infty} \left| \frac{A_n - A'_n}{A_{n+1}^{1/2}} \right| = 0. \]

The theorem then asserts that if \( \{\sqrt{A_n}\}_{n=0}^{\infty} \) and \( \{\sqrt{A'_n}\}_{n=0}^{\infty} \) satisfy the (COM) condition, and (4) holds as well, then \( A_n = A'_n \) for \( n = 1, 2, \ldots \) (\( A_0 \neq A'_0 \) is possible).
Remark 2.3. If we replace the hypothesis that \( \{ \sqrt{A_n} \}_{n=0}^{\infty} \) satisfies (COM) by the much stronger requirement
\[
\limsup_{n \to \infty} A_n^{1/n} < \infty,
\]
then we obtain a theorem of Mikusiński [Mi].

Corollary 2.4. Let \( d\mu \) be a signed Borel measure on \([0, \infty)\) with
\[
B_n = \int_0^\infty t^n |d\mu(t)| < \infty \quad (n = 0, 1, \ldots)
\]
(\( |d\mu| \) is the variation measure of \( \mu \)). If \( \{ \sqrt{B_n} \}_{n=0}^{\infty} \) satisfies the (COM) condition, then
\[
\rho = \limsup_{n \to \infty} \left( \int_0^\infty t^n d\mu(t) \right)^{1/n} = \limsup_{n \to \infty} B_n^{1/n}
\]
and \( d\mu \) is supported on \([0, \rho]\) (\( \rho = +\infty \) is not excluded).

§ 3. Proof of Theorem 2.1

Define the signed measure, \( d\sigma \), by
\[
d\sigma = d\mu - d\mu_1.
\]
We then define
\[
M_n = A_n - A'_n = \int_0^\infty t^n d\sigma(t).
\]
Also, let \( B_n = \int_0^\infty t^n |d\sigma(t)| \leq A_n + A'_n \). Define the function
\[
\Phi(z) = \int_0^\infty e^{tz} d\sigma(t) \quad \text{for } z \in \mathbb{C}_- = \{ z : \text{Re}(z) \leq 0 \}.
\]
\( \Phi \) is easily seen to be analytic in \( \mathbb{C}_- \), and differentiable in \( \overline{\mathbb{C}_-} \), with
\[
\Phi^{(n)}(z) = \int_0^\infty t^n e^{tz} d\sigma(t).
\]
We see that \( \Phi^{(n)}(0) = M_n \) and \( |\Phi^{(n)}(z)| \leq B_n \), for all \( n = 0, 1, 2, \ldots \) and for all \( z \in \overline{\mathbb{C}_-} \).

Define another function, \( F(z) \), via
\[
F(z) = \sum_{n=0}^{\infty} \frac{M_n}{n!} z^n.
\]
\( F \) is an entire function, and is of exponential type \( \rho \) (i.e. \( |F(z)| \leq C e^{(\rho+\epsilon)|z|} \) for all \( \epsilon > 0 \)).

Now, we observe that \( F^{(n)}(0) = \Phi^{(n)}(0) = M_n \) (\( n = 0, 1, 2, \ldots \)). We know that \( |\Phi^{(n)}(z)| \leq B_n \). Also,
\[
F^{(n)}(z) = \sum_{k=0}^{\infty} \frac{M_{n+k}}{k!} z^k,
\]
and therefore
\[
|F^{(n)}(z)| \leq C_R R^n \quad (n = 0, 1, \ldots) \quad \text{for all } R > \rho \text{ and } |z| \leq 2.
\]
Hence, we can conclude that on the closed disk \( \{ z : |z + 1| \leq 1 \} \),
\[
|\Phi^{(n)}(z) - F^{(n)}(z)| \leq B_n + C_R R^n \leq C'_R B_n R^n.
\]

By Proposition 1.6, \( \Phi - F \equiv 0 \) on \( \{ z : |z + 1| \leq 1 \} \); that is, \( F \) is an entire extension of \( \Phi \).

We next let \( F_\delta(z) = e^{-(\rho+\delta)z}F(z) \). This function is bounded on the imaginary axis (by \( B_0 \)), bounded on the non-negative real axis (by a constant dependent on \( \delta \)) and of exponential type \( \rho \). Hence, applying the Phragměn-Lindelőf theorem \[Mar, \text{vol. 2, p. 214}\], we can conclude that \( |F_\delta(z)| \) is bounded on all of \( \overline{\mathbb{C}_+} = \{ z : \text{Re}(z) \geq 0 \} \) by \( B_0 \). Taking limits (as \( \delta \to 0^+ \)), we see that
\[
|F(z)| \leq B_0 e^{\rho \text{Re}(z)} \text{ on } \overline{\mathbb{C}_+}.
\]

Define the function \( G(z) \) by setting
\[
G(z) = \frac{F(z)e^{-\rho z}}{1 + z}.
\]
Clearly, \( G \) is analytic on \( \mathbb{C}_+ \) and is square summable on the imaginary axis. Thus, we can apply the Paley-Wiener theorem \[PW, \text{p. 8, Theorem V}\] to conclude that
\[
G(z) = \int_{-\infty}^{0} \Psi(t) e^{iz} \, dt \text{ for some } \Psi \in L^2((-\infty, 0)).
\]
We will assume that \( \Psi \) is defined for all real numbers (by setting \( \Psi(t) = 0 \) for all \( t > 0 \)), and we will also extend our signed measure, \( d\sigma \), to the entire real line (by requiring \( |d\sigma((-\infty, 0))| = 0 \)).

On the imaginary axis, we have two representations for \( G \):
\[
\int_{-\infty}^{\infty} \Psi(t) e^{iyt} \, dt = G(iy) = \frac{e^{-i\rho y} F(iy)}{1 + iy}.
\]
So, we can conclude that
\[
\int_{-\infty}^{\infty} \Psi(t - \rho) e^{iyt} \, dt = \frac{1}{1 + iy} \int_{-\infty}^{\infty} e^{iyt} d\sigma(t).
\]
Using the notation \( \hat{f} \) for the Fourier transform of \( f \) (i.e. \( \hat{f}(x) = \int_{-\infty}^{\infty} f(y) e^{iyx} \, dy \)), and letting \( \Psi_\rho(t) = \Psi(t - \rho) \), we arrive at
\[
\hat{\Psi}_\rho(y) = \gamma(y) \hat{d\sigma}(y),
\]
where \( \gamma \) is the function
\[
\gamma(x) = \begin{cases} 
e x & \text{if } x \leq 0, \\ 0 & \text{if } x > 0. \end{cases}
\]
Hence,
\[
\Psi_\rho(y) = (\gamma * d\sigma)(y),
\]
so

$$\Psi(t - \rho) = \int_{t}^{\infty} e^{t-x} d\sigma(x).$$

Thus, for all $t > \rho$, we have $\int_{t}^{\infty} e^{-x} d\sigma(x) = 0$, which implies that the total variation of $d\sigma$ on $(\rho, \infty)$ is 0; i.e., $d\sigma = d\mu - d\mu_1$ is supported on $[0, \rho]$.

Conversely, if we are given a logarithmically convex sequence of positive numbers, $\{A_n\}_0^{\infty}$, with $\sum_{n=0}^{\infty} \sqrt{A_n/A_{n+1}} < \infty$, we set $A_{-1} = A_0$. A version of Proposition 1.6 for the half-plane $|K|$ shows that there is some function, $f \not\equiv 0$, analytic in $\mathbb{C}_+$ and continuous on $0 \in \mathbb{C}_+$, with

$$\sup_{x \geq 0} \int_{-\infty}^{\infty} |f^{(n)}(x + iy)|^2 dy \leq A_{n-1}^2 (n = 0, 1, 2, ...),$$

and $f^{(n)}(0) = 0 (n = 0, 1, 2, ...)$). Applying the same Paley-Wiener theorem, we obtain

$$f(z) = \int_{0}^{\infty} \phi(t)e^{-tz} dt \quad \text{for some } \phi \in L^2((0, \infty)),$$

as well as

$$f^{(n)}(z) = \int_{0}^{\infty} \phi(t)(-t)^n e^{-tz} dt.$$

Applying Plancherel’s theorem, we find

$$\int_{-\infty}^{\infty} |f^{(n)}(iy)|^2 dy = 2\pi \int_{-\infty}^{\infty} t^{2n} |\phi(t)|^2 dt.$$

So, $\int_{0}^{\infty} t^{2n} |\phi(t)|^2 \leq A_{n-1}^2$. Hence, by the Cauchy-Schwarz inequality,

$$\int_{0}^{\infty} t^n |\phi(t)| dt = \int_{0}^{1} t^n |\phi(t)| dt + \int_{1}^{\infty} t^{n+1} |\phi(t)| \frac{dt}{t} \leq A_{n-1} + A_n \leq KA_n,$$

for some constant $K$.

Letting $d\sigma = \frac{1}{t} \phi(t) dt$, and then defining $d\mu = d\sigma^+$ and $d\mu_1 = d\sigma^-$, we get the desired measures.

\section*{§4. Some applications}

In this section we examine some consequences of Theorem 2.1. The first result is a discrete Phragmén-Lindelöf type theorem.

\textbf{Theorem 4.1.} Let $f(z)$ be analytic in $\mathbb{C}_+ = \{ z : \Re(z) > 0 \}$ and continuous in $\overline{\mathbb{C}_+}$. Define

$$A_n = \sup_{0 \leq x \leq n} |f(x + iy)| \quad \text{for } n=1, 2, ... .$$

Assume $A_n < \infty$ for all $n$. 
If \( \{ \sqrt{A_n} \}_{0}^{\infty} \) satisfies the (COM) condition, and \( \sup_{n \in \mathbb{N}} |f(n)| < \infty \), then \( f(z) \) is bounded on \( \mathbb{C}_+ \).

Proof. Consider the function
\[
g(z) = f(z) \left( \frac{1 - e^{-z}}{z} \right)
\]
and let
\[
B_n = \sup_{0 \leq x \leq n} \left( \int_{-\infty}^{\infty} |g(x + iy)|^2 dy \right)^{\frac{1}{2}}.
\]
We find that \( B_n \leq 4A_n \). Using the Paley-Wiener theorem for the strip \( [PW, p. 7, \text{Theorem IV}] \), we have
\[
g(x + iy) = \int_{-\infty}^{\infty} \phi(\sigma)e^{\sigma(x+iy)} d\sigma,
\]
where this integral converges for all \( x > 0 \). Applying Plancherel’s theorem yields
\[
\int_{-\infty}^{\infty} |g(n + iy)|^2 dy = 2\pi \int_{-\infty}^{\infty} e^{2n\sigma} |\phi(\sigma)|^2 d\sigma \leq B_n^2 \leq 16A_n^2.
\]
Setting \( \Psi(t) = \phi(\log t) \) we get
\[
g(1 + z) = \int_{0}^{\infty} t^2 \Psi(t) dt \quad (Re \; z > -1).
\]
Also \( (n \geq 0) \),
\[
C_n = \int_{0}^{\infty} t^n |\Psi(t)| dt = \int_{-\infty}^{\infty} e^{(n+1)t} |\phi(t)| dt
\]
\[
= \int_{-\infty}^{0} e^{nt} |\phi(t)| e^{-t} dt + \int_{0}^{\infty} e^{(n+2)t} |\phi(t)| e^{-t} dt
\]
\[
\leq \left( \frac{1}{2} \int_{-\infty}^{0} e^{2nt} |\phi(t)|^2 dt \right)^{\frac{1}{2}} + \left( \frac{1}{2} \int_{0}^{\infty} e^{2(n+2)t} |\phi(t)|^2 dt \right)^{\frac{1}{2}}
\]
\[
\leq B_n + \frac{B_{n+2}}{2\sqrt{\pi}} \leq \frac{4}{\sqrt{\pi}}A_{n+2}.
\]
Thus \( \{ \sqrt{C_n} \}_{0}^{\infty} \) also satisfies the (COM) condition. Since \( \sup_{n \in \mathbb{N}} |g(n)| \leq \sup_{n \in \mathbb{N}} |f(n)| \), Corollary 2.4 yields \( \Psi(t) = 0 \; (t > 1) \) and \( \phi(t) = 0 \; (t > 0) \). Therefore
\[
f(z) = \frac{z}{1 - e^{-z}} \int_{-\infty}^{0} \phi(\sigma)e^{\sigma z},
\]
which is bounded on every strip \( \{ 0 < Re \; z < a \} \), satisfies \( |f(z)| = O(|z|) \; (|z| \to \infty) \).
Applying the Phragmén-Lindelöf theorem \([Mar, vol. 2, p. 214]\), we see that \( f \) is bounded on \( \mathbb{C}_+ \).

**Theorem 4.2.** Let \( f(z) \) be analytic in \( \mathbb{C}_+ \) and continuous on \( \overline{\mathbb{C}_+} \). Assume
\[
|f(iy)| \leq M \text{ for all real } y \quad \text{and} \quad \sup_{n \in \mathbb{N}} |f(n)| < \infty.
\]
If \(|f(z)| \leq C e^{p|z|\log|z|} \) for some \(p < 2\) and some constant \(C\), then \(|f(z)| \leq M\) for all \(z \in \mathbb{C}_+\).

Proof. Consider the function

\[ h_1(z) = f(z)e^{i \frac{\pi}{2} z^{-p}} \]

in the first quadrant: \(Q_1 = \{ z : 0 \leq \text{Arg}(z) \leq \frac{\pi}{2} \}\). This function is bounded on \(\partial Q_1\), and in \(Q_1\)

\[ |h_1(z)| = O(e^{|z|^{1+\epsilon}}) \quad \text{for} \quad |z| \to \infty, \epsilon > 0. \]

Applying the same Phragmén-Lindelöf theorem \([Mar]\) gives that \(h_1(z)\) is bounded in \(Q_1\). Using similar estimates on \(h_2(z) = f(z)e^{-i \frac{\pi}{2} z^{-p}}\) in the fourth quadrant \(Q_4 = \{ z : -\frac{\pi}{2} \leq \text{Arg}(z) \leq 0 \}\) gives that \(h_2(z)\) is bounded in \(Q_4\). These estimates yield

\[ |f(x + iy)| \leq C \exp\{px \log|x + iy| - (py)\tan^{-1}\left(\frac{y}{x}\right) + \frac{p}{2} \pi |y|\} \]

in \(\mathbb{C}_+\). Now, consider the function

\[ g(z) = f(z)e^{-\delta z^{\frac{p}{2}}} \quad \text{for any} \quad \delta > 0. \]

If

\[ D_n = \sup_{0 \leq x \leq n} |g(z)|, \]

then by a straightforward computation it is not hard to see that

\[ D_n \leq C_0 e^{\pi n} n^{2n}. \]

So, in fact, the sequence \(\{\sqrt{D_n}\}_0^\infty\) satisfies the (COM) condition. This implies that \(|g(z)|\) is bounded in \(\mathbb{C}_+\) by \(M\). Letting \(\delta \to 0^+\), we get

\[ |f(z)| \leq M \quad \text{on} \quad \mathbb{C}_+. \]

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