HANKEL OPERATORS
ON THE BERGMAN SPACE OF THE UNIT BALL

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Abstract. We characterize the bounded holomorphic functions $f, g$ in the unit ball of $\mathbb{C}^n$ for which the operator $H_f^* H_g$ is compact. For $n = 1$ the result was obtained by Axler and Gorkin in 1988 and by Zheng in 1989.

1. Introduction

Let $B_n$ be the open unit ball in $\mathbb{C}^n$ with $dv$ the normalized volume measure on $B_n$. When $n = 1$ we will denote by $\mathbb{D}$ the unit disc in $\mathbb{C}$. The Bergman space $L^2_a(B_n)$ is the closed subspace of $L^2(B_n, dv)$ consisting of holomorphic functions. Let $P$ be the orthogonal projection of $L^2(B_n, dv)$ onto $L^2_a(B_n)$. For $f \in L^2(B_n, dv)$ the Toeplitz operator $T_f : L^2_a(B_n) \to L^2_a(B_n)$ and the Hankel operator $H_f : L^2_a(B_n) \to (L^2_a(B))^{\perp}$ are defined by

$$T_f(g) = P(fg) \quad \text{and} \quad H_f(g) = fg - P(fg),$$

respectively. In fact, these operators are densely defined for bounded holomorphic functions and extended to all functions in $L^2_a(B_n)$.

In 1986 S. Axler ([4], [5]) proved that for $f \in L^2_a(\mathbb{D})$ the Hankel operator $H_f : L^2_a(\mathbb{D}) \to (L^2_a(\mathbb{D}))^{\perp}$ is bounded if and only if $f$ is a Bloch function. Moreover, $H_f$ is compact if and only if $f$ in the little Bloch space. Many generalizations of these results have been found since then (see e.g. [3], [7], [8]).

In view of the relation

$$T_f^* T_g - T_g T_f^* = H_f^* H_g$$

for $f, g \in L^2_a(B)$, the problem of characterization of the functions $f, g$ for which the operator $H_f^* H_g$ is compact seems natural. In the case of the unit disc the problem was proposed by S. Axler [4] and solved by S. Axler and P. Gorkin [6] and independently by D. Zheng [14]. They have proved the following
Theorem A. Let \( f, g \) be bounded holomorphic functions on \( \mathbb{D} \). Then the following statements are equivalent:

(a) \( H_g^* H_f \) is compact,

(b) \( \lim_{|z| \to 1} (1 - |z|^2) \min \{|f'(z)|, |g'(z)|\} = 0 \),

(c) \( \lim_{|w| \to 1} \int_{\mathbb{D}} |f(z) - f(w)||g(z) - g(w)|dv(z) = 0 \),

(d) either \( f \) or \( g \) is constant on each Gleason part (except \( \mathbb{D} \)) of the maximal ideal space of \( H_\infty(\mathbb{D}) \).

It was also noticed that each of the conditions (b), (c) implies the compactness of \( H_g^* H_f \) under the weaker and more natural assumption that \( f, g \) are Bloch functions. Moreover, these sufficient conditions for compactness of \( H_g^* H_f \) are still valid if \( \mathbb{D} \) is replaced by the unit ball \( B_n, n > 1 \).

Here we show that in the case of \( \mathbb{D} \) (b) (c) is also a necessary condition for compactness of \( H_g^* H_f \) under the assumption that \( f, g \) are Bloch functions. The main result of this paper is a characterization of the functions \( f, g \) bounded and holomorphic on \( B_n \) for which the operator \( H_g^* H_f \) is compact.

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2. Preliminaries

Let \( \langle z, w \rangle \) denote the inner product in \( \mathbb{C}^n \) given by

\[
\langle z, w \rangle = \sum_{j=1}^{n} z_j \bar{w}_j, \quad z = (z_1, ..., z_n), \quad w = (w_1, ..., w_n).
\]

Let \( \text{Aut}(B_n) \) be the group of all biholomorphic maps of \( B_n \) into \( B_n \). It is known that \( \text{Aut}(B_n) \) is generated by the unitary operators on \( \mathbb{C}^n \) and the involutions \( \varphi_a \) of the form

\[
\varphi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{1/2} Q_a z}{1 - \langle z, a \rangle},
\]

where \( a \in B_n, P_a \) is the orthogonal projection into the space spanned by \( a \), i.e.

\[
P_a z = \frac{\langle z, a \rangle a}{|a|^2}, \quad |a|^2 = \langle a, a \rangle,
\]

\[
P_0 z = 0.
\]

For fixed \( a \in B_n \) and \( r, 0 < r < 1 \), define

\[
E_n(a, r) = \varphi_a(rB_n).
\]

Since \( \varphi_a \) is an involution, \( z \in E_n(a, r) \) if and only if \( |\varphi_a(z)| < r \).

As in [11] we say that a function \( f \in C^2(B_n) \) is \( M \)-harmonic in \( B_n \) if \( \tilde{\Delta} f(z) = 0 \) for every \( z \in B_n \). The operator \( \tilde{\Delta} \) is defined by

\[
(\tilde{\Delta} f)(z) = \Delta(f \circ \varphi_a(0)),
\]

where \( \Delta \) is the ordinary Laplacian. We will need to use the following recent results on \( M \)-harmonic functions.
Theorem B ([2]). Suppose that $f$ and $g$ are nonconstant holomorphic functions in $B_n$, and that $fg$ is $M$-harmonic.

(a) When $n = 2$, this cannot happen. 
(b) When $n \geq 3$, then there exist 
(i) an integer $m$, $2 \leq m \leq n - 1$, 
(ii) entire functions $\varphi : \mathbb{C}^{n-1} \to \mathbb{C}$ and $\psi : \mathbb{C}^{n-m} \to \mathbb{C}$, such that
\[
\langle \varphi(z), \psi(z) \rangle = \varphi\left(\frac{z_2}{1 - z_1}, \ldots, \frac{z_m}{1 - z_1}\right), \quad \langle g(z), \psi(z) \rangle = \psi\left(\frac{z_{m+1}}{1 - z_1}, \ldots, \frac{z_n}{1 - z_1}\right).
\]

Moreover, $f(B_n) = \varphi(\mathbb{C}^{n-1})$, $g(B_n) = \psi(\mathbb{C}^{n-m})$, and $(fg)(B_n) = \mathbb{C}$ or $\mathbb{C}\setminus\{0\}$. (The latter case occurs only when both $\varphi$ and $\psi$ omit the value 0.)

Theorem C ([1]). Assume that for $f \in L^1(B_n, dv)$ the linear operator $T_0$ is defined by
\[
(T_0 f)(z) = \int_{B_n} (f \circ \varphi_2) dv, \quad z \in B_n.
\]

(a) If $f \in L^\infty(B_n)$ and $T_0 f = f$ then $\Delta f = 0$.
(b) If $f \in L^1(B_n, dv)$, $n < 12$ and $T_0 f = f$, then $\Delta f = 0$; this fails for all $n \geq 12$.

We say that a holomorphic function $f$ is a Bloch function if
\[
\|f\|_B = \sup\{|\nabla_x f| (1 - |z|^2); \ z \in B_n\} < \infty.
\]

Bloch functions on bounded homogeneous domains in $\mathbb{C}^n$ were defined by R. Timoney [12] and studied by many authors (e.g., [9], [11], [12]). One of the most important properties of the space of Bloch functions on $B_n$, $n \geq 2$, is that (3) is equivalent to the following conditions (for some $K \geq 0$):
\[
|\langle \nabla_z f, z \rangle| (1 - |z|^2) \leq K, \\
|\langle \nabla_z f, \bar{x} \rangle| (1 - |x|^2) \leq K \text{ for all } x \in \mathbb{C}^n \text{ satisfying } |x| = 1 \text{ and } \langle x, \bar{z} \rangle = 0
\]

for all $z \in B_n$. This means that $f$ is a Bloch function on $B_n$ if and only if the radial derivative of $f$ is $O(1/(1 - |z|^2))$ and the directional derivatives of $f$ in directions perpendicular to the radial direction are $O(1/(1 - |z|^2)^{1/2})$.

3. Main results

We start with the following

Lemma 1. Let $f, g$ be bounded holomorphic functions on $B_n$. If the operator $H^*_g H_f$ is compact then
\[
\lim_{|a| \to 1^{-}} (1 - |a|^2) \min\{|\nabla_a f|, |\nabla_a g|\} = 0.
\]

Proof. For $a \in B_n$ define
\[
k_a(z) = \frac{(1 - |a|^2)^{(n+1)/2}}{(1 - \langle a, z \rangle)^{n+1}}, \quad z \in B_n.
\]

Because for $f \in L^2(B_n)$
\[
\langle f, k_a \rangle = (1 - |a|^2)^{\frac{n+1}{2}} \int_{B_n} \frac{f(z)}{(1 - \langle a, z \rangle)^{n+1}} dv(z) = (1 - |a|^2)^{\frac{n+1}{2}} f(a)
\]
and holomorphic bounded functions are dense in \( L^2_0(B_n) \), \( k_a \) tends weakly to zero in \( L^2_0 \) for \( |a| \to 1^- \). Hence the compactness of \( H^*_g H^*_f \) implies
\[
0 = \lim_{|a| \to 1^-} \langle H^*_g H^*_f k_a, k_a \rangle = \lim_{|a| \to 1^-} \langle H^*_f k_a, H^*_g k_a \rangle
\]
\[
= \lim_{|a| \to 1^-} \int_{B_n} (\bar{f} \circ \varphi_a - \bar{f}(a)) (g \circ \varphi_a - g(a)) dv.
\]
Suppose that (5) does not hold. Then there exists a sequence \( \{a_m\} \) in \( B \) such that \( \lim_{m \to \infty} |a_m| = 1 \) and
\[
\lim_{m \to \infty} (1 - |a_m|^2) |\nabla_{a_m} f| = a > 0 \quad \text{and} \quad \lim_{m \to \infty} (1 - |a_m|^2) |\nabla_{a_m} g| = b > 0.
\]
Since the families of functions
\[
\{f \circ \varphi_a - f(a); \ a \in B_n \} \quad \text{and} \quad \{g \circ \varphi_a - g(a); \ a \in B_n \}
\]
are compact, there exists a subsequence of \( \{a_m\} \) (which will be denoted also by \( \{a_m\} \)) and holomorphic functions \( F, G \) such that
\[
f \circ \varphi_a - f(a) \xrightarrow{m \to \infty} F \quad \text{and} \quad g \circ \varphi_a - g(a) \xrightarrow{m \to \infty} G
\]
uniformly on compact subsets of \( B_n \).

Now let \( w \in B_n \) be fixed. Then also
\[
f \circ \varphi_{a_m} \circ \varphi_w - f(a_m) \xrightarrow{m \to \infty} f \circ \varphi_w \quad \text{and} \quad g \circ \varphi_{a_m} \circ \varphi_w - g(a_m) \xrightarrow{m \to \infty} G \circ \varphi_w
\]
uniformly on compacta.

Let \( a'_m = \varphi_{a_m}(w) \), \( m \in \mathbb{N} \). There exists a unique unitary transformation \( U_m \) for which \( \varphi_{a_m} \circ \varphi_w = \varphi_{a'_m} \circ U_m \) [12, p.29].

It follows from the relation [12, p.26]
\[
1 - |\varphi_a(w)|^2 = \frac{(1 - |a|^2)(1 - |w|^2)}{|1 - \langle w, a \rangle|^2}
\]
that \( |a'_m| \to 1^- \) if \( |a_m| \to 1^- \). Thus by the compactness of \( H^*_g H^*_f \)
\[
0 = \lim_{m \to \infty} \int_{B_n} (\bar{f} \circ \varphi_{a'_m} - \bar{f}(a'_m))(g \circ \varphi_{a'_m} - g(a'_m)) dv
\]
\[
= \lim_{m \to \infty} \int_{B_n} (\bar{f} \circ \varphi_{a'_m} \circ U - \bar{f}(a'_m))(g \circ \varphi_{a'_m} \circ U - g(a'_m)) dv
\]
\[
= \lim_{m \to \infty} \int_{B_n} (\bar{f} \circ \varphi_{a_m} \circ \varphi_w - \bar{f}(\varphi_{a_m} \circ \varphi_w)(0))(g \circ \varphi_{a_m} \circ \varphi_w - g(\varphi_{a_m} \circ \varphi_w)(0)) dv
\]
In view of boundedness of \( f, g \) and uniform convergence on compacta we get
\[
\lim_{m \to \infty} \int_{B_n} [(\bar{f} \circ \varphi_{a_m} \circ \varphi_w(z) - \bar{f}(a_m)) - (\bar{f}(\varphi_{a_m} \circ \varphi_w)(0) - \bar{f}(a_m))] \times [(g \circ \varphi_{a_m} \circ \varphi_w(z) - g(a_m)) - (g(\varphi_{a_m} \circ \varphi_w)(0) - g(a_m))] dv(z)
\]
\[
= \int_{B_n} (\bar{F} \circ \varphi_w(z) - \bar{F}(\varphi_w(0))(G \circ \varphi_w(z) - G(\varphi_w(0)) dv(z) = 0.
\]
The last equality can be rewritten in the form
\[
\int_{B_n} \bar{F} \circ \varphi_w(z) G \circ \varphi_w(z) dv(z) = \bar{F}(w) G(w).
\]
Now Theorem C implies that $\tilde{F} G$ is $\mathcal{M}$-harmonic. Because $F, G$ are bounded the Ahern-Rudin theorem (Theorem B) and Liouville's theorem imply that either $F$ or $G$ is constant on $B_n$. Suppose that $F$ is constant. Then by the chain rule and the symmetry of the matrix $\varphi_a'(0) = (1 - |a|^2) P_a - (1 - |a|^2)^{1/2} Q_a$ we get

$$0 = |\nabla_0 F|^2 = \lim_{m \to \infty} |\nabla_0 (f \circ \varphi_{a_m})|^2 = \lim_{m \to \infty} \langle \nabla_{a_m} f \varphi_{a_m}'(0), \nabla_{a_m} f \varphi_{a_m}'(0) \rangle$$

$$= \lim_{m \to \infty} \langle \varphi_{a_m}'(0) \nabla_{a_m} f, \varphi_{a_m}'(0) \nabla_{a_m} f \rangle$$

$$= \lim_{m \to \infty} \left( (1 - |a_m|^2)^2 |P_{\bar{a}} \nabla_{a_m} f|^2 + (1 - |a_m|^2)^2 |Q_{\bar{a}} \nabla_{a_m} f|^2 \right)$$

$$\geq \lim_{m \to \infty} (1 - |a_m|^2)^2 |\nabla_{a_m} f|^2.$$ 

Because the sequence $(1 - |a_m|^2) |\nabla_{a_m} g|$ is bounded for bounded and holomorphic $g$, the last inequality implies that

$$\lim_{m \to \infty} (1 - |a_m|^2)^2 |\nabla_{a_m} g||\nabla_{a_m} f| = 0$$

which contradicts (6).

Note that for the unit disc the same proof still goes under the assumption that $f, g$ are Bloch functions. In this case the equality (8) follows from the Lebesgue’s dominated theorem. In fact, if $f$ is a Bloch function on $D$ then [4, p.320]

$$\int_D |f \circ \varphi_{a_m}(z) - f(a_m)| dv(z) \leq \|f\|_B \int_D |\ln(1 - |z|)| dv(z) \leq c\|f\|_B .$$

Thus we have

**Lemma 2.** If $f, g$ are Bloch functions on $D$ such that the operator $H_{\bar{g}}^* H_f$ is compact then

$$\lim_{|a| \to 1} (1 - |a|^2) \min\{|f'(a)|, |g'(z)|\} = 0 .$$

Analyzing the proof of Lemma 1 one can easily get

**Lemma 3.** If $f, g$ are bounded holomorphic functions on $B_n$ and the operator $H_{\bar{g}}^* H_f$ is compact then

$$(9) \quad \lim_{|a| \to 1} (1 - |a|^2) \min\{|P_{\bar{a}} \nabla_{a} f|, |P_{\bar{a}} \nabla_{a} g|\} = 0$$

and

$$(9') \quad \lim_{|a| \to 1} (1 - |a|^2)^{1/2} \min\{|Q_{\bar{a}} \nabla_{a} f|, |Q_{\bar{a}} \nabla_{a} g|\} = 0 .$$

Our main result is the following
Theorem. Let $f, g$ be bounded holomorphic functions on $B_n$ and $0 < r < 1$. Then the following statements are equivalent:

(a) \[ H^*_H \mathcal{F} \text{ is compact ;} \]
(b) \[ \lim_{|a| \to 1} (1 - |a|^2) \min \{|P_a \nabla_a f|, |P_a \nabla_g g|\} = \lim_{|a| \to 1} (1 - |a|^2)^{1/2} \min \{|Q_a \nabla_a f|, |Q_a \nabla_g g|\} = 0 ; \]
(c) \[ \lim_{|a| \to 1} \int_{rB_n} |f \circ \varphi_a - f(a)||g \circ \varphi_a - g(a)|dv = 0 ; \]
(d) \[ \lim_{|a| \to 1} \int_{E_n(a, r)} |f - f(a)||g - g(a)|dv = 0 ; \]
(e) \[ \lim_{|a| \to 1} \int_{B_n} |f \circ \varphi_a - f(a)||g \circ \varphi_a - g(a)|dv = 0 . \]

Proof. (a)\(\Rightarrow\)(b). By Lemma 3.

(b)\(\Rightarrow\)(c). Suppose $f$ and $g$ are bounded holomorphic functions on $B_n$ for which statement (b) holds. Let \( \{a_m\} \) be a sequence of points in $B_n$ such that $\lim_{m \to \infty} |a_m| = 1$ and
\[ F = \lim_{m \to \infty} (f \circ \varphi_{a_m} - f(a_m)) \quad \text{and} \quad G = \lim_{m \to \infty} (g \circ \varphi_{a_m} - g(a_m)) , \]
where the convergence is uniform on compact subsets of $B_n$ and $F, G$ are holomorphic functions on $B_n$. We will show that
\[ \int_{rB_n} |\nabla_z F||\nabla_z G|dv(z) = 0 . \]

Applying the definitions of $F, G$ and changing the variables $w = \varphi_{a_m}(z)$ we obtain
\[ \int_{rB_n} |\nabla_z F||\nabla_z G|dv(z) = \lim_{m \to \infty} \int_{rB_n} |\nabla \varphi_{a_m}(z) F \varphi'_{a_m}(z)||\nabla \varphi_{a_m}(z) G \varphi'_{a_m}(z)|dv(z) \]
\[ = \lim_{m \to \infty} \int_{E_n(a_m, r)} |\nabla w F \varphi'_{a_m}(\varphi_{a_m}(w))||\nabla w G \varphi'_{a_m}(\varphi_{a_m}(w))|\frac{(1 - |a_m|^2)^{n+1}}{|1 - \langle w, a_m \rangle^2|^2 + 2}\ dv(w) . \]

Let $\varphi_{a_m}(w) = \zeta_m$. To calculate $\varphi'_{a_m}(\zeta_m)$ notice that for fixed $w$ and $a_m$ the biholomorphic mapping $\varphi_w \circ \varphi_{a_m} \circ \varphi_{a_m}$ is a unitary operator, say $U_m$ [12, p.29]. Hence
\[ \varphi'_{a_m}(\zeta_m) = \varphi'_w(0)U_m \varphi'_{a_m}(\zeta_m) . \]

Let $\|A\|$ denote the standard norm of the linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$. Because for every $z \in B_n$
\[ \varphi'_z(z) = -(1 - |z|^2)^{-1}P_z - (1 - |z|^2)^{-1/2}Q_z , \]
we have the following estimate:
\[ \|\varphi'_{a_m}(\zeta_m)\| \leq \frac{2}{1 - |\zeta_m|^2} . \]

Moreover, in view of (7) we get
\[ \|\varphi'_{a_m}(\zeta_m)\| \leq \frac{2|1 - \langle w, a_m \rangle^2}{(1 - |a_m|^2)(1 - |w|^2)} . \]
\[
\int_{rB_n} |\nabla F||\nabla G| \, dv(z)
\]
\[
\leq 4 \lim_{{m \to \infty}} \int_{{E_n(a_m,r)}} (1 - |w|^2)^{-2} |\varphi'_w(0)\nabla_w f||\varphi'_w(0)\nabla_w g| \left(1 - |a_m|^2\right)^{n-1} \left|1 - \frac{1}{w, a_m}\right|^{2n-2} \, dv(w)
\]
\[
\leq 4 \lim_{{m \to \infty}} \int_{{E_n(a_m,r)}} (1 - |w|^2)^{-2} \left| (1 - |w|^2) P_w \nabla_w f + (1 - |w|^2)^{1/2} Q_w \nabla_w f \right| \\
\times \left| (1 - |w|^2) P_w \nabla_w g + (1 - |w|^2)^{1/2} Q_w \nabla_w g \right| \left(1 - |a_m|^2\right)^{n-1} \left|1 - \frac{1}{w, a_m}\right|^{2n-2} \, dv(w)
\]
\[
\leq 2^{n+1} \lim_{{m \to \infty}} \int_{{E_n(a_m,r)}} |P_w \nabla_w f||P_w \nabla_w g|(1 - |w|^2)^{-n+1} \, dv(w)
\]
\[
+ 2^{n+1} \lim_{{m \to \infty}} \int_{{E_n(a_m,r)}} |P_w \nabla_w f||Q_w \nabla_w g|(1 - |w|^2)^{-n+1/2} \, dv(w)
\]
\[
+ 2^{n+1} \lim_{{m \to \infty}} \int_{{E_n(a_m,r)}} |Q_w \nabla_w f||P_w \nabla_w g|(1 - |w|^2)^{-n+1/2} \, dv(w)
\]
\[
+ 2^{n+1} \lim_{{m \to \infty}} \int_{{E_n(a_m,r)}} |Q_w \nabla_w f||Q_w \nabla_w g|(1 - |w|^2)^{-n} \, dv(w)
\]

To finish the proof of (10) it is enough to show that each of the four limits on the right-hand side of the last inequality is zero. Indeed, we have

\[
\int_{{E_n(a_m,r)}} |P_w \nabla_w f||P_w \nabla_w g|(1 - |w|^2)^{-n+1} \, dv(w)
\]
\[
\leq \sup_{{w \in E_n(a_m,r)}} (1 - |w|^2)^2 |P_w \nabla_w f||P_w \nabla_w f| \int_{{E_n(a_m,r)}} (1 - |w|^2)^{-n-1} \, dv(w)
\]
\[
= \frac{2^{n+1}}{(1 - r^2)^{n+1}} \sup_{{w \in E_n(a_m,r)}} (1 - |w|^2)^2 |P_w \nabla_w f||P_w \nabla_w f|
\]

Let \( \{\zeta_m\} \) be the sequence in \( B_n \) such that: \( \zeta_m \in E(a_m, r) \) and

\[
\sup_{{w \in E(a_m,r)}} (1 - |w|^2)^2 |P_w \nabla_w f||P_w \nabla_w f| = (1 - |\zeta_m|^2)^2 |P_{\zeta_m} \nabla_{\zeta_m} f||P_{\zeta_m} \nabla_{\zeta_m} f|
\]

Then \( \lim_{{m \to \infty}} |\zeta_m| = 1 \) and (b) implies that the last expression tends to 0. The same reasoning applies to the remaining cases.

Now (10) implies that at least one of the functions \( F, G \) must be identically 0 on \( rB_n \). Hence

\[
\lim_{{m \to \infty}} \int_{{rB_n}} |f \circ \varphi_{a_m} - f(a_m)||g \circ \varphi_{a_m} - g(a_m)| \, dv = \int_{{rB_n}} |FG| \, dv = 0
\]

(c) \( \iff \) (d) A change-of-variables yields

\[
\int_{{E_n(a,r)}} |f - f(a)||g - g(a)| \, dv
\]
\[
= \int_{{rB_n}} |f \circ \varphi_a - f(a)||g \circ \varphi_a - g(a)| \frac{(1 - |a|^2)^{n+1}}{|1 - \langle z, a \rangle|^2(n+1)} \, dv.
\]
Because
\[ |E_n(a,r)| = \left( \frac{1 - |a|^2}{1 - r^2|a|^2} \right)^{n+1} \]
the desired equivalence follows from the inequalities
\[ \frac{1}{r^{2n}(1 + r^2)^{2(n+1)}} \leq \frac{1}{|E_n(a,r)|} \frac{1}{(1 - \langle z, a \rangle^2)^{2(n+1)}} \leq \frac{1}{r^{2n}(1 - r^2)^{n+1}}, \quad z \in rB_n. \]

(c) \implies (e) Let \( \{a_m\} \) be a sequence such that \( \lim_{m \to \infty} |a_m| = 1 \) and \( F, G \) be as in the proof of the implication (b) \implies (c). Then (c) implies that
\[ \int_{rB_n} |F||G|dv = 0. \]
This in turn implies that either \( F \) or \( G \) is identically zero on \( rB_n \), and hence on \( B_n \).

(c) \implies (a) It is enough to proceed analogously to the proof of Theorem 2 of [14]. \qed

Remark. Notice that the functions \( f, g \) defined in assertion (ii) of Theorem B cannot be Bloch functions. Hence in view of Theorem C, when \( n < 12 \) our theorem holds for Bloch functions instead of bounded holomorphic functions.

REFERENCES


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