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ENUMERATIONS, COUNTABLE STRUCTURES AND TURING DEGREES

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ABSTRACT. It is proven that there is a family of sets of natural numbers which has enumerations in every Turing degree except for the recursive degree. This implies that there is a countable structure which has representations in all but the recursive degree. Moreover, it is shown that there is such a structure which has a recursively represented elementary extension.

1. INTRODUCTION

In the following we are concerned with countable structures in a recursive language. Researchers have investigated how one could measure the intuitive idea of *information content* of such structures and tried to relate each one of them to a Turing degree [2], [3], [1]. The natural starting point is to look at the collection of *representations*. Let $\mathfrak{A}$ be a structure. If $\mathfrak{B}$ is an isomorphic structure with universe ω, then $\mathfrak{B}$ is called a representation of $\mathfrak{A}$ (written $\mathfrak{B} \simeq \mathfrak{A}$). $D(\mathfrak{B})$, its open diagram, can be regarded as a subset of ω so that it has a Turing degree, and one can look at the collection of degrees $\{deg(D(\mathfrak{B})) : \mathfrak{B} \simeq \mathfrak{A}\}$. A first guess for capturing the complexity of $\mathfrak{A}$ would be to let its degree be the least element of this collection, especially in the light of the following theorem [2, Theorem 4.1]:

Theorem 1.1 (Knight)**.** *Let $\mathfrak{A}$ be a structure in a relational language. Then exactly one of the following holds: (1) For any $\mathbf{d} > deg(D(\mathfrak{A}))$, there is a representation $\mathfrak{B}$ of $\mathfrak{A}$ such that $deg(D(\mathfrak{B})) = \mathbf{d}$. (2) There is a finite subset S of the universe of $\mathfrak{A}$ such that all permutations of the universe which fix S are automorphisms of $\mathfrak{A}$.*

But this idea fails. For example, Richter [3, Theorem 3.3] shows that for any countable order $\mathfrak{C}$ which has no recursive representation the collection $\{deg(D(\mathfrak{B})) : \mathfrak{B} \simeq \mathfrak{C}\}$ has no least element. Therefore more involved concepts have been tried to assign degrees to structures [1].

Now for the particular problems addressed in this paper. Steffen Lempp asked (unpublished): Does a structure with representations in all non-recursive degrees have a recursive representation? Julia Knight asked some related questions: With a binary relation $R \subseteq \omega^2$ associate a family of subsets of ω given by $\mathcal{F}_R := \{R_n : n \in \omega\}$, where $R_n := \{x : (n, x) \in R\}$; say that R is an enumeration of $\mathcal{F}_R$. She

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asked (also unpublished): If a family $\mathcal{F}$ has the feature that for every non-recursive set X, $\mathcal{F}$ has an enumeration recursive in X, does $\mathcal{F}$ have a recursive enumeration? Similarly, if for every non-recursive set X, $\mathcal{F}$ has an enumeration r.e. in X, does $\mathcal{F}$ have an r.e. enumeration?

In the next section we give some positive results on Knight's questions under extra hypotheses. In Section 3 we prove that the answer to Knight's questions is negative, by constructing a single suitable family. This implies that Lempp's question also has a negative answer, as is shown in the last section. The same finding is obtained in [4], but by another approach. We close by discussing the difference.

The notation is quite standard and follows [5]. All sets considered are subsets of ω, the set of natural numbers. We call countable collections of subsets of ω *families*. Let φ be a Gödel-numbering of the partial recursive functions (of varying arity) and $W_i := \mathrm{Rng}(\varphi_i)$ be an enumeration of the recursively enumerable sets as usual. $W_{i,s}$ is the set of numbers enumerated in W_i by stage s. We use recursive bijections $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot,\cdot\rangle$ between ω and ω^2, ω and ω^3, respectively. We also use projections $(.)_{.1}$ and $(.)_{.2}$ so that, for example, $(\langle a,b\rangle)_{.1} = a$. Let $\langle x, A\rangle$ be the set $\{\langle x,a\rangle : a \in A\}$. Similarly, $A_{.2} := \{(a)_{.2} : a \in A\}$. We let $A + x = \{a + x : a \in A\}$ and $A - x = \{b : b + x \in A\}$.

Fix an effective listing Ω of recursively enumerable enumerations of all families with r.e. enumerations:

$$\Omega^{(e)} := \{(i,x) : \langle i,x\rangle \in W_e\},$$

and write $\Omega_i^{(e)}$ for $\{x : \langle i,x\rangle \in W_e\}$, the i-th set of the enumeration $\Omega^{(e)}$. Define $\mathcal{C}^{(e)} := \{\Omega_i^{(e)} : i \in \omega\}$ to be the family enumerated by $\Omega^{(e)}$.

$D : \omega \to 2^\omega$ denotes the canonical enumeration of the family of finite sets; write D_n for the n-th finite set. Then the binary predicates $x \in D_n$ and $x = |D_n|$ are recursive.

2. Positive results

In this section we give conditions on a family which ensure that the implications of Knight's two questions hold. These are given in Theorems 2.3 and 2.4 and are due to Julia Knight. Jockusch gave a proof of Theorem 2.4 which also showed the following: There is a property which families with a recursive (r.e.) enumeration share with families that have, for all non-recursive degrees $\mathbf{d}$, an enumeration recursive (r.e.) in $\mathbf{d}$. These seem to be the only positive statements possible about such families.

What is this property? By a rather straightforward forcing construction it follows that if a set of natural numbers is recursive (r.e.) in all non-recursive degrees, then it is recursive (r.e.). Therefore, the members of a family are recursive (recursively enumerable) if the family has, for all non-recursive degrees $\mathbf{d}$, an enumeration recursive (r.e.) in $\mathbf{d}$. Hence such families $\mathcal{F}$ are fully described by the index set

$$I_r(\mathcal{F}) := \{i : (\exists A \in \mathcal{F})(\varphi_i = \chi_A)\}$$

or, respectively,

$$I_{re}(\mathcal{F}) := \{i : (\exists A \in \mathcal{F})(W_i = A)\}.$$

Both the index set I_r of a family with a recursive enumeration and the index set I_{re} of a family with a recursively enumerable enumeration are Σ^0_3 in the arithmetical hierarchy. Here is the coincidence:

Theorem 2.1. *Let $\mathcal{F}$ be a family. If, for every non-recursive degree* $\mathbf{d}$*, $\mathcal{F}$ has an enumeration recursive in* $\mathbf{d}$*, then its index set $I_r(\mathcal{F})$ is Σ^0_3.*

Theorem 2.2 (Jockusch)**.** *Let $\mathcal{F}$ be a family. If, for every non-recursive degree* $\mathbf{d}$*, $\mathcal{F}$ has an enumeration recursively enumerable in* $\mathbf{d}$*, then its index set $I_{re}(\mathcal{F})$ is Σ^0_3.*

Towards giving a sufficient condition under which the implication of her first question holds, Julia Knight defines an *extension function* for a family $\mathcal{F}$ to be a (possibly partial) function $f\colon 2^{<\omega} \to \omega$ such that if $\sigma \in 2^{<\omega}$ and there exists a set $A \in \mathcal{F}$ such that $\chi_A \supseteq \sigma$, then $\varphi_{f(\sigma)} = \chi_A$ for some such A. We mention two facts: Any family with a recursive enumeration has a partial recursive extension function, and so does a family containing all finite sets.

We prove Theorems 2.1 and 2.2 simultaneously with the following two.

Theorem 2.3 (Knight)**.** *Let $\mathcal{F}$ be a family which has, for all non-recursive* $\mathbf{d}$*, an enumeration recursive in* $\mathbf{d}$. *If $\mathcal{F}$ has a partial recursive extension function, then $\mathcal{F}$ has a recursive enumeration.*

Proof (of Theorems 2.3 and 2.1). Let $\mathcal{F}$ be a family which has, for any non-recursive set X, an enumeration recursive in X. We construct a generic set D, attempting to meet the following requirements and expecting to fail. Below, this failure will be exploited to prove the statements of the two theorems for $\mathcal{F}$ separately.

$$R_e\colon\ \varphi_e^D \text{ is not the characteristic function of an enumeration of } \mathcal{F}.$$

The set D will be Cohen-generic. The set of forcing conditions is $2^{<\omega}$ and the partial order is given by $\subseteq$. We use the old-fashioned notion of a *complete forcing sequence* (c.f.s.), where $p_{n+1} \supseteq p_n$, with p_{n+1} entering the nth dense set in some countable collection. D is the set with characteristic function $\bigcup_{n\in\omega} p_n$.

Fix a condition $p \in 2^{<\omega}$ and $e \in \omega$. We consider the following four possibilities for p and e, showing how extensions of p may force satisfaction of R_e in each case.

P1. For some $q \supseteq p, n, x \in \omega$, $q \Vdash \varphi_e^D(n,x)\downarrow \neq 0, 1$, or $q \Vdash \varphi_e^D(n,x)\uparrow$.
We include q in the c.f.s., thereby satisfying R_e.

P2. For some $q \supseteq p$ and some n, for all $q' \supseteq q$ there exist x and $r_0, r_1 \supseteq q'$ such that $r_i \Vdash \varphi_e^D(n,x) = i$.
For each $A \in \mathcal{F}$ the set
$$D^1_A := \{r : r \supseteq q \Rightarrow (\exists x)(r \Vdash \varphi_e^D(n,x) \neq \chi_A(x))\}$$
is dense. We add q to the c.f.s. and enter the sets D^1_A. Then the requirement R_e is satisfied.

We write $q \Vdash E_n = A$ if for all x and all $q' \supseteq q$ there is an $r \supseteq q'$ such that $r \Vdash \varphi_e^D(n,x)\downarrow = \chi_A(x)$. Note that for q and n there is at most one set A such that $q \Vdash E_n = A$.

P3. For some $q \supseteq p$, $n \in \omega$ and $B \notin \mathcal{F}$ we have $q \Vdash E_n = B$.
By putting q into the c.f.s., we meet R_e.

P4. Not P2 but there exist $A \in \mathcal{F}$ and $q \supseteq p$ such that for all $q' \supseteq q$ and all n, if $q' \Vdash E_n = B$ then $A \neq B$.

Since P2 does not hold, the set

$$D_n^2 := \{q' : q' \supseteq q \Rightarrow (\exists B)(q' \vdash E_n = B)\}$$

is dense for each n. By including q in the c.f.s. and entering the sets D_n^2, we meet R_e.

If for all $p \in 2^{<\omega}$ and $e \in \omega$ one of the cases P1,...,P4 holds, then the forcing construction yields a generic (and so non-recursive) set D, in which $\mathcal{F}$ has no recursive enumeration, contrary to the assumption on $\mathcal{F}$.

So let $p \in 2^{<\omega}$ and $e \in \omega$ be such that none of the cases P1,...,P4 hold. It follows that if $q \vdash E_n = A$, then $A \in \mathcal{F}$ for all $q \supseteq p$, n and $A \subseteq \omega$. Moreover, all elements of $\mathcal{F}$ occur in this way.

We first complete the proof of Theorem 2.3. Let f be a partial recursive extension function for $\mathcal{F}$. Fix $n \in \omega$ and $p \subseteq q \in 2^{<\omega}$. Say that $\sigma \in 2^{<\omega}$ *has a q-computation* if there is an $r \supseteq q$ such that $r \vdash \varphi_e^D(n, x) = \sigma(x)$ for all $x \in dom(\sigma)$. We make the following observations:

- $q \vdash E_n = A$ and $q' \supseteq q$ implies $q' \vdash E_n = A$.
- The set $\{\sigma \in 2^{<\omega} : \sigma$ has a q-computation$\}$ is r.e. (uniformly in q).
- By definition, $q \vdash E_n = A$ if and only if any $\sigma \in 2^{<\omega}$ has a q-computation if and only if $\sigma \subseteq \chi_A$.
- Since p, e do not satisfy P1, for any $q' \supseteq q$ there is a $\sigma \in 2^{<\omega}$ which has a q'-computation.
- The previous two items imply that the predicate $q \supseteq p \Rightarrow (\forall A)(q \not\vdash E_n = A)$ in q and n is r.e.
- Since p, e do not satisfy P3, if $q \supseteq p$ and $q \not\vdash E_n = A$ for any $A \in \mathcal{F}$, then for any $\sigma \in 2^{<\omega}$ with a q-computation there is a set $A \in \mathcal{F}$ such that $\sigma \subseteq \chi_A$.

We form a recursive enumeration R of $\mathcal{F}$, using pairs (q, n) as indices, where $q \in 2^{<\omega}$ with $q \supseteq p$ and $n \in \omega$. Since there is a recursive bijection between $P \times \omega$ and ω, where $P = \{q \in 2^{<\omega} : q \supseteq p\}$, this suffices. Let $(\sigma_q^{(n)})_{n\in\omega}$ be an effective enumeration of $\{\sigma \in 2^{<\omega} : \sigma$ has a q-computation$\}$ by the second observation. We choose this enumeration so that, additionally, for every σ with a q-computation there are infinitely many i such that $\sigma_q^{(i)} = \sigma$. Define a partial recursive function a_q by $a_q(0) := 0$ and $a_q(n+1) := \mu m > n . \sigma_q^{(m)} \supseteq \sigma_q^{(a_q(n))}$. By the fourth observation, let $h : \omega \to 2^{<\omega} \times \omega$ be a recursive function with range $\{(q, n) : q \supseteq p$ and $(\forall A)(q \not\vdash E_n = A)\}$.

$R_{(q,n)}$ is defined by

$$R_{(q,n)} := \begin{cases} \bigcup_{0\le i\le m} \sigma_q^{(a_q(i))} \cup \varphi_{f(\sigma_q^{a_q(m)})} & \text{if } m \text{ is least such that } h(m) = (q, n), \\ \\ \bigcup_{i\in\omega} \sigma_q^{(a_q(i))} & \text{otherwise.} \end{cases}$$

Fix $q \supseteq p$ and $n \in \omega$. By definition of the functions h, there is m satisfying the first case if and only if $(\forall A)(q \not\vdash E_n = A)$. If $q \vdash E_n = A$, then the choice of the enumeration $(\sigma_q^{(n)})_{q\in 2^{<\omega}, n\in\omega}$ and the function a guarantees $R_{(q,n)} = A$. If $q \not\vdash E_n = A$ for any A, then by the fact that f is an extension function for $\mathcal{F}$, it follows that $R_{(q,n)} \in \mathcal{F}$.

Since for every $A \in \mathcal{F}$ there are q and n such that $q \vdash E_n = A$, it follows that R is an enumeration of $\mathcal{F}$, and Theorem 2.3 is proved.

We turn to the proof of Theorem 2.1. As mentioned above, if none of the cases holds for p and e, then we have

$$\mathcal{F} = \{X : (\exists q \supseteq p)(\exists n)(q \vdash E_n = X)\}.$$

The ternary relation $(q \vdash E_n = X) \wedge (\chi_X = \varphi_i)$ (in q, n and i), when restricted to i such that φ_i is the characteristic function of a set, is Π^0_2 so that $I_r(\mathcal{F})$ is Σ^0_3. □

Theorem 2.4 (Knight)**.** *Let $\mathcal{F}$ be a family such that for all non-recursive X, $\mathcal{F}$ has an enumeration r.e. in X. If $\mathcal{F}$ contains all finite sets, then $\mathcal{F}$ has an r.e. enumeration.*

Proof of Theorems 2.4 and 2.2. As above, we construct a generic set D, attempting to meet the following requirements and expecting to fail.

R_e. W_e^D is not an enumeration of $\mathcal{F}$.

We use the same forcing notion as was used for the previous proof. Fix $p \in 2^{<\omega}$ and $e \in \omega$. We consider the following three cases.

C1. For some $q \supseteq p$ and some $n \in \omega$, for all $q' \supseteq q$, there exist $x \in \omega$ and $r_0, r_1 \supseteq q'$ such that $r_0 \vdash (n, x) \in W_e^D$ and $r_1 \vdash (n, x) \notin W_e^D$.
For each $A \in \mathcal{F}$ the set

$$\begin{aligned} D^1_A := \{r : r \supseteq q \Rightarrow (\exists x)[&(r \vdash (n,x) \in W_e^D \wedge x \notin A) \text{ or} \\ &(r \vdash (n,x) \notin W_e^D \wedge x \in A)]\} \end{aligned}$$

is dense. We put q into the c.f.s. and enter the sets D^1_A. Requirement R_e is satisfied.

We write $q \vdash E_n = A$ if for all x, if $x \in A$, then for all $q' \supseteq q$ there is $r \supseteq q'$ such that $r \vdash (n, x) \in W_e^D$, and if $x \notin A$ then $q \vdash (n, x) \notin W_e^D$.

C2. Case C1 does not hold, but $q \vdash E_n = B$ for some $q \supseteq p$, $n \in \omega$ and $B \notin \mathcal{F}$.
Requirement R_e is met by putting q in the c.f.s.

C3. Case C1 does not hold, but there exists $A \in \mathcal{F}$ such that for all $q' \supseteq q$ and all n, if $q' \vdash E_n = B$, then $A \neq B$.
Since Case C1 does not hold, the set

$$D^2_n := \{q' : q' \supseteq q \Rightarrow (\exists B)(q' \vdash E_n = B)\}$$

is dense for all n. We meet the requirement R_n by including q in the c.f.s. and entering all sets D^2_n.

If for all $p \in 2^{<\omega}$ and $e \in \omega$ one of the cases C1,C2, or C3 holds, then the forcing construction yields a generic (and hence non-recursive) set D, in which $\mathcal{F}$ has no r.e. enumeration. This contradicts the assumption of both Theorem 2.2 and Theorem 2.4. So let $p \in 2^{<\omega}$ and $e \in \omega$ be such that none of the cases C1,C2,C3 holds. We show that the index set I_{re} of $\mathcal{F}$ is Σ^0_3. As in the proof of Theorem 2.3, for all $q \supseteq p$, n, and $A \subseteq \omega$, if $q \vdash E_n = A$, then $A \in \mathcal{F}$, and all members of $\mathcal{F}$ occur in this way. Therefore we have

$$\{i : W_i \in \mathcal{F}\} = \{i : (\exists q \supseteq p)(\exists n)(q \vdash E_n = W_i)\}.$$

The relation "$q \vdash E_n = W_i$" is Π^0_2, and so the index set of $\mathcal{F}$ is Σ^0_3. This completes the proof of Theorem 2.2. To complete the proof of Theorem 2.4, note that since $\mathcal{F}$ includes all finite sets and only contains r.e. sets, by a theorem of Yates [7, Theorem 8], $\mathcal{F}$ has an r.e. enumeration. □

3. A family of finite sets

In this section we first define a family $\mathcal{C}$ which has no r.e. enumeration. Then we show that for every non-recursive set X there is an enumeration of $\mathcal{C}$ which is recursive in X, and finally that every non-recursive degree contains an enumeration of $\mathcal{C}$. This corrects an earlier statement in [6, p 187]. In particular, we apologize for connecting the error with Martin Kummer.

Let r be the partial recursive function defined by

$$r(e) := (\mu\langle i, x, s\rangle . \langle e, x\rangle \in \Omega^{(e)}_{i,s})_{.1}.$$

Informally, the value of $r(e)$ is the first index i to be found such that there is a number of the form $\langle e, x\rangle \in \Omega^{(e)}_i$; if there is no such i then $r(e)$ is not defined. Let the family $\mathcal{C}$ be defined by

$$\mathcal{C} := \{\langle e, A\rangle : A \text{ is finite}, e \in \omega\} - \{\langle e, \omega\rangle \cap \Omega^{(e)}_{r(e)} : r(e)\downarrow\}.$$

$\mathcal{C}$ does not have an r.e. enumeration; for suppose $\Omega^{(e_0)}$ is an enumeration of $\mathcal{C}$. Then $r(e_0)$ is defined, and the set $\Omega^{(e_0)}_{r(e_0)} = \langle e, \omega\rangle \cap \Omega^{(e_0)}_{r(e_0)}$ is not a member of $\mathcal{C}$.

Let X be an arbitrary non-recursive set. To see how to construct an enumeration S^X of $\mathcal{C}$ such that $S^X \leq_T X$ we use the following lemma.

Lemma 3.1. *Uniformly in i and recursively in X there is a finite set $A^X_i \leq_T X$ such that $W_i \neq A^X_i$.*

Let g be a partial recursive function such that $g(e)\downarrow$ if and only if $r(e)\downarrow$ and if $r(e)\downarrow$ then $W_{g(e)} = (\langle e, \omega\rangle \cap \Omega^{(e)}_{r(e)})_{.2}$. Let h be a partial recursive function such that $h(e, a)\downarrow$ if and only if $g(e)\downarrow$ and if $g(e)\downarrow$ then $W_{h(e,a)} = W_{g(e)} - a$.

Define

$$S^X_{\langle n,t,e\rangle} := \langle e, B^X(n, t, e)\rangle,$$

where

$$B^X(n,t,e) := \begin{cases} D_n \cup (s_0 + A^X_{h(e,s_0)}) & \text{if there is } s > t, \max(D_n)+1 \\ & \text{such that } g_s(e)\downarrow \text{ and } W_{g(e),s} = D_n, \\ & \text{and } s_0 \text{ is the least such,} \\ D_n & \text{otherwise.} \end{cases}$$

By the lemma, S^X is recursive in X. We claim that S^X is an enumeration of $\mathcal{C}$.

"$\subseteq$". First of all, it follows from the lemma that all sets $B^X(n, t, e)$ are finite and so the family enumerated by S^X is contained in $\{\langle e, A\rangle : A \text{ is finite}\}$.

Suppose $r(e)\downarrow$ and $S^X_{\langle n,t,e\rangle} = \langle e, \omega\rangle \cap \Omega^{(e)}_{r(e)}$. It follows that $B^X(n, t, e) = W_{g(e)}$. There has to be a stage $s > t, \max(D_n) + 1$ such that $W_{g(e),s} = D_n$, because otherwise $B^X(n, t, e) = D_n \neq W_{g(e)}$, a contradiction. Let s_0 be the least such s. Then $B^X(n, t, e) = D_n \cup (s_0 + A^X_{h(e,s_0)}) = W_{g(e)}$, so that $A^X_{h(e,s_0)} = W_{h(e,s_0)}$, a contradiction.

"$\supseteq$". Let $e, n \in \omega$ such that $C = \langle e, D_n\rangle \in \mathcal{C}$. We want to find a number z such that $S^X_z = C$. Let t be such that $W_{g(e),s} \neq D_n$ for all $s > t$. (If there is no such t then $C \notin \mathcal{C}$.) By definition, $B^X(n, t, e) = D_n$, so that $S^X_{\langle n,t,e\rangle} = C$.

We have constructed an enumeration of $\mathcal{C}$ which is recursive in X. Simple coding is sufficient to obtain an enumeration T-equivalent to X: Choose $A, B \in \mathcal{C}$ so that

$A - B \neq \emptyset$. Define another enumeration of $\mathcal{C}$,

$$P_{2n}^X := S_n^X, \quad P_{2n+1}^X := \begin{cases} A & \text{if } n \in X, \\ B & \text{if } n \notin X. \end{cases}$$

Clearly, P^X is recursive in X and enumerates $\mathcal{C}$. Let $x \in A - B$. Then $z \in X$ if and only $x \in P_{2z+1}^X$, so that X is recursive in P^X.

Proof of Lemma 3.1. Let $y, n : \omega^2 \to \omega$ be two recursive one-one functions such that their ranges are disjoint and cover ω. Let

$$\alpha(s, x) := \begin{cases} y(s, x) & \text{if } x \in X, \\ n(s, x) & \text{otherwise,} \end{cases}$$

so that $\alpha \leq_T X$. We construct the set A_i^X in stages uniformly in i and recursively in X. In the course of the construction, numbers a_x may become defined.

Stage 0. Set $x := 0$. $A_{i,0}^X$ is empty.

Stage $s+1$. If $W_{i,s} \neq A_{i,s}^X$, pass to the next stage. Otherwise enumerate $\alpha(s, x)$ in A_i^X, let $a_x := s$ and increase x by one.

End of construction.

The set A_i^X is (by construction) r.e. in X. It is also recursive in X: By inspection, A_i^X only contains numbers $\alpha(a, b)$. If $y(s, x) \in A_i^X$, then $y(s, x) \in A_{i,s+1}^X$ and the same holds for numbers $n(s, x)$. This together with the choice of y and n is sufficient.

Claim 1. *The set A_i^X is finite.*

Suppose A_i^X is infinite. During the construction of A_i^X infinitely many numbers a_x are defined. By induction on x, it follows that $A_{i,a_x}^X = \{\alpha(a_j, j) : j \leq x\}$, and therefore $W_{i,a_x} = \{\alpha(a_j, j) : j < x\}$ for all $x \in \omega$. Thus, $W_i = \{\alpha(a_x, x) : x \in \omega\}$. Now, $x \in X$ if and only if $(\exists t)(y(t, x) \in W_i)$, and $x \notin X$ if and only if $(\exists t)(n(t, x) \in W_i)$. This means that X is recursive, a contradiction.

Claim 2. *A_i^X is different from W_i.*

By the previous claim it is sufficient to consider the case when W_i is finite. Let s_0 be the least number such that $W_{i,s_0} = W_i$. Then either $W_{i,s_0} \neq A_{i,s_0}^X$, in which case $W_{i,s_0} \neq A_{i,s_0}^X = A_i^X$; or $W_{i,s_0} = A_{i,s_0}^X$, so that at stage $s_1 = s_0 + 1$ a new number is enumerated in A_i^X, whence $W_i = W_{i,s_1} = A_{i,s_0}^X \neq A_i^X$.

Lemma 3.1 is proved. $\square$

4. Application to Lempp's question

Let $\mathcal{F}$ be a family. With $\mathcal{F}$ we associate the following countable structure $\mathfrak{A}_\mathcal{F}$ in the language $L = (S, Z, I)$, where S is a binary predicate symbol and Z and I are unary predicate symbols. The universe of $\mathfrak{A}_\mathcal{F}$ is $\mathcal{F} \times \omega \times \omega$. For every $A \in \mathcal{F}$, set $Z((A, x, 0))$ and $S((A, x, n), (A, x, n+1))$. Set $I((A, x, n))$ if and only if $n \in A$. Thus, countably many S-chains $(A, x, 0), (A, x, 1), (A, x, 2), \ldots$ are associated with every $A \in \mathcal{F}$, and in every chain I holds of the n-th member if $n \in A$.

Theorem 4.1. *Let* $\mathbf{d}$ *be a Turing degree and* $\mathcal{F}$ *be a family. Then* $\mathcal{F}$ *has an enumeration recursive in* $\mathbf{d}$ *if and only if the structure* $\mathfrak{A}_\mathcal{F}$ *has a representation recursive in* $\mathbf{d}$.

Proof. "⇒". Let Q^X be an enumeration of $\mathcal{F}$ which is recursive in $X \in \mathbf{d}$. Define R^X to be another X-recursive enumeration of $\mathcal{F}$ by $R^X_{\langle n,i\rangle} := Q^X_i$. A representation of $\mathfrak{A}_{\mathcal{F}}$ is given by $\mathfrak{B}$, where $Z_{\mathfrak{B}}(\langle x,0\rangle)$, $S_{\mathfrak{B}}(\langle x,n\rangle,\langle x,n+1\rangle)$, and $I_{\mathfrak{B}}(\langle x,n\rangle)$ if and only if $x \in R^X_n$. These predicates are recursive in R^X, and therefore $D(\mathfrak{B})$ is recursive in X.

"⇐". Let $\mathfrak{B}$ be a representation of $\mathfrak{A}_{\mathcal{F}}$ whose open diagram is recursive in $X \in \mathbf{d}$. The set $Z := \{x : Z_{\mathfrak{B}}(x)\}$ is recursive in X. With each $x \in Z$ we associate the set $S_x := \{n : I_{\mathfrak{B}}(f^{(n)}(x))\}$, where $f(x)$ is the unique y such that $S_{\mathfrak{B}}(x,y)$, and $f^{(n)}(x)$ denotes the n-fold application of f to x. The sets S_x are uniformly in $x \in Z$ recursive in X, and, by definition of $\mathfrak{A}_{\mathcal{F}}$, $\mathcal{F} = \{S_x : x \in Z\}$. This suffices. □

Corollary 4.2 (Slaman [4])**.** *There is a structure which has representations only in the non-recursive degrees.*

Proof. Apply the theorem to the family $\mathcal{C}$ defined in the previous section to obtain representations of $\mathfrak{A}_{\mathcal{C}}$ below any non-recursive T-degree. Apply Theorem 1.1 to obtain representations in all non-recursive T-degrees. □

Slaman remarked (private communication) that the construction given in [4] yields a structure which is not elementarily equivalent to any recursively represented structure. Informally, the reason for this is as follows. Essentially, the construction proceeds in such a way that the final outcome of the actions taken to diagonalize against the recursive representations can be read off the theory of the structure (which is called $\mathfrak{M}$):

He writes ([4, Section 2.1]): "We will ensure that either $R^{-1}(T_i)^{\mathfrak{R}_i} = \emptyset$, or $\langle T_i, <_L\restriction T_i\rangle^{\mathfrak{R}_i}$ is not isomorphic to $\langle T_i, <_L\restriction T_i\rangle^{\mathfrak{M}}$, or there is a p in $R^{-1}(T_i)^{\mathfrak{R}_i}$ such that $\zeta(p)^{\mathfrak{R}_i}$ is not maximal, or there is a p in $R^{-1}(T_i)^{\mathfrak{R}_i}$ such that $\zeta(p)^{\mathfrak{R}_i}$ is infinite. Since none of these disjuncts apply to $\mathfrak{M}$, we will thus ensure that $\mathfrak{M}$ has no recursive presentation."

Fix $i \in \omega$. The first and third disjunct can be directly formalized in the language $\mathfrak{L}$ of the structure. If the second disjunct holds, then the construction [4, Section 2.2.1] yields a tree $\langle T_i, <_L\restriction T_i\rangle^{\mathfrak{M}}$ which is finite. Therefore this tree is described by a sentence in the theory of $\mathfrak{M}$. The fourth disjunct also cannot be formalized in $\mathfrak{L}$, but, provided the first three disjuncts are not true, the strategy used in [4, Section 2.2.2] results in

$$(\exists p)(R(p, s^i(0)) \wedge (\forall x)(R(p,x) \rightarrow (\exists y)(x <_T y \wedge R(p,y))))$$

being true in $\mathfrak{R}_i$. This is a formula in $\mathfrak{L}$, and (by the same strategy) not true in $\mathfrak{M}$. Hence there is no recursively represented structure which is elementarily equivalent to $\mathfrak{M}$.

The structure obtained from the family $\mathcal{C}$ by Theorem 4.1 is of a different kind:

Theorem 4.3. *There is a structure which has representations only in the non-recursive degrees and has a recursively represented elementary extension.*

Proof. Let us look at the following family:

$$\mathcal{D} := \{\langle e, A\rangle : A \text{ is finite}, e \in \omega\}.$$

Obviously, $\mathcal{D}$ has a recursive enumeration, and $\mathfrak{A}_{\mathcal{D}}$ has a recursive representation. At the same time, the structure $\mathfrak{A}_{\mathcal{C}}$ from the proof of Corollary 4.2 is contained in $\mathfrak{A}_{\mathcal{D}}$, and they are elementarily equivalent:

It suffices to show that for any formula $(\exists x)(\phi)$ in variables $x_1, \ldots, x_n$, if

$$\mathfrak{A}_{\mathcal{D}} \models (\exists x)(\phi)[x_1 := c_1, \ldots, x_n := c_n]$$

and $c_1, \ldots c_n \in \mathfrak{A}_{\mathcal{C}}$, then there is $c \in \mathfrak{A}_{\mathcal{C}}$ such that

$$\mathfrak{A}_{\mathcal{D}} \models \phi[x := c, x_1 := c_1, \ldots, x_n := c_n].$$

Note that if $\mathfrak{A}_{\mathcal{D}} \models (\exists x)(\phi)[x_1 := c_1, \ldots, x_n := c_n]$, then there is a d in $\mathfrak{A}_{\mathcal{D}}$ such that either

$$\begin{aligned}\mathfrak{A}_{\mathcal{D}} \models & \phi[x := d, x_1 := c_1, \ldots, x_n := c_n] \\ & \wedge \neg S(d, c_1) \wedge \neg S(c_1, d) \wedge \ldots \wedge \neg S(d, c_n) \wedge \neg S(c_n, d),\end{aligned}$$

or

$$\begin{aligned}\mathfrak{A}_{\mathcal{D}} \models & \phi[x := d, x_1 := c_1, \ldots, x_n := c_n] \\ & \wedge (S(d, c_1) \vee S(c_1, d) \vee \ldots \vee S(d, c_n) \vee S(c_n, d)).\end{aligned}$$

In the former case, choose c from $\mathfrak{A}_{\mathcal{C}}$ outside of the S-chains of $\mathfrak{A}_{\mathcal{C}}$ which $c_1, \ldots, c_n$ belong to such that it satisfies Z and I in the same way d does. In the latter case, d is already part of an S-chain which is contained in $\mathfrak{A}_{\mathcal{C}}$, and so an element of $\mathfrak{A}_{\mathcal{C}}$. $\square$

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