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ORDINARY DIFFERENTIAL INEQUALITIES AND QUASIMONOTONICITY IN ORDERED TOPOLOGICAL VECTOR SPACES

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Abstract. A well known comparison theorem on ordinary differential inequalities with quasimonotone right-hand side $f(t,x)$ was carried over by Volkmann (1972) to (pre)ordered topological vector spaces. We prove that the quasimonotonicity of f is a *necessary* condition here if f is continuous. Then it is shown that quasimonotonicity can be verified by considering only a *few* positive continuous linear functionals in the definition (for instance in ℓ_∞ by taking coordinate functionals).

1. Introduction

Quasimonotonicity has its origins in initial value problems

$$u(t_0) = x_0, \quad u'(t) = f(t, u(t)) \quad (t_0 \le t \le t_1) \tag{1}$$

and corresponding differential inequalities, when certain theorems were carried over from the scalar case to problems in $\mathbb{R}^n$. First, Müller (1927) [3] proved the existence of a solution to (1) between given lower and upper solutions, and Kamke [1] established extremal solutions. They referred to the componentwise ordering in $\mathbb{R}^n$, and they assumed that, roughly speaking, each component $f_i(t, x_1, ..., x_n)$ is monotone increasing in every x_j with $j \ne i$. Later on, Walter [12] called such functions quasimonotone increasing in x. In some applications the term cooperative is used.

Finally, Volkmann (1972) [8] gave the general definition of quasimonotonicity in topological vector spaces preordered by a cone, which makes use of the dual cone. He carried over a very useful comparison theorem on ordinary differential inequalities where the right-hand side $f(t,x)$ is quasimonotone increasing in x. Our purpose is to prove that, conversely, the quasimonotonicity of f is a *necessary* condition here in case f is continuous. Then we will show that quasimonotonicity can be verified by considering only some small *subset* of the dual cone.

A generalization of Müller's theorem to preordered Banach spaces is given in [7]. Note that (1) need not have any local solution if f is only assumed to be continuous and bounded. It should be pointed out that in some ordered Banach spaces the quasimonotonicity of f implies the existence of solutions to (1), in fact extremal solutions, without any Lipschitz or compactness condition. Some of these results

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are based on the fixed point theorem of Lemmert [2]. See the survey [10]. Finally, quasimonotonicity is also significant for systems of parabolic differential equations, cf. [6], [12], and for fixed points of discontinuous functions, cf. [4].

2. Notations

Let E be a Hausdorff topological vector space, and let $K \subset E$ be a cone (i.e. K is closed, convex, nonempty, and satisfies $\lambda x \in K$ for all $\lambda \geq 0,\ x \in K$). By the definition

$$x \leq y \iff y - x \in K \quad (x, y \in E),$$

a preordering (a reflexive transitive relation) on E is given; this preordering is an ordering (also antisymmetric) iff K is strict (i.e. $x, -x \in K \Longrightarrow x = 0_E$). Now E is said to be preordered or ordered, respectively. We write

$$x \ll y \iff y \gg x \iff y - x \in \operatorname{Int} K \quad (x, y \in E),$$

where $\operatorname{Int} K$ denotes the interior of K. The dual cone of K is

$$K^* = \{\varphi \in E_{\mathbb{R}}^* :\ \varphi(x) \geq 0 \text{ for all } x \in K\},$$

where $E_{\mathbb{R}}^*$ denotes the dual of $E_{\mathbb{R}}$ (E regarded as a real space).

A function $f : G \to E$ on $G \subset E$ is said to be *quasimonotone increasing* if for all $x, y \in G$ and all $\varphi \in K^*$ the implication

$$x \leq y,\ \varphi(x) = \varphi(y) \Longrightarrow \varphi(f(x)) \leq \varphi(f(y)) \tag{2}$$

holds. Finally, (P) will denote the following property, where now $f : D \to E$ has domain $D \subset \mathbb{R} \times E$.

(P) If $v, w : [t_0, t_1] \to E$ $(t_0 < t_1)$ are any differentiable functions such that $\operatorname{graph} v$, $\operatorname{graph} w \subset D$, $v(t_0) \ll w(t_0)$ and

$$v'(t) - f(t, v(t)) \ll w'(t) - f(t, w(t)) \quad (t_0 \leq t \leq t_1), \tag{3}$$

then $v(t) \ll w(t)$ for $t_0 \leq t \leq t_1$.

3. (P) implies quasimonotonicity

Roughly speaking, property (P) and quasimonotonicity are equivalent.

Theorem 1. *Let E be a Hausdorff topological vector space preordered by a cone K with $\operatorname{Int} K \neq \emptyset$. Suppose $D \subset \mathbb{R} \times E$ is such that for every $(t, x) \in D$ there exist $\varepsilon > 0$ and a neighborhood G of x satisfying $[t, t + \varepsilon) \times G \subset D$. Assume $f : D \to E$ is continuous. Then (P) holds if and only if for each $t \in \mathbb{R}$ the function $x \mapsto f(t, x)$ is quasimonotone increasing.*

Note that the quasimonotonicity of f in x always implies (P), by Volkmann [8], without any assumptions on D and f. Simon and Volkmann [5] prove the converse in case E is a Banach space. Here we give a proof for the general case, which is nevertheless much shorter.

Proof. Assume (P). Suppose $(t_0, x), (t_0, y) \in D$ and $\varphi \in K^*$ satisfy

$$x \leq y, \quad \varphi(x) = \varphi(y).$$

Fix any $p \in \text{Int } K$. For each $t_1 \geq t_0$, define $v, w : [t_0, t_1] \to E$ by

$$v(t) = x + (t - t_0)(f(t_0, x) - p),$$
$$w(t) = y + (t_1 - t_0)\, p + (t - t_0)(f(t_0, y) + p)$$

$(t_0 \leq t \leq t_1)$. Choosing $t_1 > t_0$ sufficiently close to t_0, we may assume graph v, graph $w \subset D$, and

$$f(t, v(t)) - f(t_0, x) + p \in \text{Int } K,$$
$$f(t_0, y) + p - f(t, w(t)) \in \text{Int } K$$

for $t_0 \leq t \leq t_1$. For the left sides may be regarded as continuous functions of (t_1, t), which assign p to (t_0, t_0). Thus we have

(4) $$v'(t) \ll f(t, v(t)), \quad w'(t) \gg f(t, w(t)) \quad (t_0 \leq t \leq t_1),$$

hence (3), and clearly $v(t_0) \ll w(t_0)$. By (P), this implies $v(t_1) \leq w(t_1)$. We conclude that $\varphi(v(t_1)) \leq \varphi(w(t_1))$, hence

$$\varphi(f(t_0, x) - p) \leq \varphi(f(t_0, y) + 2p),$$

and, letting $p \to 0_E$, $\varphi(f(t_0, x)) \leq \varphi(f(t_0, y))$. Thus the quasimonotonicity of f is verified. □

Remark 1. If we replace (3) by (4) in (P) then Theorem 1 still holds, by the preceding proof. Moreover, if the three $\ll$ signs in (P) are replaced by $\leq$ then this modified property $(\text{P}_{\leq})$ also implies the quasimonotonicity of f in x, under the assumptions of Theorem 1. Note that the converse is valid if, in addition, E is a normed space and f satisfies a local Lipschitz condition with respect to x; cf. [9].

4. Another characterization of quasimonotonicity

To verify quasimonotonicity, it suffices to show (2) only for a few $\varphi \in K^*$ if these φ, regarded as supporting functionals of K, provide sufficiently many supporting points.

Theorem 2. *Let E be a Hausdorff topological vector space preordered by a cone K with* Int $K \neq \emptyset$. *Let $S \subset K^*$ be such that the set*

(5) $$\{x \in K : \varphi(x) = 0 \text{ for some nontrivial } \varphi \in S\}$$

is dense in the boundary of K. Suppose $G \subset E$ is open, and $f : G \to E$ is continuous. If the implication (2) *holds for all $x, y \in G$ and all $\varphi \in S$, then f is quasimonotone increasing.*

This result was motivated by Walter [11], [12, Theorem 12 XII] where the ordered Banach space $E = \ell_\infty(A)$ is underlying; cf. Remark 2 and [8, Beispiel 3].

Proof. By Theorem 1, it suffices to verify (P) rewritten for autonomous f. Suppose $v, w : [t_0, t_1] \to G$ $(t_0 < t_1)$ are differentiable functions which satisfy $v(t_0) \ll w(t_0)$ and

(6) $$v'(t) - f(v(t)) \ll w'(t) - f(w(t)) \quad (t_0 \leq t \leq t_1).$$

Put

$$u(t) = w(t) - v(t) \quad (t_0 \leq t \leq t_1).$$

To prove $u(t) \in \text{Int } K$ for $t_0 \le t \le t_1$, assume the contrary. Since $u(t_0) \in \text{Int } K$, there exists $t_2 \in (t_0, t_1]$ such that

$$u(t) \in \text{Int } K \quad (t_0 \le t < t_2) \tag{7}$$

and such that $u(t_2)$ lies on the boundary of K.

Choosing some $t_3 \in [t_0, t_2)$ sufficiently close to t_2, we may write

$$f(w(t_2)) - f(v(t_2)) \ll \frac{u(t_2) - u(t_3)}{t_2 - t_3},$$

by (6). Since G is open and f is continuous, there is a neighborhood Δ of 0_E such that $w(t_2) + d \in G$ and

$$f(w(t_2) + d) - f(v(t_2)) \le \frac{u(t_2) + d - u(t_3)}{t_2 - t_3} \tag{8}$$

for all $d \in \Delta$. We now choose $d \in \Delta$ such that $u(t_2) + d$ is a member of the set (5). Consequently, $u(t_2) + d \in K$, and $\varphi(u(t_2) + d) = 0$ for some nontrivial $\varphi \in S$. Substituting $u(t_2) + d = w(t_2) + d - v(t_2)$, we conclude that

$$\varphi(f(v(t_2))) \le \varphi(f(w(t_2)) + d),$$

by the restricted hypothesis (2). Thus the value of φ at the left side of (8) is nonnegative, hence $0 \le \varphi(u(t_2) + d) - \varphi(u(t_3))$, and therefore $\varphi(u(t_3)) \le 0$. But this contradicts (7), and (P) is verified. □

Remark 2. Suppose E is the real Banach space ℓ_∞ ordered by its natural strict cone

$$K = \{x \in l_\infty : \, x_n \ge 0 \text{ for all } n \in \mathbb{N}\},$$

where $x = (x_n)_{n=1}^\infty$. Then $\text{Int } K \ne \emptyset$, and the set $S = \{\delta_n : n \in \mathbb{N}\}$ of all coordinate functionals $\delta_n : E \to \mathbb{R}$, $x \mapsto x_n$, satisfies the hypothesis of Theorem 2.

We mention some other ordered Banach spaces consisting of functions $A \to \mathbb{R}$, such that their natural cone has interior points and such that the corresponding set of all evaluation functionals $x \mapsto x(\alpha)$ $(\alpha \in A)$ may be taken in Theorem 2, similarly: $\ell_\infty(A)$ (A any set), $C^k(A)$ or $BV(A)$ ($A \subset \mathbb{R}$ any compact interval having interior points, $k \in \mathbb{N}_0$), or $BC(A)$ (A any topological space).

Remark 3. The following example shows that we cannot omit the continuity condition on f in Theorem 2. Put $E = \ell_\infty$ and K, S as in Remark 2. Define $f : E \to E$ by $f(x) = (f_n(x))_{n=1}^\infty$ where

$$f_n(x) = 1 \ \text{ if } x_n = 0 \text{ and } x \in K, \quad f_n(x) = -2 \ \text{ otherwise}$$

$(n \in \mathbb{N},\, x \in E)$. Then (2) holds for all $x, y \in E$ and all $\varphi \in S$. On the other hand, the functions $v, w : [0, 2] \to E$,

$$v(t) = 0_E, \quad w(t) = (1 + 1/n - t)_{n=1}^\infty \quad (0 \le t \le 2),$$

satisfy $v(0) \ll w(0)$ and $v'(t) \ll f(v(t))$ as well as $w'(t) \gg f(w(t))$ for $0 \le t \le 2$. But even $v(t) \le w(t)$ is false if $1 < t \le 2$. Thus f cannot be quasimonotone increasing. This can also be verified by taking $x = 0_E$, $y = (1/n)_{n=1}^\infty$ and φ as Banach limit in (2).

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