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DUNFORD-PETTIS COMPOSITION OPERATORS ON H^1 IN SEVERAL VARIABLES

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ABSTRACT. A bounded composition operator C_ϕ on $H^1(B)$, where B is the unit ball in $\mathbb{C}^n$, is Dunford-Pettis if and only if the radial limit function ϕ^* of ϕ takes values on the unit sphere S only on a set of surface measure zero. A similar theorem holds on bounded strongly pseudoconvex domains with smooth boundary.

A bounded operator from one Banach space to another is said to be Dunford-Pettis or completely continuous if it takes weak Cauchy sequences to norm convergent sequences. This is equivalent to the requirement that the operator take weakly null sequences to norm null sequences. In [1] Cima and Matheson showed that a composition operator on H^1 of the unit disk is Dunford-Pettis if and only if the inducing map $\phi\colon \mathbb{D} \to \mathbb{D}$ has radial limits of modulus one only on a set of measure zero. It is the purpose of this note to prove a similar theorem for composition operators on H^1 of the unit ball B in $\mathbb{C}^n$. A version for H^1 on strongly pseudoconvex domains with smooth boundary will also be presented. The restricton to the Hardy spaces H^1 is natural, since the Dunford-Pettis operators on reflexive Banach spaces coincide with the compact operators.

If $\phi\colon B \to B$ is a holomorphic map and if f is holomorphic on B, then the composition $C_\phi f = f \circ \phi$ is also holomorphic. However, in contrast to the one variable case, C_ϕ may not induce a bounded operator on the Hardy space $H^1(B)$, so it will be necessary to assume this here. A Carleson measure criterion for this was noticed by MacCluer in [6] (cf. also [2, Theorem 3.35]).

For a function g defined on B, g^* will denote the limit function

$$g^*(z) = \lim_{r\to 1} g(rz)$$

on the closed unit ball $\overline{B}$, at least where this radial limit exists. Normalized surface measure on S will be denoted by σ. By standard results (cf. [5]) the radial limit functions ϕ^*, f^*, and $(f\circ\phi)^*$ exist σ-almost everywhere on S when $\phi\colon B \to B$ is a holomorphic map and $f \in H^1(B)$.

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Theorem 1. *Let $\phi\colon B \to B$ be a holomorphic map which induces a bounded composition operator C_ϕ on $H^1(B)$. Then C_ϕ is Dunford-Pettis if and only if the set*

$$E = \{\, \zeta \in S \mid \phi^*(\zeta) \in S \,\}$$

has σ-measure zero.

Although the structure of the proof is the same as in the one-variable case, there are several technical difficulties to overcome. On the unit disk, a special role was played by powers of the identity function. In the ball the identity function will be replaced by an inner function u. Constructions and properties of inner functions on B can be found in [5] or [7]. In particular, u has a continuous extension to no point of the unit sphere S. This raises the question as to the relationship between $u^* \circ \phi^*$ and $(u \circ \phi)^*$. The following two lemmas, which can be found in [6] as Corollary 1.4 and Lemma 1.6, provide the answer. The boundedness of C_ϕ, expressed as a Carleson condition, plays a crucial role in the proof of Lemma A, which in turn is used to prove Lemma B.

Lemma A. *If C_ϕ is bounded on $H^1(B)$, then ϕ^* cannot carry a set of positive σ-measure on S onto a set of zero σ-measure on S.*

Lemma B. *If C_ϕ is bounded on $H^1(B)$ and $f \in H^1(B)$, then $(f \circ \phi)^* = f^* \circ \phi^*$ σ-almost everywhere on S.*

Proof of Theorem 1. First assume that C_ϕ is Dunford-Pettis. Let u be an inner function on B with $u(0) = 0$. Then (u^n) is an orthonormal sequence, hence weakly null, in $L^2(d\sigma)$. Thus

$$\lim_{n\to\infty} \int_S u^n \overline{g}\, d\sigma = 0$$

for all $g \in L^2(d\sigma)$. Since $L^\infty(d\sigma) \subset L^2(\sigma)$, it follows that (u^n) is a weakly null sequence in $L^1(d\sigma)$, and so, by the Hahn-Banach theorem, in $H^1(B)$. Thus $(u^n \circ \phi)$ converges to zero in norm, and consequently the set

$$F = \{\, \zeta \in S \mid |(u \circ \phi)^*(\zeta)| = 1 \,\}$$

has σ-measure zero. Because of Lemma B the set

$$G = \{\, \zeta \in S \mid |u^* \circ \phi^*| = 1 \,\}$$

has the same measure as F. Clearly $G \subset E$. On the other hand, $E \setminus G$ consists of those $\zeta \in E$ such that $u^*(\phi^*(\zeta))$ either does not exist or has modulus less than one. Hence the set $E \setminus G$ is mapped by ϕ^* into the set where u^* either does not exist or has modulus less than one. Since u is an inner function, the latter set has σ-measure zero, and thus Lemma A shows that $E \setminus G$ has σ-measure zero. Therefore $\sigma(E) = 0$.

Conversely, suppose $\sigma(E) = 0$. Let (f_n) be a weakly null sequence in $H^1(B)$. In particular, $\lim_{n\to\infty} f_n(z) = 0$ for each $z \in B$. Since C_ϕ is bounded, $(f_n \circ \phi)$ is also a weakly null sequence. Hence $\lim_{n\to 0} f_n^* \circ \phi^*(\zeta) = 0$ for σ-almost every $\zeta \in S$. By Lemma B, $f_n^* \circ \phi^* = (f_n \circ \phi)^*$ σ-almost everywhere on S. Thus $(f_n \circ \phi)^*$ converges to zero σ-almost everywhere. Now a theorem of Dunford and Pettis [3, p. 295] shows that $\|f_n \circ \phi\|_1 \to 0$. This completes the proof. $\square$

With almost no extra effort the theorem can be seen to hold in $H^1(\Omega)$ when Ω is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary. Indeed, a Carleson measure characterization for boundedness follows from a theorem of Hörmander [4]. The analogs of Lemmas A and B follow from this by essentially the same argument as on the unit sphere. A construction of an inner function u in Ω can be found in [5]. Although the sequence u^n may no longer be orthogonal with respect to surface measure σ, the argument given above can be salvaged by a simple device. Let a be a point in Ω and let $\gamma = u(a)$. Then the inner function $v = \frac{u-\gamma}{1-\overline{\gamma}u}$ vanishes at a. Let $P_a(z)$ denote the Poisson kernel for the point a in Ω. Since P_a reproduces holomorphic functions, the sequence v^n is orthonormal in $L^2(P_a d\sigma)$, and so is weakly null in this Hilbert space. On the other hand, there are positive constants c and C such that $c \le P_a(z) \le C$ for all $z \in \partial\Omega$. Consequently $L^2(d\sigma)$ and $L^2(P_a d\sigma)$ are isomorphic as Banach spaces, and thus v^n is weakly null in $L^2(d\sigma)$. The rest of the argument then proceeds as above, proving the following theorem.

Theorem 2. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary, and let σ denote the usual surface measure on $\partial\Omega$. Let $\phi\colon \Omega \to \Omega$ be a holomorphic map which induces a bounded composition operator C_ϕ on $H^1(\Omega)$. Then C_ϕ is Dunford-Pettis if and only if the set*

$$E = \{\, \zeta \in \partial\Omega \mid \phi^*(\zeta) \in \partial\Omega \,\}$$

has σ-measure zero.

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