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A NOTE ON THE REDUCIBILITY OF AUTOMORPHISMS OF THE KLEIN CURVE AND THE η-INVARIANT OF MAPPING TORI

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ABSTRACT. We give a characterization for the reducibility of automorphisms of the genus 3 Klein curve in terms of the η-invariant of finite order mapping tori.

1. INTRODUCTION

The automorphism group of the Klein curve S of genus 3, defined by the equation

$$x^3y + y^3z + z^3x = 0,$$

is known to be isomorphic to $G = PSL_2(F_7)$ [3]. This is a finite group of order 168 and has the presentation

$$\langle s, t \mid s^2 = t^3 = (st)^7 = [s,t]^4 = 1\rangle,$$

where $[s,t]$ denotes the commutator of s and t. Then G can be embedded in the mapping class group $\mathcal{M}_3$ of genus 3. This is a Hurwitz group acting on an oriented closed surface of genus 3 and hereafter we shall consider S as a smooth 2-manifold.

On the other hand, for an element φ of G we can construct the mapping torus M_φ corresponding to φ. We now endow M_φ with the metric which is induced from the product of the standard metric on S^1 and the metric of S such that φ acts on S as an isometry. Then the reducibility of the automorphism φ (see section 3 for the definition) is characterized via a spectral invariant of M_φ.

Theorem 1. *An automorphism φ of the Klein curve is reducible if and only if the η-invariant of the signature operator on mapping torus M_φ vanishes.*

From our earlier work [8], the η-invariant mentioned above is determined by the action of automorphisms on the first integral homology group $H = H_1(S; \mathbf{Z})$. More precisely, it is described via Atiyah's canonical 2-cocycle [1]. This cocycle is equal to Meyer's signature cocycle [7] up to scalar multiple, so that to prove Theorem 1 we really compute the latter one for each conjugacy class of G. The proof of Theorem 1 is given in section 3.

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TABLE 1. Conjugacy classes of $PSL_2(F_7)$

Class g	1	2	3	4	7_1	7_2
$\|Cl(g)\|$	1	21	56	42	24	24

2. $PSL_2(F_7)$, SIGNATURE COCYCLE AND η-INVARIANT

Let $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $t = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ be the generators of $PSL_2(F_7)$. It is easy to verify that $s^2 = t^3 = (st)^7 = [s,t]^4 = 1$. The conjugacy classes of G are well-known (Table 1, see also [4]). In Table 1, each conjugacy class is denoted by the order of its elements. Here the subscripts mean the different classes.

Now we review the definition of Meyer's signature cocycle briefly. See [7] for more detail. For $A, B \in \mathrm{Sp}(2h; \mathbf{Z})$, let $V_{A,B} = \{(u,v) \in H \oplus H \mid (A^{-1} - E)u + (B - E)v = 0\}$, where E is the identity matrix. Then we define a pairing map $\langle\ ,\ \rangle : V_{A,B} \times V_{A,B} \to \mathbf{Z}$ by $\langle (u_1, v_1), (u_2, v_2) \rangle = (u_1 + v_1) \cdot (E - B)v_2$, where $\cdot$ is the intersection number. It is easy to see that $\langle\ ,\ \rangle$ is a symmetric bilinear form on $V_{A,B}$. We define $\tau_h(A,B)$ to be the signature of $(V_{A,B}, \langle\ ,\ \rangle)$. This is determined for conjugacy classes.

We choose a symplectic basis α_i, β_i $(i = 1,2,3)$ of S as usual. Let $\rho : \mathcal{M}_3 \to \mathrm{Sp}(6; \mathbf{Z})$ be the epimorphism induced by the action of $\mathcal{M}_3$ on H. Then the images of each generator of $PSL_2(F_7)$ are given by the following matrices [6]:

$$\rho(s) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Moreover to compute the η-invariant we need matrices $\rho([s,t])$ and $\rho(st)$. Using the above Sp-representations, from the definition we can directly calculate Meyer's cocycle τ_3 at level 3.

As for the definition of the η-invariant of the signature operator, see the original paper [2]. To be brief, it measures the extent to which the Hirzebruch signature formula fails for a non-closed $4k$-dimensional Riemannian manifold whose metric is a product near its boundary.

Let M_{φ^n} $(1 \le n \le m)$ be a mapping torus determined by $\varphi \in PSL_2(F_7) \subset \mathcal{M}_3$, where m is the order of φ. Namely it is the identification space $S \times [0,1]/(p,0) \sim (\varphi^n(p), 1)$. Then the η-invariant of M_{φ^n} is given by

$$\eta(M_{\varphi^n}) = -\sum_{k=1}^{n-1} \tau_3(A, A^k) + \frac{n}{m} \sum_{k=1}^{m-1} \tau_3(A, A^k),$$

where $A = \rho(\varphi)$ (see [8]). From this formula we can compute the η-invariant of mapping torus corresponding to each element of G.

Proposition 2. *For each element of $PSL_2(F_7)$, explicit lists of the η-invariant of mapping tori are given in Table 2.*

TABLE 2. Signature cocycles and η-invariants for $PSL_2(F_7)$

Order m	$\tau_3(A, A^k)$ $(1 \le k \le m-1)$	$\eta(M_{\varphi^n})$ $(1 \le n \le m)$
1	0	0
2	0	0
3	0	0
4	0	0
7_1	-2 $(k = 1, 5)$	-2 $(n = 1, 2, 4)$
	-6 $(k = 2, 4)$	2 $(n = 3, 5, 6)$
	2 $(k = 3)$	0 $(n = 7)$
	0 $(k = 6)$	

Remark 3. For the conjugacy classes of order 7, we present the values of the signature cocycle and the η-invariant only for the class 7_1, since a representative of 7_2 is given by the inverse of 7_1. Then each value of 7_1 and 7_2 have the opposite sign.

3. Proof of Theorem 1

An essential 1-submanifold of S is a disjoint union of simple closed curves in S each component of which does not bound a 2-disk in S, and no two components of which are homotopic. An automorphism of S is reducible if it has a representative which leaves some essential 1-submanifold of S invariant.

Proof. For a surface of odd genus σ, a finite cyclic action on it is reducible if and only if the order of a generator is less than $2\sigma + 1$ [5]. Therefore Theorem 1 immediately follows from Proposition 2. □

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