PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 7, July 1998, Pages 2005–2012
S 0002-9939(98)04252-X

HANKEL OPERATORS ON THE BERGMAN SPACE OF THE UNIT BALL

MARIA NOWAK

(Communicated by Theodore W. Gamelin)

ABSTRACT. We characterize the bounded holomorphic functions f, g in the unit ball of $\mathbb{C}^n$ for which the operator $H_{\bar{g}}^* H_{\bar{f}}$ is compact. For $n = 1$ the result was obtained by Axler and Gorkin in 1988 and by Zheng in 1989.

1. INTRODUCTION

Let B_n be the open unit ball in $\mathbb{C}^n$ with dv the normalized volume measure on B_n . When $n = 1$ we will denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$. The Bergman space $L_a^2(B_n)$ is the closed subspace of $L^2(B_n, dv)$ consisting of holomorphic functions. Let P be the orthogonal projection of $L^2(B_n, dv)$ onto $L_a^2(B_n)$. For $f \in L^2(B_n, dv)$ the Toeplitz operator T_f : $L_a^2(B_n) \to L_a^2(B_n)$ and the Hankel operator $H_f : L_a^2(B_n) \to (L_a^2(B))^\perp$ are defined by

$$T_f(g) = P(fg) \quad \text{and} \quad H_f(g) = fg - P(fg),$$

respectively. In fact, these operators are densely defined for bounded holomorphic functions and extended to all functions in $L_a^2(B_n)$.

In 1986 S.Axler ([4], [5]) proved that for $f \in L_a^2(\mathbb{D})$ the Hankel operator $H_{\bar{f}} : L_a^2(\mathbb{D}) \to (L_a^2(\mathbb{D}))^\perp$ is bounded if and only if f is a Bloch function. Moreover, $H_{\bar{f}}$ is compact if and only if f in the little Bloch space. Many generalizations of these results have been found since then (see e.g. [3], [7], [8]).

In view of the relation

$$(1) \qquad T_f^* T_g - T_g T_f^* = H_{\bar{g}}^* H_{\bar{f}} \quad \text{for } f, g \in L_a^2(B)$$

the problem of characterization of the functions f, g for which the operator $H_{\bar{g}}^* H_{\bar{f}}$ is compact seems natural. In the case of the unit disc the problem was proposed by S. Axler [4] and solved by S. Axler and P. Gorkin [6] and independently by D. Zheng [14]. They have proved the following

Received by the editors December 6, 1995 and, in revised form, December 10, 1996.
1991 *Mathematics Subject Classification.* Primary 30H05, 47B38.
This work was supported in part by KBN grant No. 2 PO3A- 002-08.

Theorem A. *Let* f, g *be bounded holomorphic functions on* $\mathbb{D}$. *Then the following statements are equivalent*:

(a) $H_{\bar{g}}^* H_{\bar{f}}$ *is compact* ,

(b) $\lim_{|z|\to 1^-}(1-|z|^2)\min\{|f'(z)|,|g'(z)|\}=0$,

(c) $\lim_{|w|\to 1^-}\int_{\mathbb{D}}|f(z)-f(w)||g(z)-g(w)|dv(z)=0$,

(d) *either* f *or* g *is constant on each Gleason part* (*except* $\mathbb{D}$) *of the maximal ideal space of* $H^\infty(\mathbb{D})$.

It was also noticed that each of the conditions (b), (c) implies the compactness of the operator $H_{\bar{g}}^* H_{\bar{f}}$ under the weaker and more natural assumption that f, g are Bloch functions. Moreover, these sufficient conditions for compactness of $H_{\bar{g}}^* H_{\bar{f}}$ are still valid if $\mathbb{D}$ is replaced by the unit ball B_n , $n > 1$.

Here we show that in the case of $\mathbb{D}$ (b) ((c)) is also a necessary condition for compactness of $H_{\bar{g}}^* H_{\bar{f}}$ under the assumption that f, g are Bloch functions. The main result of this paper is a characterization of the functions f, g bounded and holomorphic on B_n for which the operator $H_{\bar{g}}^* H_{\bar{f}}$ is compact.

The author wishes to express her thanks to J. Arazy for pointing out the reference [1] and for helpful conversations.

2. Preliminaries

Let $\langle z, w\rangle$ denote the inner product in $\mathbb{C}^n$ given by

$$\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j \ , \qquad z=(z_1,...,z_n)\ ,\ w=(w_1,...,w_n)\ .$$

Let $\mathrm{Aut}(B_n)$ be the group of all biholomorphic maps of B_n into B_n . It is known that $\mathrm{Aut}(B_n)$ is generated by the unitary operators on $\mathbb{C}^n$ and the involutions φ_a of the form

$$\varphi_a(z) = \frac{a - P_a z - (1-|a|^2)^{1/2} Q_a z}{1-\langle z, a\rangle}\ , \tag{2}$$

where $a \in B_n$, P_a is the orthogonal projection into the space spanned by a, i.e.

$$P_a z = \frac{\langle z, a\rangle a}{|a|^2}\ , \quad |a|^2 = \langle a, a\rangle\ ,$$
$$P_0 z = 0$$

and $Q_a = I - P_a$.

For fixed $a \in B_n$ and $r,\ 0 < r < 1$, define

$$E_n(a, r) = \varphi_a(rB_n)\ .$$

Since φ_a is an involution, $z \in E_n(a, r)$ if and only if $|\varphi_a(z)| < r$.

As in [11] we say that a function $f \in C^2(B_n)$ is $\mathcal{M}$- harmonic in B_n if $\widetilde{\Delta} f(z) = 0$ for every $z \in B_n$. The operator $\widetilde{\Delta}$ is defined by

$$(\widetilde{\Delta} f)(z) = \Delta(f\circ\varphi_z)(0)\ ,$$

where Δ is the ordinary Laplacian. We will need to use the following recent results on $\mathcal{M}$-harmonic functions.

Theorem B ([2]). *Suppose that f and g are nonconstant holomorphic functions in B_n, and that $f\bar{g}$ is $\mathcal{M}$-harmonic.*

(a) *When $n = 2$, this cannot happen.*

(b) *When $n \geq 3$, then there exist*

(i) *an integer m, $2 \leq m \leq n-1$,*

(ii) *entire functions $\varphi : \mathbb{C}^{m-1} \to \mathbb{C}$ and $\psi : \mathbb{C}^{n-m} \to \mathbb{C}$, such that*

$$f(Uz) = \varphi\left(\frac{z_2}{1-z_1}, ..., \frac{z_m}{1-z_1}\right), \; g(Uz) = \psi\left(\frac{z_{m+1}}{1-z_1}, ..., \frac{z_n}{1-z_1}\right).$$

Moreover, $f(B_n) = \varphi(\mathbb{C}^{m-1})$, $g(B_n) = \psi(\mathbb{C}^{n-m})$, and $(f\bar{g})(B_n) = \mathbb{C}$ or $\mathbb{C} \setminus \{0\}$. (The latter case occurs only when both φ and ψ omit the value 0.)

Theorem C ([1]). *Assume that for $f \in L^1(B_n, dv)$ the linear operator T_0 is defined by*

$$(T_0 f)(z) = \int_{B_n} (f \circ \varphi_z) dv, \quad z \in B_n.$$

(a) *If $f \in L^\infty(B_n)$ and $T_0 f = f$ then $\widetilde{\Delta} f = 0$.*

(b) *If $f \in L^1(B_n, dv)$, $n < 12$ and $T_0 f = f$, then $\widetilde{\Delta} f = 0$; this fails for all $n \geq 12$.*

We say that a holomorphic function f is a Bloch function if

$$\|f\|_{\mathcal{B}} = \sup\{|\nabla_z f|(1-|z|^2);\ z \in B_n\} < \infty. \tag{3}$$

Bloch functions on bounded homogeneous domains in $\mathbb{C}^n$ were defined by R. Timoney [12] and studied by many authors.(e.g. [9], [11], [12]). One of the most important properties of the space of Bloch functions on B_n, $n \geq 2$, is that (3) is equivalent to the following conditions (for some $K \geq 0$):

$$|\langle \nabla_z f, \bar{z} \rangle|(1-|z|^2) \leq K,$$

$$|\langle \nabla_z f, \bar{x} \rangle|(1-|z|^2) \leq K \text{ for all } x \in \mathbb{C}^n \text{ satisfying } |x| = 1 \text{ and } \langle x, \bar{z} \rangle = 0$$

for all $z \in B_n$. This means that f is a Bloch function on B_n if and only if the radial derivative of f is $O(1/(1-|z|^2))$ and the directional derivatives of f in directions pependicular to the radial direction are $O(1/(1-|z|^2)^{1/2})$.

3. Main results

We start with the following

Lemma 1. *Let f, g be bounded holomorphic functions on B_n. If the operator $H_{\bar{g}}^* H_{\bar{f}}$ is compact then*

$$\lim_{|a| \to 1^-} (1-|a|^2) \min\{|\nabla_a f|, |\nabla_a g|\} = 0. \tag{5}$$

Proof. For $a \in B_n$ define

$$k_a(z) = \frac{(1-|a|^2)^{(n+1)/2}}{(1-\langle z, a \rangle)^{n+1}}, \quad z \in B_n.$$

Because for $f \in L_a^2(B_n)$

$$\langle f, k_a \rangle = (1-|a|^2)^{\frac{n+1}{2}} \int_{B_n} \frac{f(z)}{(1-\langle a, z \rangle)^{n+1}} dv(z) = (1-|a|^2)^{\frac{n+1}{2}} f(a)$$

and holomorphic bounded functions are dense in $L^2_a(B_n)$, k_a tends weakly to zero in L^2_a for $|a|\to 1^-$. Hence the compactness of $H^*_{\bar g}H_{\bar f}$ implies

$$\begin{aligned}0&=\lim_{|a|\to1^-}\langle H^*_{\bar g}H_{\bar f}k_a,k_a\rangle=\lim_{|a|\to1^-}\langle H_{\bar f}k_a,H_{\bar g}k_a\rangle\\&=\lim_{|a|\to1^-}\int_{B_n}(\bar f\circ\varphi_a-\bar f(a))(g\circ\varphi_a-g(a))dv\ .\end{aligned}$$

Suppose that (5) does not hold. Then there exists a sequence $\{a_m\}$ in B such that $\lim_{m\to\infty}|a_m|=1$ and

$$\lim_{m\to\infty}(1-|a_m|^2)|\nabla_{a_m}f|=a>0\quad\text{and}\quad\lim_{m\to\infty}(1-|a_m|^2)|\nabla_{a_m}g|=b>0\ . \tag{6}$$

Since the families of functions

$$\{f\circ\varphi_a-f(a);\ a\in B_n\ \}\quad\text{and}\quad\{g\circ\varphi_a-g(a);\ a\in B_n\ \}$$

are compact, there exists a subsequence of $\{a_m\}$ (which will be denoted also by $\{a_m\}$) and holomorphic functions F,G such that

$$f\circ\varphi_{a_m}-f(a_m)\xrightarrow[m\to\infty]{}F\quad\text{and}\quad g\circ\varphi_{a_m}-g(a_m)\xrightarrow[m\to\infty]{}G$$

uniformly on compact subsets of B_n .

Now let $w\in B_n$ be fixed. Then also

$$f\circ\varphi_{a_n}\circ\varphi_w-f(a_m)\xrightarrow[m\to\infty]{}F\circ\varphi_w\quad\text{and}\quad g\circ\varphi_{a_n}\circ\varphi_w-g(a_m)\xrightarrow[m\to\infty]{}G\circ\varphi_w$$

uniformly on compacta.

Let $a'_m=\varphi_{a_m}(w)$, $m\in\mathbb{N}$. There exists a unique unitary transformation U_m for which $\varphi_{a_m}\circ\varphi_w=\varphi_{a'_m}\circ U_m$ [12, p.29].

It follows from the relation [12, p.26]

$$1-|\varphi_a(w)|^2=\frac{(1-|a|^2)(1-|w|^2)}{|1-\langle w,a\rangle|^2} \tag{7}$$

that $|a'_m|\to1^-$ if $|a_m|\to1^-$. Thus by the comapctness of $H^*_{\bar g}H_{\bar f}$

$$\begin{aligned}0&=\lim_{m\to\infty}\int_{B_n}(\bar f\circ\varphi_{a'_m}-\bar f(a'_m))(g\circ\varphi_{a'_m}-g(a'_m))dv\\&=\lim_{m\to\infty}\int_{B_n}(\bar f\circ\varphi_{a'_m}\circ U-\bar f(a'_m))(g\circ\varphi_{a'_m}\circ U-g(a'_m))dv\\&=\lim_{m\to\infty}\int_{B_n}(\bar f\circ\varphi_{a_m}\circ\varphi_w-\bar f(\varphi_{a_m}\circ\varphi_w)(0))(g\circ\varphi_{a_m}\circ\varphi_w-g(\varphi_{a_m}\circ\varphi_w)(0))dv\end{aligned}$$

In view of boundedness of $f,\ g$ and uniform convergence on compacta we get

$$\begin{aligned}&\lim_{m\to\infty}\int_{B_n}[(\bar f\circ\varphi_{a_m}\circ\varphi_w(z)-\bar f(a_m))-(\bar f(\varphi_{a_m}\circ\varphi_w)(0)-\bar f(a_m))]\\&\qquad\times[(g\circ\varphi_{a_m}\circ\varphi_w(z)-g(a_m))-(g(\varphi_{a_m}\circ\varphi_w)(0)-g(a_m))]dv(z)\\&=\int_{B_n}(\bar F\circ\varphi_w(z)-\bar F\circ\varphi_w(0))(G\circ\varphi_w(z)-G\circ\varphi_w(0))dv(z)=0.\end{aligned} \tag{8}$$

The last equality can be rewritten in the form

$$\int_{B_n}\bar F\circ\varphi_w(z)G\circ\varphi_w(z)dv(z)=\bar F(w)G(w)\ .$$

Now Theorem C implies that $\bar{F}G$ is $\mathcal{M}$-harmonic. Because F, G are bounded the Ahern-Rudin theorem (Theorem B) and Liouville's theorem imply that either F or G is constant on B_n . Suppose that F is constant. Then by the chain rule and the symmetry of the matrix $\varphi_a'(0) = (1-|a|^2)P_a - (1-|a|^2)^{1/2}Q_a$ we get

$$\begin{aligned} 0 = |\nabla_0 F|^2 &= \lim_{m\to\infty} |\nabla_0(f\circ\varphi_{a_m})|^2 = \lim_{m\to\infty} \langle \nabla_{a_m} f \varphi_{a_m}'(0), \nabla_{a_m} f \varphi_{a_m}'(0)\rangle \\ &= \lim_{m\to\infty} \langle \varphi_{\bar{a}_m}'(0)\nabla_{a_m} f, \varphi_{\bar{a}_m}'(0)\nabla_{a_m} f\rangle \\ &= \lim_{m\to\infty} \big((1-|a_m|^2)^2 |P_{\bar{a}_m}\nabla_{a_m} f|^2 + (1-|a_m|^2)|Q_{\bar{a}_m}\nabla_{a_m} f|^2\big) \\ &\geq \lim_{m\to\infty} (1-|a_m|^2)^2 |\nabla_{a_m} f|^2 . \end{aligned}$$

Because the sequence $(1-|a_m|^2)|\nabla_{a_m} g|$ is bounded for bounded and holomorphic g , the last inequality implies that

$$\lim_{m\to\infty} (1-|a_m|^2)^2 |\nabla_{a_m} g||\nabla_{a_m} f| = 0$$

which contradicts (6). □

Note that for the unit disc the same proof still goes under the assumption that f, g are Bloch functions. In this case the equality (8) follows from the Lebesgue's dominated theorem. In fact, if f is a Bloch function on $\mathbb{D}$ then [4, p.320]

$$\int_{\mathbb{D}} |f\circ\varphi_{a_m}(z) - f(a_m)| dv(z) \leq \|f\|_{\mathcal{B}} \int_{\mathbb{D}} |\ln(1-|z|)| dv(z) \leq c\|f\|_{\mathcal{B}} \ .$$

Thus we have

Lemma 2. *If* f, g *are Bloch functions on* $\mathbb{D}$ *such that the operator* $H_{\bar{g}}^* H_{\bar{f}}$ *is compact then*

$$\lim_{|a|\to 1^-} (1-|a|^2) \min\{|f'(a)|, |g'(z)|\} = 0 \ .$$

Analyzing the proof of Lemma 1 one can easily get

Lemma 3. *If* f, g *are bounded holomorphic functions on* B_n *and the operator* $H_{\bar{g}}^* H_{\bar{f}}$ *is compact then*

$$\lim_{|a|\to 1^-} (1-|a|^2) \min\{|P_{\bar{a}}\nabla_a f|, |P_{\bar{a}}\nabla_a g|\} = 0 \tag{9}$$

and

$$\lim_{|a|\to 1^-} (1-|a|^2)^{1/2} \min\{|Q_{\bar{a}}\nabla_a f|, |Q_{\bar{a}}\nabla_a g|\} = 0 \ . \tag{9'}$$

Our main result is the following

Theorem. *Let* f, g *be bounded holomorphic functions on* B_n *and* $0 < r < 1$. *Then the following statements are equivalent*

(a) $H^*_{\bar{g}} H_{\bar{f}}$ *is compact ;*

(b)
$$\lim_{|a|\to 1}(1-|a|^2)\min\{|P_{\bar{a}}\nabla_a f|,|P_{\bar{a}}\nabla_a g|\}$$
$$= \lim_{|a|\to 1}(1-|a|^2)^{1/2}\min\{|Q_{\bar{a}}\nabla_a f|,|Q_{\bar{a}}\nabla_a g|\} = 0\ ;$$

(c)
$$\lim_{|a|\to 1}\int_{rB_n}|f\circ\varphi_a - f(a)||g\circ\varphi_a - g(a)|dv = 0\ ;$$

(d)
$$\lim_{|a|\to 1}\frac{1}{|E_n(a,r)|}\int_{E_n(a,r)}|f-f(a)||g-g(a)|dv = 0\ ;$$

(e)
$$\lim_{|a|\to 1}\int_{B_n}|f\circ\varphi_a - f(a)||g\circ\varphi_a - g(a)|dv = 0\ .$$

Proof. (a)$\Longrightarrow$(b). By Lemma 3.

(b)$\Longrightarrow$(c). Suppose f and g are bounded holomorphic functions on B_n for which statement (b) holds. Let $\{a_m\}$ be a sequence of points in B_n such that $\lim_{m\to\infty}|a_m| = 1$ and

$$F = \lim_{m\to\infty}(f\circ\varphi_{a_m} - f(a_m)) \quad \text{and} \quad G = \lim_{m\to\infty}(g\circ\varphi_{a_m} - g(a_m))\ ,$$

where the convergence is uniform on compact subsets of B_n and F, G are holomorphic functions on B_n . We will show that

$$\int_{rB_n}|\nabla_z F||\nabla_z G|dv(z) = 0\ . \tag{10}$$

Applying the definitions of F, G and changing the variables $w = \varphi_{a_m}(z)$ we obtain

$$\int_{rB_n}|\nabla_z F||\nabla_z G|dv(z) = \lim_{m\to\infty}\int_{rB_n}|\nabla_{\varphi_{a_m}(z)} f\varphi'_{a_m}(z)||\nabla_{\varphi_{a_m}(z)} g\varphi'_{a_m}(z)|dv(z)$$
$$= \lim_{m\to\infty}\int_{E_n(a_m,r)}|\nabla_w f\varphi'_{a_m}(\varphi_{a_m}(w))||\nabla_w g\varphi'_{a_m}(\varphi_{a_m}(w))|\frac{(1-|a_m|^2)^{n+1}}{|1-\langle w,a_m\rangle|^{2n+2}}dv(w)\ .$$

Let $\varphi_{a_m}(w) = \zeta_m$. To calculate $\varphi'_{a_m}(\zeta_m)$ notice that for fixed w and a_m the biholmorphic maping $\varphi_w\circ\varphi_{a_m}\circ\varphi_{\zeta_m}$ is a unitary operator , say U_m [12, p.29]. Hence

$$\varphi'_{a_m}(\zeta_m) = \varphi'_w(0)U_m\varphi'_{\zeta_m}(\zeta_m)\ .$$

Let $\|A\|$ denote the standard norm of the linear operator $A:\ \mathbb{C}^n\to\mathbb{C}^n$. Because for every $z\in B_n$

$$\varphi'_z(z) = -(1-|z|^2)^{-1}P_z - (1-|z|^2)^{-1/2}Q_z\ ,$$

we have the following estimate:

$$\|\varphi'_{\zeta_m}(\zeta_m)\| \le \frac{2}{1-|\zeta_m|^2}\ .$$

Moreover, in view of (7) we get

$$\|\varphi'_{\zeta_m}(\zeta_m)\| \le \frac{2|1-\langle w,a_m\rangle|^2}{(1-|a_m|^2)(1-|w|^2)}\ .$$

Hence

$$\begin{aligned}
&\int_{rB_n} |\nabla_z F||\nabla_z G| dv(z) \\
&\leq 4 \lim_{m\to\infty} \int_{E_n(a_m,r)} (1-|w|^2)^{-2} |\varphi'_{\bar{w}}(0)\nabla_w f||\varphi'_{\bar{w}}(0)\nabla_w g| \frac{(1-|a_m|^2)^{n-1}}{|1-\langle w, a_m\rangle|^{2n-2}} dv(w) \\
&\leq 4 \lim_{m\to\infty} \int_{E_n(a_m,r)} (1-|w|^2)^{-2} \left|(1-|w|^2)P_{\bar{w}}\nabla_w f + (1-|w|^2)^{1/2} Q_{\bar{w}}\nabla_w f\right| \\
&\qquad \times \left|(1-|w|^2)P_{\bar{w}}\nabla_w g + (1-|w|^2)^{1/2} Q_{\bar{w}}\nabla_w g\right| \frac{(1-|a_m|^2)^{n-1}}{|1-\langle w, a_m\rangle|^{2n-2}} dv(w) \\
&\leq 2^{n+1} \lim_{m\to\infty} \int_{E_n(a_m,r)} |P_{\bar{w}}\nabla_w f||P_{\bar{w}}\nabla_w g|(1-|w|^2)^{-n+1} dv(w) \\
&+ 2^{n+1} \lim_{m\to\infty} \int_{E_n(a_m,r)} |P_{\bar{w}}\nabla_w f||Q_{\bar{w}}\nabla_w g|(1-|w|^2)^{-n+1/2} dv(w) \\
&+ 2^{n+1} \lim_{m\to\infty} \int_{E_n(a_m,r)} |Q_{\bar{w}}\nabla_w f||P_{\bar{w}}\nabla_w g|(1-|w|^2)^{-n+1/2} dv(w) \\
&+ 2^{n+1} \lim_{m\to\infty} \int_{E_n(a_m,r)} |Q_{\bar{w}}\nabla_w f||Q_{\bar{w}}\nabla_w g|(1-|w|^2)^{-n} dv(w) \ .
\end{aligned}$$

To finish the proof of (10) it is enough to show that each of the four limits on the right-hand side of the last inequality is zero. Indeed, we have

$$\begin{aligned}
&\int_{E_n(a_m,r)} |P_{\bar{w}}\nabla_w f||P_{\bar{w}}\nabla_w g|(1-|w|^2)^{-n+1} dv(w) \\
&\leq \sup_{w\in E_n(a_m,r)} (1-|w|^2)^2 |P_{\bar{w}}\nabla_w f||P_{\bar{w}}\nabla_w f| \int_{E_n(a_m,r)} (1-|w|^2)^{-n-1} dv(w) \\
&= \frac{r^{2n}}{(1-r^2)^{n+1}} \sup_{w\in E_n(a_m,r)} (1-|w|^2)^2 |P_{\bar{w}}\nabla_w f||P_{\bar{w}}\nabla_w f| \ .
\end{aligned}$$

Let $\{\zeta_m\}$ be the sequence in B_n such that: $\zeta_m \in E(a_m, r)$ and

$$\sup_{w\in E(a_m,r)} (1-|w|^2)^2 |P_{\bar{w}}\nabla_w f||P_{\bar{w}}\nabla_w f| = (1-|\zeta_m^2)^2 |P_{\bar{\zeta}_m}\nabla_{\zeta_m} f||P_{\bar{\zeta}_m}\nabla_{\zeta_m} f| \ .$$

Then $\lim_{m\to\infty}|\zeta_m| = 1$ and (b) implies that the last expression tends to 0 . The same reasoning applies to the remaining cases.

Now (10) implies that at least one of the functions F, G must be identically 0 on rB_n . Hence

$$\lim_{m\to\infty} \int_{rB_n} |f\circ\varphi_{a_m} - f(a_m)||g\circ\varphi_{a_m} - g(a_m)| dv = \int_{rB_n} |FG| dv = 0 \ .$$

(c)$\Longleftrightarrow$(d) A change-of-variables yields

$$\begin{aligned}
&\int_{E_n(a,r)} |f-f(a)||g-g(a)| dv \\
&\qquad = \int_{rB_n} |f\circ\varphi_a(z) - f(a)||g\circ\varphi_a(z) - g(a)| \frac{(1-|a|^2)^{n+1}}{|1-\langle z,a\rangle|^{2(n+1)}} dv.
\end{aligned}$$

Because

$$|E_n(a,r)| = \left(\frac{1-|a|^2}{1-r^2|a|^2}\right)^{n+1}$$

the desired equivalence follows from the inequalities

$$\frac{1}{r^{2n}(1+r^2)^{2(n+1)}} \le \frac{1}{|E_n(a,r)|}\frac{(1-|a|^2)^{(n+1)}}{|1-\langle z,a\rangle|^{2(n+1)}} \le \frac{1}{r^{2n}(1-r^2)^{n+1}}\ , \quad z \in rB_n\ .$$

(c)$\Longrightarrow$(e) Let $\{a_m\}$ be a sequence such that $\lim_{m\to\infty}|a_m| = 1$ and F, G be as in the proof of the implication (b)$\Longrightarrow$(c). Then (c) implies that

$$\int_{rB_n} |F||G|dv = 0\ .$$

This in turn implies that either F or G is identically zero on rB_n , and hence on B_n .

(e)$\Longrightarrow$(a) It is enough to proceed analogously to the proof of Theorem 2 of [14]. □

Remark. Notice that the functions f, g defined in assertion (ii) of Theorem B cannot be Bloch functions. Hence in view of Theorem C, when $n < 12$ our theorem holds for Bloch functions instead of bounded holomorphic functions.

References

[1] P.Ahern, M.Flores, and W.Rudin, *An Invariant Volume-Mean-Value Property*, J. Funct. Anal. **111** (1993), 380-397. MR **94b:**31002

[2] P.Ahern and W.Rudin, *$\mathcal{M}$-harmonic product*, Indag. Mathem., N.S., **2 (2)** (1991), 141-147. MR **92g:**32013

[3] J. Arazy, S.Fisher and J Peetre, *Hankel operators on weighted Bergman spaces*, Amer.J. Math. **110** (1988), 989-1054. MR **90a:**47067

[4] S.Axler, *The Bergman space , Bloch space,and the commutators of multiplication operators*, Duke Math. J. **53** (1986), 315-332. MR **87m:**47064

[5] S.Axler, *Bergman spaces and their operators, Surveys of some recent results on operator theory*, Pitman research notes in mathematics series, **No 171** (1988), 1-50. MR **90b:**47048

[6] S.Axler and P.Gorkin, *Algebras on the disk and doubly commuting multiplication operators*, Trans. Amer. Math. Soc. **309** (1988), 711-723. MR **90a:**46133

[7] D.Békollé, C.A. Berger, L.A.Coburn, and Kehe Zhu, *BMO in the Bergman metric on bounded symmetric domains*, J. Funct. Anal. **93** (1990), 310-350. MR **91j:**32034

[8] Huiping Li and D.H. Luecking, *BMO on strongly pseudoconvex domains: Hankel operators, Duality and $\bar{\partial}$- estimates*, Trans. Amer. Math. Soc. **346(2)** (1994). MR **95b:**47027

[9] M.Jevtić and M.Pavlović, *On $\mathcal{M}$- harmonic Bloch space*, Proc. Amer. Math.Soc. **123 (5)** (1995), 1385- 1392. MR **95i:**32009

[11] Z.J.Lou, *Characterizations of Bloch functions on the unit ball of $\mathbb{C}^n$*, Kodai Math. J. **16** (1993), 74-78. MR **94b:**32007

[10] W.Rudin, *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Springer-Verlag, 1980. MR **82i:**32002

[12] R.M. Timoney, *Bloch functions in several complex variables*, Bull. London Math. Soc. **12** (1980), 241-267. MR **83b:**32004

[13] R.M. Timoney, *Bloch functions in several complex variables, II*, J.Reine Angew. Math. **319** (1980), 1-22. MR **83b:**32005

[14] D.Zheng, *Hankel operators and Toeplitz operators on the Bergman Space*, J. Funct. Anal. **83** (1989), 89-120. MR **91b:**47057

Instytut Matematyki UMCS , pl. Marii Curie-Skłodowskiej 1,20-031 Lublin, Poland
E-mail address: nowakm@golem.umcs.lublin.pl