A NOTE ON THE REDUCIBILITY OF AUTOMORPHISMS OF THE KLEIN CURVE AND THE $\eta$-INVARIANT OF MAPPING TORI

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Abstract. We give a characterization for the reducibility of automorphisms of the genus 3 Klein curve in terms of the $\eta$-invariant of finite order mapping tori.

1. INTRODUCTION

The automorphism group of the Klein curve $S$ of genus 3, defined by the equation

$$x^3y + y^3z + z^3x = 0,$$

is known to be isomorphic to $G = PSL_2(F_7)$ [3]. This is a finite group of order 168 and has the presentation

$$\langle s, t \mid s^2 = t^3 = (st)^7 = [s, t]^4 = 1 \rangle,$$

where $[s, t]$ denotes the commutator of $s$ and $t$. Then $G$ can be embedded in the mapping class group $\mathcal{M}_3$ of genus 3. This is a Hurwitz group acting on an oriented closed surface of genus 3 and hereafter we shall consider $S$ as a smooth 2-manifold.

On the other hand, for an element $\varphi$ of $G$ we can construct the mapping torus $M_\varphi$ corresponding to $\varphi$. We now endow $M_\varphi$ with the metric which is induced from the product of the standard metric on $S^1$ and the metric of $S$ such that $\varphi$ acts on $S$ as an isometry. Then the reducibility of the automorphism $\varphi$ (see section 3 for the definition) is characterized via a spectral invariant of $M_\varphi$.

Theorem 1. An automorphism $\varphi$ of the Klein curve is reducible if and only if the $\eta$-invariant of the signature operator on mapping torus $M_\varphi$ vanishes.

From our earlier work [8], the $\eta$-invariant mentioned above is determined by the action of automorphisms on the first integral homology group $H = H_1(S; \mathbb{Z})$. More precisely, it is described via Atiyah’s canonical 2-cocycle [1]. This cocycle is equal to Meyer’s signature cocycle [7] up to scalar multiple, so that to prove Theorem 1 we really compute the latter one for each conjugacy class of $G$. The proof of Theorem 1 is given in section 3.
For each element of Proposition 2. mapping torus corresponding to each element of \( G \), \( \langle \langle \cdot, \cdot \rangle \rangle \) is a symmetric bilinear form on \( V_{A,B} \). We define \( \tau_{A,B}(\cdot, \cdot) \) to be the signature of \( (V_{A,B}, \langle \cdot, \cdot \rangle) \). This is determined for conjugacy classes.

We choose a symplectic basis \( \alpha_{i}, \beta_{i} \) \((i = 1, 2, 3)\) of \( S \) as usual. Let \( \rho : \mathcal{M}_{3} \to \text{Sp}(6; \mathbb{Z}) \) be the epimorphism induced by the action of \( \mathcal{M}_{3} \) on \( H \). Then the images of each generator of \( \text{PSL}_{2}(F_{7}) \) are given by the following matrices [6]:

\[
\rho(s) = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 0 & -1 \\
-1 & -1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad \rho(t) = \begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & -1 & 0 \\
-1 & -1 & 1 & 1 & 1 & 0
\end{pmatrix}.
\]

Moreover to compute the \( \eta \)-invariant we need matrices \( \rho([s, t]) \) and \( \rho(st) \). Using the above \( \text{Sp} \)-representations, from the definition we can directly calculate Meyer’s cocycle \( \tau_{3} \) at level 3.

As for the definition of the \( \eta \)-invariant of the signature operator, see the original paper [2]. To be brief, it measures the extent to which the Hirzebruch signature formula fails for a non-closed 4\(k\)-dimensional Riemannian manifold whose metric is a product near its boundary.

Let \( M_{\varphi^{n}} \) \((1 \leq n \leq m)\) be a mapping torus determined by \( \varphi \in \text{PSL}_{2}(F_{7}) \subset \mathcal{M}_{3} \), where \( m \) is the order of \( \varphi \). Namely it is the identification space \( S \times [0, 1]/(p, 0) \sim (\varphi^{n}(p), 1) \). Then the \( \eta \)-invariant of \( M_{\varphi^{n}} \) is given by

\[
\eta(M_{\varphi^{n}}) = - \sum_{k=1}^{n-1} \tau_{3}(A, A^{k}) + \frac{n}{m} \sum_{k=1}^{m-1} \tau_{3}(A, A^{k}),
\]

where \( A = \rho(\varphi) \) (see [8]). From this formula we can compute the \( \eta \)-invariant of mapping torus corresponding to each element of \( G \).

**Proposition 2.** For each element of \( \text{PSL}_{2}(F_{7}) \), explicit lists of the \( \eta \)-invariant of mapping tori are given in Table 2.
Table 2. Signature cocycles and $\eta$-invariants for $PSL_2(F_7)$

<table>
<thead>
<tr>
<th>Order $m$</th>
<th>$\tau_3(A, A^k)\ (1 \leq k \leq m-1)$</th>
<th>$\eta(M_{\varphi^n})\ (1 \leq n \leq m)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7_1</td>
<td>$-2\ (k = 1, 5)$</td>
<td>$-2\ (n = 1, 2, 4)$</td>
</tr>
<tr>
<td></td>
<td>$-6\ (k = 2, 4)$</td>
<td>$2\ (n = 3, 5, 6)$</td>
</tr>
<tr>
<td></td>
<td>$2\ (k = 3)$</td>
<td>$0\ (n = 7)$</td>
</tr>
<tr>
<td></td>
<td>$0\ (k = 6)$</td>
<td></td>
</tr>
</tbody>
</table>

Remark 3. For the conjugacy classes of order 7, we present the values of the signature cocycle and the $\eta$-invariant only for the class $7_1$, since a representative of $7_2$ is given by the inverse of $7_1$. Then each value of $7_1$ and $7_2$ have the opposite sign.

3. Proof of Theorem 1

An essential 1-submanifold of $S$ is a disjoint union of simple closed curves in $S$ each component of which does not bound a 2-disk in $S$, and no two components of which are homotopic. An automorphism of $S$ is reducible if it has a representative which leaves some essential 1-submanifold of $S$ invariant.

Proof. For a surface of odd genus $\sigma$, a finite cyclic action on it is reducible if and only if the order of a generator is less than $2\sigma + 1$ [5]. Therefore Theorem 1 immediately follows from Proposition 2.

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References


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