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RELATIVE TO ANY NONRECURSIVE SET

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ABSTRACT. There is a countable first order structure $\mathfrak{M}$ such that for any set of integers X, X is not recursive if and only if there is a presentation of $\mathfrak{M}$ which is recursive in X.

1. Introduction

It is a well-known truth that if a set $R \subseteq \omega$ is recursive relative to every nonrecursive subset of ω, then R is recursive (see [Kleene and Post, 1954]). Similarly, if R is recursive in every element of a co-meager set, then R is recursive; if R is recursive in every element of a set of positive measure, then R is recursive; and if R is recursive in every element of a nonempty $\mathbf{\Pi}^0_1$ set P, then R is recursive in the reals appearing in the definition of P. In fact, one can consider a wide variety of notions of forcing $\mathbb{P}$, and conclude that if R is recursive in every generic set (extending the condition p), then R is recursive in $\mathbb{P}$ (and p). Thus, in wide generality, there is no $R \subseteq \omega$ which constitutes information common to all nonrecursive reals or to all generic reals.

Even so, one need not abandon the possibility that there is a mathematical object which instantiates the property of being nonrecursive. The following question of S. Lempp suggests one such possibility.

Question 1.1 (Lempp)**.** Does there exist a countable first order structure $\mathfrak{M}$ such that, for every X, X is nonrecursive if and only if there is a presentation of $\mathfrak{M}$ which is recursive in X?

In Section 2, we answer Question 1.1 by showing that there is such an $\mathfrak{M}$. In Theorem 2.1, we construct an $\mathfrak{M}$ which has no recursive presentation and is uniformly-recursively presented relative to every nonrecursive real.

Independently and shortly before our obtaining Theorem 2.1, Wehner [Wehner, 1996] answered a question of J. Knight, and obtained an example of a similar type. In the following, say that a family of sets $\mathcal{F}$ is uniformly-recursively enumerable if there is a recursively enumerable set of pairs W such that

$$(\forall F)\,[F \in \mathcal{F} \iff (\exists i)[F = \{n : (i,n) \in W\}]]\,.$$

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Theorem 1.2 (Wehner). *There is a family $\mathcal{F}$ of subsets of ω which is not uniformly-recursively enumerable, such that for all X, X is nonrecursive if and only if there is a uniformly-recursive-in-X enumeration of $\mathcal{F}$.*

Shortly after our obtaining Theorem 2.1, A. McAllister showed that Theorem 2.1 can be derived from Wehner's theorem. The Σ_1-types of the elements of McAllister's model correspond to the sets in Wehner's family. Conversely, it is not difficult to deduce Wehner's theorem by analyzing the Σ_1-types realized in our model.

2. The model

Here, we present the central result of this paper.

Theorem 2.1. *There is a countable model $\mathfrak{M}$ and there is a recursive function $M : X \mapsto M(X)$, which maps subsets of ω to presentations of countable first order structures with the following properties.*

1. *For all X, if X is not recursive, then $M(X)$ is a presentation of $\mathfrak{M}$.*
2. *There is no recursive presentation of $\mathfrak{M}$.*

In this section, we will specify the language of $\mathfrak{M}$ and describe the general features of $\mathfrak{M}$, construct $\mathfrak{M}$, and then prove that $\mathfrak{M}$ has the required properties.

2.1. **Describing $\mathfrak{M}$.** The language of $\mathfrak{M}$ consists of a single constant symbol 0, a unary function symbol s, and three binary relation symbols R, $<_T$ and $<_L$. We will write $R(x, y)$, $x <_T y$, and $x <_L y$. Let $\mathcal{L}$ denote this language.

In $\mathfrak{M}$, we use 0 and s to represent the natural numbers with successor. That is, we let $s^i(0)^{\mathfrak{M}}$ denote the ith iterate of $s^{\mathfrak{M}}$ applied to $0^{\mathfrak{M}}$, and set $s^{\mathfrak{M}}$ to be the identity on all points not in $\{s^i(0) : i \in \omega\}^{\mathfrak{M}}$. We use $<_T^{\mathfrak{M}}$ to represent a countable family of nonempty binary trees, and informally let $T_i^{\mathfrak{M}}$ denote the ith tree $\{x : x >_T s^i(0)\}^{\mathfrak{M}}$ with the ordering given by $<_T^{\mathfrak{M}}$. We let $<_L^{\mathfrak{M}}$ uniformly order the levels of the $T_i^{\mathfrak{M}}$'s.

Finally, we use $R^{\mathfrak{M}}$ to pick out paths within $T^{\mathfrak{M}}$, as follows . Let $R^{-1}(T_i)^{\mathfrak{M}}$ denote $\{p : R(p, s^i(0))\}^{\mathfrak{M}}$. For each p in $R^{-1}(T_i)^{\mathfrak{M}}$, we let $\zeta(p)^{\mathfrak{M}}$ denote the set $\{x : R(p, x)\}^{\mathfrak{M}}$. We will ensure that $\zeta(p)^{\mathfrak{M}}$ is a maximal finite path in $T_i^{\mathfrak{M}}$. Further, we will ensure that there are maximal finite paths in $T_i^{\mathfrak{M}}$, and ensure that for each maximal finite path ζ in $T_i^{\mathfrak{M}}$ there are infinitely many p such that $\zeta = \zeta(p)^{\mathfrak{M}}$ and so $R^{-1}(T_i)^{\mathfrak{M}}$ is infinite.

In Figure 1, we show a finite subset of the diagram of $\mathfrak{M}$. In this picture, $\zeta(p)^{\mathfrak{M}}$ is equal to the collection of points below x in the sense of $<_T^{\mathfrak{M}}$.

Let $\langle \mathfrak{R}_i, i \in \omega \rangle$ be a recursive enumeration of the recursively-enumerably presented $\mathcal{L}$-structures.

We will ensure that either $R^{-1}(T_i)^{\mathfrak{R}_i} = \emptyset$, or $\langle T_i, <_L \restriction T_i \rangle^{\mathfrak{R}_i}$ is not isomorphic to $\langle T_i, <_L \restriction T_i \rangle^{\mathfrak{M}}$, or there is a p in $R^{-1}(T_i)^{\mathfrak{R}_i}$ such that $\zeta(p)^{\mathfrak{R}_i}$ is not maximal, or there is a p in $R^{-1}(T_i)^{\mathfrak{R}_i}$ such that $\zeta(p)^{\mathfrak{R}_i}$ is infinite. Since none of these disjuncts apply to $\mathfrak{M}$, we will thus ensure that $\mathfrak{M}$ has no recursive presentation.

Although we have been discussing $\mathfrak{M}$, we will only define $\mathfrak{M}$ at the end of our proof. During the proof, we will define a Σ_1^0 function M, mapping subsets X of ω to relatively recursively enumerable presentations of models $M(X)$, interpreting the language $\mathcal{L}$. Since a recursively enumerable presentation of an infinite structure can be uniformly converted to a recursive presentation of the same structure, we can conclude that there is a uniformly recursive function mapping X to a recursive presentation of $M(X)$.

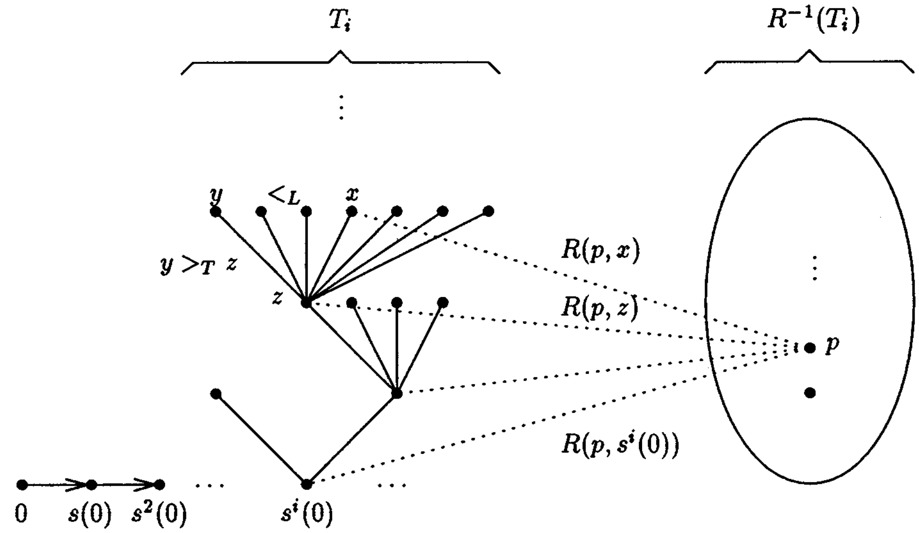

FIGURE 1. Part of the diagram of $\mathfrak{M}$

Our intention is to ensure that if X is not recursive, then for all i and all p in $R^{-1}(s^i(0))^{M(X)}$, $\zeta(p)^{M(X)}$ is a maximal finite path in $T_i^{M(X)}$. To diagonalize against all of the recursive isomorphism types, we may make $T_i^{M(X)}$ infinite, and so $T_i^{M(X)}$ may have an infinite path as well. For arbitrary X, we cannot avoid the possibility that $\zeta(p)^{M(X)}$ may be infinite, so we ensure that if $\zeta(p)^{M(X)}$ is infinite, then X is recursive. To this end, we ensure that $T_i^{\mathfrak{M}}$ is recursively isomorphic to a recursive tree with at most one infinite path, which would necessarily be recursive. Then, we make the function M sufficiently injective so that for each $p \in \omega$, there can be at most one X such that $\zeta(p)^{M(X)}$ is an infinite path in $T_i^{M(X)}$. Then, this X is recursive.

2.2. **Constructing $\mathfrak{M}$.** We divide our construction of $M(X)$ into two parts, the part that is recursively enumerable without reference to X and and the part that is recursively enumerable with essential references to X.

2.2.1. *The part of $M(X)$ which does not refer to X.* We begin by recursively enumerating the diagram of a model M, which will interpret 0, s, $<_T$, and $<_L$. Only the interpretation of R within $M(X)$ will depend on X.

We let 0 and s be a recursive presentation of the natural numbers with 0 and the successor function such that the domain of s is a recursive coinfinite set.

We enumerate $<_T$ above $s^i(0)$ while monitoring the ith recursive model $\mathfrak{R}_i$ of the same signature as $\mathfrak{M}$. In the following, we use the suffix "$[s]$" to denote a finite approximation given by what has been enumerated by stage s. For example, $\mathfrak{R}_i[s]$ is the finite subset of the diagram of $\mathfrak{R}_i$ enumerated by stage s, and $T_i^M[s]$ is the finite subtree of T_i^M which we have enumerated by stage s. Now, we enumerate the remaining part of M as follows.

1. (a) While there is a no x within $\mathfrak{R}_i[s]$ such that $\mathfrak{R}_i[s] \models R(x, s^i(0))$, we do not enumerate any successors of $s^i(0)$.

(b) If there is an element x within $\mathfrak{R}_i[s]$ such that $\mathfrak{R}_i[s] \models R(x, s^i(0))$, then let p_0 be the first such in the enumeration of $\mathfrak{R}_i$.

2. (a) While either the reduced structure $\langle T_i, <_L\restriction T_i\rangle^{\mathfrak{R}_i}[s]$ is not isomorphic to $\langle T_i, <_L\restriction T_i\rangle^M[s]$ or the set $\zeta(p_0)^{\mathfrak{R}_i}[s]$ is not a maximal chain in $T_i^{\mathfrak{R}_i}[s]$, we do not enumerate any further elements into T_i^M.

   (b) Otherwise, we let π denote the unique isomorphism mapping $\langle T_i, <_L\restriction T_i\rangle^M[s]$ to $\langle T_i, <_L\restriction T_i\rangle^{\mathfrak{R}_i}[s]$. We let $z_i^M[s]$ be the maximal element of $T_i^M[s]$ such that $\pi(z_i^M[s])$ is the maximal element of $\zeta(p_0)^{\mathfrak{R}_i}[s]$. We enumerate 2^s many immediate $<_T$-successors of $z_i^M[s]$ into T_i and extend $<_L^M$ so that it linearly orders these new elements of T_i^M. At the next stage, we return to 2(a).

The effect of our enumeration is that either the structures $\langle T_i, <_L\restriction T_i\rangle^{\mathfrak{R}_i}$ and $\langle T_i, <_L\restriction T_i\rangle^M$ are not isomorphic, or they are isomorphic by a recursive isomorphism $\pi : \langle T_i, <_L\restriction T_i\rangle^M \to \langle T_i, <_L\restriction T_i\rangle^{\mathfrak{R}_i}$, T_i^M is an infinite finitely-branching tree, and $\pi^{-1}\left(\zeta(p_0^{\mathfrak{R}_i})\right)$ is isomorphic to its unique branch.

Note, we may safely assume that our enumeration of M proceeds so that the universe of M is a co-infinite recursive subset of ω.

2.2.2. *The part of $M(X)$ which refers to X.* We begin by fixing some notation. Let $\preceq$ be the lexicographic ordering of $2^{<\omega}$, and for σ in 2^s let $|\sigma|_\preceq$ be the number of elements of 2^s which are less than or equal to σ under $\preceq$.

We define $R^{M(X)}$ by recursively enumerating a collection of pairs $\langle\sigma, \langle p, z\rangle\rangle$ into W_R, where σ is a finite binary sequence, meant to be an initial segment of X, and $\langle p, z\rangle$ is a pair, meant to belong to $R^{M(X)}$. Thus, $\langle p, z\rangle$ is an element of $R^{M(X)}$ if and only if there is a finite initial segment σ of X such that $\langle\sigma, \langle p, z\rangle\rangle$ is an element of W_R. During stage s, we proceed as follows.

1. (a) While there is a no x within $\mathfrak{R}_i[s]$ such that $\mathfrak{R}_i[s] \models R(x, s^i(0))$, we execute only Step 3, below.

   (b) If there is an element x within $\mathfrak{R}_i[s]$ such that $\mathfrak{R}_i[s] \models R(x, s^i(0))$, then let p_0 be the first such in the enumeration of $\mathfrak{R}_i$, and proceed to Step 2.

2. (a) While either the reduced structure $\langle T_i, <_L\restriction T_i\rangle^{\mathfrak{R}_i}[s]$ is not isomorphic to $\langle T_i, <_L\restriction T_i\rangle^M[s]$ or $\zeta(p_0)^{\mathfrak{R}_i}[s]$ is not a maximal chain in $T_i^{\mathfrak{R}_i}[s]$, we execute only Step 3.

   (b) Otherwise, we let π denote the unique isomorphism mapping $\langle T_i, <_L\restriction T_i\rangle^M[s]$ to $\langle T_i, <_L\restriction T_i\rangle^{\mathfrak{R}_i}[s]$. We let $z_i^M[s]$ be the maximal element of $T_i^M[s]$ such that $\pi(z_i^M[s])$ is the maximal element of $\zeta(p_0)^{\mathfrak{R}_i}[s]$. For each σ in 2^s, let k_σ be the $|\sigma|_\preceq$th immediate successor of $z_i^M[s]$ (added during stage s, as above); for each p such that $R^\sigma(p, z_i^M[s])$ holds during stage s, enumerate the pair $\langle\sigma, \langle p, k_\sigma\rangle\rangle$ into W_R.

3. For each maximal chain ζ in $T_i^M[s]$ and for each σ in 2^s, we choose a number p^* greater than any number which has previously been mentioned in the construction, and for each point z in ζ enumerate $\langle\sigma, \langle p, z\rangle\rangle$ into W_R.

Definition 2.2. A countable extension M^* of M to an $\mathfrak{L}$-structure is *appropriate* if the following conditions hold.

1. 0, s, $<_T$ and $<_L$ have identical interpretations in M and M^*.
2. For each p and z in M^*, if $M^* \models R(p, z)$, then $p \in M^* \setminus M$.
3. For each p in $M^* \setminus M$, $\{z : M^* \models R(p, z)\}$ is a maximal path in one of the T_i^M.

By virtue of our construction, for every X, $M(X)$ is an appropriate extension of $\mathfrak{M}$.

2.3. **Verifying the theorem.**

Lemma 2.3. *Suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two appropriate extensions of M such that for k equal to 1 or 2,*

1. *for every i and every maximal finite path ζ in T_i, there are infinitely many p in $\mathfrak{M}_k$ such that $\zeta(p)^{\mathfrak{M}_k} = \zeta$;*
2. *and there is no p in $\mathfrak{M}_k$ such that $\zeta(p)^{\mathfrak{M}_k}$ is infinite.*

Then, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic.

Proof. Each of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ consists of M and infinitely many additional elements which it relates to the elements in M by its interpretation of R. Any bijection between the new elements of $\mathfrak{M}_1$ and the new elements of $\mathfrak{M}_2$ which preserves R will then be an isomorphism.

Each new element p of $\mathfrak{M}_1$ is uniquely associated with a maximal finite path $\zeta = \{z \in T_i : R(p,z)\}$ contained in some T_i. Further, for each such ζ there are infinitely many p such that $\zeta = \{z \in T_i : R^{\mathfrak{M}_1}(p,z)\}$. The same conditions hold for $\mathfrak{M}_2$.

Let π be any bijection between $\mathfrak{M}_1 \setminus M$ and $\mathfrak{M}_2 \setminus M$ so that for each p in $\mathfrak{M}_1 \setminus M$,

$$\zeta(p)^{\mathfrak{M}_1} = \zeta(\pi(p))^{\mathfrak{M}_2}.$$

Since the only pairs in R are those which associate new elements outside of M to paths in the T_i, for all x and y, $R^{\mathfrak{M}_1}(x,y)$ if and only if $R^{\mathfrak{M}_2}(\pi(x),\pi(y))$. Thus, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic. □

Definition 2.4. Let $\mathfrak{M}$ be the isomorphism type of the extension of M to a model of the type mentioned in Lemma 2.3

Corollary 2.5. *If for each p in $M(X)\setminus M$, $\zeta(p)$ is finite, then $M(X)$ is isomorphic to $\mathfrak{M}$.*

Proof. Apply Lemma 2.3. □

Lemma 2.6. *There is no recursive presentation of $\mathfrak{M}$.*

Proof. Suppose that $\mathfrak{M}$ has a recursive presentation, and fix i so that $\mathfrak{R}_i$ is isomorphic to $\mathfrak{M}$. Since $R^{-1}(s^i(0))^{\mathfrak{M}}$ is not empty, we may let p_0 be the first element in the enumeration of $R^{-1}(s^i(0))^{\mathfrak{R}_i}$.

In Section 2.2.1, we defined T_i in the following way. We waited for an element to appear in $R^{-1}(\{s^i(0)\})^{\mathfrak{R}_i}$. Given that p_0 is the first such element, we then ensured that either T_i^M is finite and $\zeta(p_0)^{\mathfrak{R}_i}$ is not a maximal path in $T_i^{\mathfrak{R}_i}$, or $\zeta(p_0)^{\mathfrak{R}_i}$ is the unique infinite path in $T_i^{\mathfrak{R}_i}$.

For every element p of $R^{-1}(T_i)^{\mathfrak{M}}$, $\zeta(p)^{\mathfrak{M}}$ is a finite maximal chain in $T_i^{\mathfrak{M}}$. But then, p cannot be isomorphic to p_0, since $\zeta(p_0)^{\mathfrak{R}_i}$ is either not maximal or is infinite. □

Lemma 2.7. *For each X and $i \in \omega$, one of the following conditions holds.*

1. *For every $p \in R^{-1}(T_i)^{M(X)}$, $\zeta(p)^{M(X)}$ is a finite maximal path in T_i^M.*
2. *X is recursive.*

Proof. As mentioned above, we defined T_i^M in the following way. We waited for an element to appear in $R^{-1}(\{s^i(0)\})^{\mathfrak{R}_i}$. For the first such element, p_0, to appear in the enumeration of $\mathfrak{R}_i$, we ensured that either T_i^M is finite and either $T_i^{\mathfrak{R}_i}$ is not isomorphic to T_i^M or $\zeta(p_0)^{\mathfrak{R}_i}$ is not a maximal path in $T_i^{\mathfrak{R}_i}$, or T_i^M is infinite and $\zeta(p_0)^{\mathfrak{R}_i}$ is the unique infinite path in $T_i^{\mathfrak{R}_i}$.

Item (1) is a clear consequence of the first case. So we may assume that the second case occurs.

In Section 2.2.2, we defined R as follows. If during stage s, $\zeta(p_0)^{\mathfrak{R}_i}[s]$ increases in length to become an maximal chain in $T_i^{\mathfrak{R}_i}[s]$ and π is the isomorphism from $T_i^M[s]$ to $T_i^{\mathfrak{R}_i}[s]$, then for each σ in 2^s and each p such that $\pi(\zeta(p)^{M(\sigma)})[s]$ is equal to $\zeta(p_0)^{\mathfrak{R}_i}[s]$, we ensure that for each X extending σ, only one of the immediate predecessors of the maximum of $\zeta(p)^{M(\sigma)}[s]$ is related to p by $R^{M(X)}$, namely k_σ. Note that we choose k_σ as an injective function of σ.

Now, suppose that X and p are given so that $\pi(\zeta(p)^{M(X)}) = \zeta(p_0)^{\mathfrak{R}_i}$. We can compute X as follows. Given a number x, wait for a stage s in the construction such that $s > x$ and $\zeta(p_0)^{\mathfrak{R}_i}[s]$ is larger than ever before. During stage s, we enumerated R so that for each extension ζ of $\pi^{-1}(\zeta(p_0)^{\mathfrak{R}_i})[s]$ in T_i^M, there is at most one binary sequence σ of length s such that $R(\sigma, \langle p, k\rangle)$ holds for all of the elements k appearing in ζ. But then, there is only one such ζ such that $\pi(\zeta)$ is an initial segment of $\zeta(p_0)^{\mathfrak{R}_i}[s]$, and only one such σ for which $R(\sigma, \langle p, k\rangle)$ holds for all of the elements k appearing in that ζ. Since $\zeta(p)^{M(X)}$ is infinite, this unique σ must be an initial segment of X, and $X(x)$ is equal to $\sigma(x)$. $\square$

We can now prove Theorem 2.1. If X is not recursive, then Lemma 2.7 states that there is no p such that $\zeta(p)^{M(X)}$ is infinite. But then Corollary 2.5 states that $M(X)$ is isomorphic to $\mathfrak{M}$. Thus, we can conclude the first claim of Theorem 2.1, that we have a uniformly recursive method to present $\mathfrak{M}$ relative to any nonrecursive set X. By Lemma 2.6, there is no recursive presentation of $\mathfrak{M}$, and we can conclude the remaining claim of Theorem 2.1.

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