ENUMERATIONS, COUNTABLE STRUCTURES
AND TURING DEGREES

STEPHAN WEHNER

(Communicated by Andreas R. Blass)

Abstract. It is proven that there is a family of sets of natural numbers which
has enumerations in every Turing degree except for the recursive degree. This
implies that there is a countable structure which has representations in all but
the recursive degree. Moreover, it is shown that there is such a structure which
has a recursively represented elementary extension.

1. Introduction

In the following we are concerned with countable structures in a recursive lan-
guage. Researchers have investigated how one could measure the intuitive idea of
information content of such structures and tried to relate each one of them to a
Turing degree [2], [3], [1]. The natural starting point is to look at the collection of
representations. Let \( \mathfrak{A} \) be a structure. If \( \mathfrak{B} \) is an isomorphic structure with universe
\( \omega \), then \( \mathfrak{B} \) is called a representation of \( \mathfrak{A} \) (written \( \mathfrak{B} \simeq \mathfrak{A} \)). \( D(\mathfrak{B}) \), its open diagram,
can be regarded as a subset of \( \omega \) so that it has a Turing degree, and one can look
at the collection of degrees \( \{\deg(D(\mathfrak{B})) : \mathfrak{B} \simeq \mathfrak{A} \} \). A first guess for capturing the
complexity of \( \mathfrak{A} \) would be to let its degree be the least element of this collection,
especially in the light of the following theorem [2, Theorem 4.1]:

**Theorem 1.1** (Knight). Let \( \mathfrak{A} \) be a structure in a relational language. Then exactly one of the following holds: (1) For any \( d > \deg(D(\mathfrak{A})) \), there is a representa-
tion \( \mathfrak{B} \) of \( \mathfrak{A} \) such that \( \deg(D(\mathfrak{B})) = d \). (2) There is a finite subset \( S \) of the universe
of \( \mathfrak{A} \) such that all permutations of the universe which fix \( S \) are automorphisms of
\( \mathfrak{A} \).

But this idea fails. For example, Richter [3, Theorem 3.3] shows that for any
countable order \( \mathfrak{C} \) which has no recursive representation the collection \( \{\deg(D(\mathfrak{B})) : \mathfrak{B} \simeq \mathfrak{C} \} \) has no least element. Therefore more involved concepts have been tried to
assign degrees to structures [1].

Now for the particular problems addressed in this paper. Steffen Lempp asked
(unpublished): Does a structure with representations in all non-recursive degrees
have a recursive representation? Julia Knight asked some related questions: With
a binary relation \( R \subseteq \omega^2 \) associate a family of subsets of \( \omega \) given by \( \mathcal{F}_R := \{ R_n : n \in \omega \} \), where \( R_n := \{ x : (n, x) \in R \} \); say that \( R \) is an enumeration of \( \mathcal{F}_R \). She
asked (also unpublished): If a family \( F \) has the feature that for every non-recursive set \( X \), \( F \) has an enumeration recursive in \( X \), does \( F \) have a recursive enumeration? Similarly, if for every non-recursive set \( X \), \( F \) has an enumeration r.e. in \( X \), does \( F \) have an r.e. enumeration?

In the next section we give some positive results on Knight’s questions under extra hypotheses. In Section 3 we prove that the answer to Knight’s questions is negative, by constructing a single suitable family. This implies that Lempp’s question also has a negative answer, as is shown in the last section. The same finding is obtained in [4], but by another approach. We close by discussing the difference.

The notation is quite standard and follows [5]. All sets considered are subsets of \( \omega \), the set of natural numbers. We call countable collections of subsets of \( \omega \) families. Let \( \varphi \) be a G"odel-numbering of the partial recursive functions (of varying arity) and \( W_i := \text{Rng}(\varphi_i) \) be an enumeration of the recursively enumerable sets as usual. \( W_{i,s} \) is the set of numbers enumerated in \( W_i \) by stage \( s \). We use recursive bijections \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot, \cdot \rangle \) between \( \omega \) and \( \omega^2 \), \( \omega \) and \( \omega^3 \), respectively. We also use projections \( \langle \cdot \rangle_1 \) and \( \langle \cdot \rangle_2 \) so that, for example, \( \langle (a, b) \rangle_1 = a \). Let \( \langle x, A \rangle \) be the set \( \{ \langle x, a \rangle : a \in A \} \). Similarly, \( A_2 := \{ (a)_2 : a \in A \} \). We let \( A + x = \{ a + x : a \in A \} \) and \( A - x = \{ b : b + x \in A \} \).

Fix an effective listing \( \Omega \) of recursively enumerable enumerations of all families with r.e. enumerations:

\[
\Omega^{(c)} := \{ (i, x) : (i, x) \in W_e \},
\]

and write \( \Omega_i^{(c)} \) for \( \{ x : (i, x) \in W_e \} \), the \( i \)-th set of the enumeration \( \Omega^{(c)} \). Define \( C^{(c)} := \{ \Omega_i^{(c)} : i \in \omega \} \) to be the family enumerated by \( \Omega^{(c)} \).

\( D : \omega \to 2^\omega \) denotes the canonical enumeration of the family of finite sets; write \( D_n \) for the \( n \)-th finite set. Then the binary predicates \( x \in D_n \) and \( x = |D_n| \) are recursive.

### 2. Positive results

In this section we give conditions on a family which ensure that the implications of Knight’s two questions hold. These are given in Theorems 2.3 and 2.4 and are due to Julia Knight. Jockusch gave a proof of Theorem 2.4 which also showed the following: There is a property which families with a recursive (r.e.) enumeration share with families that have, for all non-recursive degrees \( d \), an enumeration recursive (r.e.) in \( d \). These seem to be the only positive statements possible about such families.

What is this property? By a rather straightforward forcing construction it follows that if a set of natural numbers is recursive (r.e.) in all non-recursive degrees, then it is recursive (r.e.). Therefore, the members of a family are recursive (recursively enumerable) if the family has, for all non-recursive degrees \( d \), an enumeration recursive (r.e.) in \( d \). Hence such families \( F \) are fully described by the index set

\[
I_r(F) := \{ i : (\exists A \in F)(\varphi_i = \chi_A) \}
\]

or, respectively,

\[
I_{re}(F) := \{ i : (\exists A \in F)(W_i = A) \}.
\]
Both the index set $I_r$ of a family with a recursive enumeration and the index set $I_{re}$ of a family with a recursively enumerable enumeration are $\Sigma^0_3$ in the arithmetical hierarchy. Here is the coincidence:

**Theorem 2.1.** Let $\mathcal{F}$ be a family. If, for every non-recursive degree $d$, $\mathcal{F}$ has an enumeration recursive in $d$, then its index set $I_r(\mathcal{F})$ is $\Sigma^0_3$.

**Theorem 2.2** (Jockusch). Let $\mathcal{F}$ be a family. If, for every non-recursive degree $d$, $\mathcal{F}$ has an enumeration recursively enumerable in $d$, then its index set $I_{re}(\mathcal{F})$ is $\Sigma^0_3$.

Towards giving a sufficient condition under which the implication of her first question holds, Julia Knight defines an extension function for a family $\mathcal{F}$ to be a (possibly partial) function $f: 2^{<\omega} \to \omega$ such that if $\sigma \in 2^{<\omega}$ and there exists a set $A \in \mathcal{F}$ such that $\chi_A \supseteq \sigma$, then $\varphi_f(\sigma) = \chi_A$ for some such $A$. We mention two facts: Any family with a recursive enumeration has a partial recursive extension function, and so does a family containing all finite sets.

We prove Theorems 2.1 and 2.2 simultaneously with the following two.

**Theorem 2.3** (Knight). Let $\mathcal{F}$ be a family which has, for all non-recursive $d$, an enumeration recursive in $d$. If $\mathcal{F}$ has a partial recursive extension function, then $\mathcal{F}$ has a recursive enumeration.

**Proof (of Theorems 2.3 and 2.1).** Let $\mathcal{F}$ be a family which has, for any non-recursive set $X$, an enumeration recursive in $X$. We construct a generic set $D$, attempting to meet the following requirements and expecting to fail. Below, this failure will be exploited to prove the statements of the two theorems for $\mathcal{F}$ separately.

$R_e$: $\varphi_e^D$ is not the characteristic function of an enumeration of $\mathcal{F}$.

The set $D$ will be Cohen-generic. The set of forcing conditions is $2^{<\omega}$ and the partial order is given by $\subseteq$. We use the old-fashioned notion of a complete forcing sequence (c.f.s.), where $p_{n+1} \supseteq p_n$, with $p_{n+1}$ entering the $n$th dense set in some countable collection. $D$ is the set with characteristic function $\bigcup_{n \in \omega} p_n$.

Fix a condition $p \in 2^{<\omega}$ and $e \in \omega$. We consider the following four possibilities for $p$ and $e$, showing how extensions of $p$ may force satisfaction of $R_e$ in each case.

1. For some $q \supseteq p$, $n, x \in \omega$, $q \vdash \varphi_e^D(n,x) \downarrow \neq 0, 1$, or $q \vdash \varphi_e^D(n,x) \uparrow$.
   We include $q$ in the c.f.s., thereby satisfying $R_e$.

2. For some $q \supseteq p$ and some $n$, for all $q' \supseteq q$ there exist $x$ and $r_0, r_1 \supseteq q'$ such that $r_i \vdash \varphi_e^D(n,x) = i$.
   For each $A \in \mathcal{F}$ the set
   
   $$D_A^1 := \{ r : r \supseteq q \Rightarrow (\exists x)(r \vdash \varphi_e^D(n,x) \neq \chi_A(x))\}$$
   
   is dense. We add $q$ to the c.f.s. and enter the sets $D_A^1$. Then the requirement $R_e$ is satisfied.

3. For some $q \supseteq p$, $n \in \omega$ and $B \notin \mathcal{F}$ we have $q \vdash E_n = B$.
   By putting $q$ into the c.f.s., we meet $R_e$.

4. Not P2 but there exist $A \in \mathcal{F}$ and $q \supseteq p$ such that for all $q' \supseteq q$ and all $n$, if $q' \vdash E_n = B$ then $A \neq B$. 

Since P2 does not hold, the set
\[ D_n^2 := \{ q' : q' \supseteq q \Rightarrow (\exists B)(q' \vdash E_n = B) \} \]
is dense for each \( n \). By including \( q \) in the c.f.s. and entering the sets \( D_n^2 \), we meet \( R_e \).

If for all \( p \in 2^{\omega} \) and \( e \in \omega \) one of the cases P1,...,P4 holds, then the forcing construction yields a generic (and so non-recursive) set \( D \), in which \( F \) has no recursive enumeration, contrary to the assumption on \( F \).

So let \( p \in 2^{\omega} \) and \( e \in \omega \) be such that none of the cases P1,...,P4 hold. It follows that if \( q \vdash E_n = A \), then \( A \in F \) for all \( q \supseteq p \), \( n \) and \( A \subseteq \omega \). Moreover, all elements of \( F \) occur in this way.

We first complete the proof of Theorem 2.3. Let \( f \) be a partial recursive extension function for \( F \). Fix \( n \in \omega \) and \( p \subseteq q \in 2^{\omega} \). Say that \( \sigma \in 2^{\omega} \) has a \( q \)-computation if there is an \( r \supseteq q \) such that \( r \vdash \varphi_e^D(n, x) = \sigma(x) \) for all \( x \in dom(\sigma) \). We make the following observations:

- \( q \vdash E_n = A \) and \( q' \supseteq q \) implies \( q' \vdash E_n = A \).
- The set \( \{ \sigma \in 2^{\omega} : \sigma \text{ has a } q \text{-computation} \} \) is r.e. (uniformly in \( q \)).
- By definition, \( q \vdash E_n = A \) if and only if any \( \sigma \in 2^{\omega} \) has a \( q \)-computation if and only if \( \sigma \subseteq \chi_A \).
- Since \( p, e \) do not satisfy P1, for any \( q' \supseteq q \) there is a \( \sigma \in 2^{\omega} \) which has a \( q' \)-computation.
- The previous two items imply that the predicate \( q \supseteq p \Rightarrow (\forall A)(q \not\vdash E_n = A) \) in \( q \) and \( n \) is r.e.
- Since \( p, e \) do not satisfy P3, if \( q \supseteq p \) and \( q \not\vdash E_n = A \) for any \( A \in F \), then for any \( \sigma \in 2^{\omega} \) with a \( q \)-computation there is a set \( A \in F \) such that \( \sigma \subseteq \chi_A \).

We form a recursive enumeration \( R \) of \( F \), using pairs \((q, n)\) as indices, where \( q \in 2^{\omega} \) with \( q \supseteq p \) and \( n \in \omega \). Since there is a recursive bijection between \( P \times \omega \) and \( \omega \), where \( P = \{ q \in 2^{\omega} : q \supseteq p \} \), this suffices. Let \( (\sigma_q^{(n)})_{n \in \omega} \) be an effective enumeration of \( \{ \sigma \in 2^{\omega} : \sigma \text{ has a } q \text{-computation} \} \) by the second observation. We choose this enumeration so that, additionally, for every \( \sigma \) with a \( q \)-computation there are infinitely many \( i \) such that \( \sigma_q^{(i)} = \sigma \). Define a partial recursive function \( a_q \) by \( a_q(0) := 0 \) and \( a_q(n + 1) := \mu m > n.\sigma_q^{(m)} \supseteq \sigma_q^{(a_q(n))} \). By the fourth observation, let \( h : \omega \to 2^{\omega} \times \omega \) be a recursive function with range \( \{(q, n) : q \supseteq p \text{ and } (\forall A)(q \not\vdash E_n = A)\} \).

\( R_{(q,n)} \) is defined by
\[
R_{(q,n)} := \begin{cases} 
\bigcup_{0 \leq i \leq m} \sigma_q^{(a_q(i))} \cup \varphi_f(\sigma_q^{(a_q(m))}) & \text{if } m \text{ is least such that } h(m) = (q, n), \\
\bigcup_{i \in \omega} \sigma_q^{(a_q(i))} & \text{otherwise.}
\end{cases}
\]

Fix \( q \supseteq p \) and \( n \in \omega \). By definition of the functions \( h \), there is \( m \) satisfying the first case if and only if \( (\forall A)(q \not\vdash E_n = A) \). If \( q \vdash E_n = A \), then the choice of the enumeration \( (\sigma_q^{(n)})_{n \in 2^{\omega}} \) and the function \( a \) guarantees \( R_{(q,n)} = A \). If \( q \not\vdash E_n = A \) for any \( A \), then by the fact that \( f \) is an extension function for \( F \), it follows that \( R_{(q,n)} \in F \).

Since for every \( A \in F \) there are \( q \) and \( n \) such that \( q \vdash E_n = A \), it follows that \( R \) is an enumeration of \( F \), and Theorem 2.3 is proved.
We turn to the proof of Theorem 2.1. As mentioned above, if none of the cases holds for \( p \) and \( e \), then we have

\[ \mathcal{F} = \{ X : (\exists q \supseteq p)(\exists n)(q \vdash E_n = X) \}. \]

The ternary relation \( (q \vdash E_n = X) \land (\chi_X = \varphi_i) \) (in \( q, n \) and \( i \)), when restricted to \( i \) such that \( \varphi_i \) is the characteristic function of a set, is \( \Pi^0_2 \) so that \( I_{\mathcal{F}}(\mathcal{F}) \) is \( \Sigma^0_3 \). □

**Theorem 2.4** (Knight). Let \( \mathcal{F} \) be a family such that for all non-recursive \( X \), \( \mathcal{F} \) has an enumeration r.e. in \( X \). If \( \mathcal{F} \) contains all finite sets, then \( \mathcal{F} \) has an r.e. enumeration.

**Proof of Theorems 2.4 and 2.2.** As above, we construct a generic set \( D \), attempting to meet the following requirements and expecting to fail.

- \( R_e. \) \( W_e^D \) is not an enumeration of \( \mathcal{F} \).
- \( W \) we use the same forcing notion as was used for the previous proof. Fix \( p \in 2^{<\omega} \) and \( e \in \omega \). We consider the following three cases.

**C1.** For some \( q \supseteq p \) and some \( n \in \omega \), for all \( q' \supseteq q \), there exist \( x \in \omega \) and \( r_0, r_1 \supseteq q' \) such that \( r_0 \vdash (n, x) \in W_e^D \) and \( r_1 \vdash (n, x) \notin W_e^D \).

For each \( A \in \mathcal{F} \) the set

\[ D_1^A := \{ r : r \supseteq q \Rightarrow (\exists x)[(r \vdash (n, x) \in W_e^D \land x \notin A) \lor (r \vdash (n, x) \notin W_e^D \land x \in A)] \} \]

is dense. We put \( q \) into the c.f.s. and enter the sets \( D_1^A \). Requirement \( R_e \) is satisfied.

We write \( q \vdash E_n = A \) if for all \( x \), if \( x \in A \), then for all \( q' \supseteq q \) there is \( r \supseteq q' \) such that \( r \vdash (n, x) \in W_e^D \), and if \( x \notin A \) then \( q \vdash (n, x) \notin W_e^D \).

**C2.** Case C1 does not hold, but \( q \vdash E_n = B \) for some \( q \supseteq p \), \( n \in \omega \), and \( B \notin \mathcal{F} \). Requirement \( R_e \) is met by putting \( q \) in the c.f.s.

**C3.** Case C1 does not hold, but there exists \( A \in \mathcal{F} \) such that for all \( q' \supseteq q \) and all \( n \), if \( q' \vdash E_n = B \), then \( A \neq B \).

Since Case C1 does not hold, the set

\[ D_2^A := \{ q' : q' \supseteq q \Rightarrow (\exists B)(q' \vdash E_n = B) \} \]

is dense for all \( n \). We meet the requirement \( R_e \) by including \( q \) in the c.f.s. and entering all sets \( D_2^A \).

If for all \( p \in 2^{<\omega} \) and \( e \in \omega \) one of the cases C1,C2, or C3 holds, then the forcing construction yields a generic (and hence non-recursive) set \( D \), in which \( \mathcal{F} \) has no r.e. enumeration. This contradicts the assumption of both Theorem 2.2 and Theorem 2.4. So let \( p \in 2^{<\omega} \) and \( e \in \omega \) be such that none of the cases C1,C2,C3 holds. We show that the index set \( I_{\mathcal{F}} \) of \( \mathcal{F} \) is \( \Sigma^0_3 \). As in the proof of Theorem 2.3, for all \( q \supseteq p \), \( n \), and \( A \subseteq \omega \), if \( q \vdash E_n = A \), then \( A \in \mathcal{F} \), and all members of \( \mathcal{F} \) occur in this way. Therefore we have

\[ \{ i : W_i \in \mathcal{F} \} = \{ i : (\exists q \supseteq p)(\exists n)(q \vdash E_n = W_i) \}. \]

The relation "\( q \vdash E_n = W_i \)" is \( \Pi^0_2 \), and so the index set of \( \mathcal{F} \) is \( \Sigma^0_3 \). This completes the proof of Theorem 2.2. To complete the proof of Theorem 2.4, note that since \( \mathcal{F} \) includes all finite sets and only contains r.e. sets, by a theorem of Yates [7, Theorem 8], \( \mathcal{F} \) has an r.e. enumeration. □

3. A family of finite sets

In this section we first define a family \( C \) which has no r.e. enumeration. Then we show that for every non-recursive set \( X \) there is an enumeration of \( C \) which is recursive in \( X \), and finally that every non-recursive degree contains an enumeration of \( C \). This corrects an earlier statement in [6, p 187]. In particular, we apologize for connecting the error with Martin Kummer.

Let \( r \) be the partial recursive function defined by

\[
    r(e) := (\mu(i, x, s) : (e, x) \in \Omega_i^{(e)})_{1,1}.
\]

Informally, the value of \( r(e) \) is the first index \( i \) to be found such that there is a number of the form \( \langle e, x \rangle \in \Omega_i^{(e)} \); if there is no such \( i \) then \( r(e) \) is not defined. Let the family \( C \) be defined by

\[
    C := \{ \langle e, A \rangle : A \text{ is finite, } e \in \omega \} - \{ \langle e, \omega \rangle \cap \Omega_e^{(e)} : r(e) \downarrow \}. \tag{1}
\]

\( C \) does not have an r.e. enumeration; for suppose \( \Omega^{(e_0)} \) is an enumeration of \( C \). Then \( r(e_0) \) is defined, and the set \( \Omega_e^{(e_0)} = \langle e, \omega \rangle \cap \Omega_r^{(e_0)} \) is not a member of \( C \).

Let \( X \) be an arbitrary non-recursive set. To see how to construct an enumeration \( S^X \) of \( C \) such that \( S^X \leq_T X \) we use the following lemma.

**Lemma 3.1.** Uniformly in \( i \) and recursively in \( X \) there is a finite set \( A_i^X \leq_T X \) such that \( W_i \neq A_i^X \).

Let \( g \) be a partial recursive function such that \( g(e) \downarrow \) if and only if \( r(e) \downarrow \) and if \( r(e) \downarrow \) then \( W_{g(e)} = (\langle e, \omega \rangle \cap \Omega_e^{(e)})_2 \). Let \( h \) be a partial recursive function such that \( h(e, a) \downarrow \) if and only if \( g(e) \downarrow \) and if \( g(e) \downarrow \) then \( W_{h(e, a)} = W_{g(e)} - a \).

Define

\[
    S^X_{(n, t, e)} := \langle e, B^X(n, t, e) \rangle,
\]

where

\[
    B^X(n, t, e) := \begin{cases} 
    D_n \cup (s_0 + A^X_{h(e, s_0)}) & \text{if there is } s > t, \max(D_n) + 1 \\
    D_n & \text{such that } g_e(e) \downarrow \text{ and } W_{g(e), s} = D_n, \\
    & \text{and } s_0 \text{ is the least such}, \\
    & \text{otherwise}. 
\end{cases}
\]

By the lemma, \( S^X \) is recursive in \( X \). We claim that \( S^X \) is an enumeration of \( C \).

"\( \subseteq \)". First of all, it follows from the lemma that all sets \( B^X(n, t, e) \) are finite and so the family enumerated by \( S^X \) is contained in \( \{ \langle e, A \rangle : A \text{ is finite} \} \).

Suppose \( r(e) \downarrow \) and \( S^X_{(n, t, e)} = \langle e, \omega \rangle \cap \Omega_r^{(e)} \). It follows that \( B^X(n, t, e) = W_{g(e)} \).

There has to be a stage \( s > t, \max(D_n) + 1 \) such that \( W_{g(e), s} = D_n \), because otherwise \( B^X(n, t, e) = D_n \neq W_{g(e)} \), a contradiction. Let \( s_0 \) be the least such \( s \).

Then \( B^X(n, t, e) = D_n \cup (s_0 + A^X_{h(e, s_0)}) = W_{g(e)} \), so that \( A^X_{h(e, s_0)} = W_{h(e, s_0)} \), a contradiction.

"\( \supseteq \)". Let \( e, n \in \omega \) such that \( C = \langle e, D_n \rangle \in C \). We want to find a number \( z \) such that \( S^X_z = C \). Let \( t \) be such that \( W_{g(e), s} \neq D_n \) for all \( s > t \). (If there is no such \( t \) then \( C \notin C \).) By definition, \( B^X(n, t, e) = D_n \), so that \( S^X_{(n, t, e)} = C \).

We have constructed an enumeration of \( C \) which is recursive in \( X \). Simple coding is sufficient to obtain an enumeration \( T \)-equivalent to \( X \): Choose \( A, B \in C \) so that
\[ A - B \neq \emptyset. \] Define another enumeration of \( C \),
\[
P^X_{2n} := S^X_n, \quad P^X_{2n+1} := \begin{cases} A & \text{if } n \in X, \\ B & \text{if } n \notin X. \end{cases}
\]

Clearly, \( P^X \) is recursive in \( X \) and enumerates \( C \). Let \( x \in A - B \). Then \( z \in X \) if and only \( x \in P^X_{2i+1} \), so that \( X \) is recursive in \( P^X \).

**Proof of Lemma 3.1.** Let \( y, n : \omega^2 \to \omega \) be two recursive one-one functions such that their ranges are disjoint and cover \( \omega \). Let
\[
\alpha(s, x) := \begin{cases} y(s, x) & \text{if } x \in X, \\ n(s, x) & \text{otherwise}, \end{cases}
\]

so that \( \alpha \leq_T X \). We construct the set \( A^X_i \) in stages uniformly in \( i \) and recursively in \( X \). In the course of the construction, numbers \( a_x \) may become defined.

**Stage 0.** Set \( x := 0 \). \( A^X_{i,0} \) is empty.

**Stage \( s + 1 \).** If \( W_{i,s} \neq A^X_{i,s} \), pass to the next stage. Otherwise enumerate \( \alpha(s, x) \) in \( A^X_i \), let \( a_x := s \) and increase \( x \) by one.

End of construction.

The set \( A^X_i \) is (by construction) r.e. in \( X \). It is also recursive in \( X \): By inspection, \( A^X_i \) only contains numbers \( \alpha(a, b) \). If \( y(s, x) \in A^X_i \), then \( y(s, x) \in A^X_{i,s+1} \) and the same holds for numbers \( n(s, x) \). This together with the choice of \( y \) and \( n \) is sufficient.

**Claim 1.** The set \( A^X_i \) is finite.

Suppose \( A^X_i \) is infinite. During the construction of \( A^X_i \) infinitely many numbers \( a_x \) are defined. By induction on \( x \), it follows that \( A^X_{i,a_x} = \{ \alpha(a_j, j) : j \leq x \} \), and therefore \( W_{i,a_x} = \{ \alpha(a_j, j) : j < x \} \) for all \( x \in \omega \). Thus, \( W_i = \{ \alpha(a_x, x) : x \in \omega \} \).

Now, \( x \in X \) if and only if \( (\exists t)(y(t, x) \in W_i) \), and \( x \notin X \) if and only if \( (\exists t)(n(t, x) \in W_i) \). This means that \( X \) is recursive, a contradiction.

**Claim 2.** \( A^X_i \) is different from \( W_i \).

By the previous claim it is sufficient to consider the case when \( W_i \) is finite. Let \( s_0 \) be the least number such that \( W_{i,s_0} = W_i \). Then either \( W_{i,s_0} \neq A^X_{i,s_0} \), in which case \( W_{i,s_0} \neq A^X_{i,s_0} = A^X_i \); or \( W_{i,s_0} = A^X_{i,s_0} \), so that at stage \( s_1 = s_0 + 1 \) a new number is enumerated in \( A^X_i \), whence \( W_i = W_{i,s_1} = A^X_{i,s_0} \neq A^X_i \).

Lemma 3.1 is proved.

4. **Application to Lempp’s Question**

Let \( F \) be a family. With \( F \) we associate the following countable structure \( \mathfrak{A}_F \) in the language \( L = (S, Z, I) \), where \( S \) is a binary predicate symbol and \( Z \) and \( I \) are unary predicate symbols. The universe of \( \mathfrak{A}_F \) is \( F \times \omega \times \omega \). For every \( A \in F \), set \( Z((A, x, 0)) \) and \( S((A, x, n), (A, x, n+1)) \). Set \( I((A, x, n)) \) if and only if \( n \in A \). Thus, countably many \( S \)-chains \( (A, x, 0), (A, x, 1), (A, x, 2), \ldots \) are associated with every \( A \in F \), and in every chain \( I \) holds of the \( n \)-th member if \( n \in A \).

**Theorem 4.1.** Let \( d \) be a Turing degree and \( F \) be a family. Then \( F \) has an enumeration recursive in \( d \) if and only if the structure \( \mathfrak{A}_F \) has a representation recursive in \( d \).
Proof. “⇒”. Let \( Q^X \) be an enumeration of \( \mathcal{F} \) which is recursive in \( X \in \mathfrak{d} \). Define \( R^X \) to be another \( X \)-recursive enumeration of \( \mathcal{F} \) by \( R^X_{(n,i)} := Q^X_{(n,i)} \). A representation of \( \mathfrak{A}_\mathcal{F} \) is given by \( \mathfrak{B} \), where \( Z_{\mathfrak{B}}(x,0) \), \( S_{\mathfrak{B}}(x,n), \langle x, n + 1 \rangle \), and \( I_{\mathfrak{B}}(x,n) \) if and only if \( x \in R^X_n \). These predicates are recursive in \( R^X \), and therefore \( D(\mathfrak{B}) \) is recursive in \( X \).

“⇐”. Let \( \mathfrak{B} \) be a representation of \( \mathfrak{A}_\mathcal{F} \) whose open diagram is recursive in \( X \in \mathfrak{d} \). The set \( Z := \{ x : Z_{\mathfrak{B}}(x) \} \) is recursive in \( X \). With each \( x \in Z \) we associate the set \( S_x := \{ n : I_{\mathfrak{B}}(f^{(n)}(x)) \} \), where \( f(x) \) is the unique \( y \) such that \( S_{\mathfrak{B}}(x,y) \), and \( f^{(n)}(x) \) denotes the \( n \)-fold application of \( f \) to \( x \). The sets \( S_x \) are uniformly in \( x \in Z \) recursive in \( X \), and, by definition of \( \mathfrak{A}_\mathcal{F} \), \( \mathcal{F} = \{ S_x : x \in Z \} \). This suffices.

\( \square \)

**Corollary 4.2** (Slaman [4]). There is a structure which has representations only in the non-recursive degrees.

Proof. Apply the theorem to the family \( \mathcal{C} \) defined in the previous section to obtain representations of \( \mathfrak{A}_\mathcal{C} \) below any non-recursive \( T \)-degree. Apply Theorem 1.1 to obtain representations in all non-recursive \( T \)-degrees.

\( \square \)

Slaman remarked (private communication) that the construction given in [4] yields a structure which is not elementarily equivalent to any recursively represented structure. Informally, the reason for this is as follows. Essentially, the construction proceeds in such a way that the final outcome of the actions taken to diagonalize against the recursive representations can be read off the theory of the structure (which is called \( \mathfrak{M} \)).

He writes ([4, Section 2.1]): “We will ensure that either \( R^{-1}(T_i)^{\mathfrak{M}} = \emptyset \), or \( \langle T_i, <_L \mid T_i \rangle^{\mathfrak{M}} \) is not isomorphic to \( \langle T_i, <_L \mid T_i \rangle^{\mathfrak{M}} \), or there is a \( p \) in \( R^{-1}(T_i)^{\mathfrak{M}} \) such that \( \zeta(p)^{\mathfrak{M}} \) is not maximal, or there is a \( p \) in \( R^{-1}(T_i)^{\mathfrak{M}} \) such that \( \zeta(p)^{\mathfrak{M}} \) is infinite. Since none of these disjuncts apply to \( \mathfrak{M} \), we will thus ensure that \( \mathfrak{M} \) has no recursive presentation."

Fix \( i \in \omega \). The first and third disjunct can be directly formalized in the language \( \mathcal{L} \) of the structure. If the second disjunct holds, then the construction [4, Section 2.2.1] yields a tree \( \langle T_i, <_L \mid T_i \rangle^{\mathfrak{M}} \) which is finite. Therefore this tree is described by a sentence in the theory of \( \mathfrak{M} \). The fourth disjunct also cannot be formalized in \( \mathcal{L} \), but, provided the first three disjuncts are not true, the strategy used in [4, Section 2.2.2] results in

\[
(\exists p)(R(p,s^1(0)) \land (\forall x)(R(p,x) \rightarrow (\exists y)(x <_T y \land R(p,y))))
\]

being true in \( \mathfrak{R}_i \). This is a formula in \( \mathcal{L} \), and (by the same strategy) not true in \( \mathfrak{M} \). Hence there is no recursively represented structure which is elementarily equivalent to \( \mathfrak{M} \).

The structure obtained from the family \( \mathcal{C} \) by Theorem 4.1 is of a different kind:

**Theorem 4.3.** There is a structure which has representations only in the non-recursive degrees and has a recursively represented elementary extension.

Proof. Let us look at the following family:

\[
\mathcal{D} := \{(e,A) : A \text{ is finite}, e \in \omega \}.
\]

Obviously, \( \mathcal{D} \) has a recursive enumeration, and \( \mathfrak{A}_\mathcal{D} \) has a recursive representation. At the same time, the structure \( \mathfrak{A}_\mathcal{C} \) from the proof of Corollary 4.2 is contained in \( \mathfrak{A}_\mathcal{D} \), and they are elementarily equivalent.
It suffices to show that for any formula \((\exists x)(\phi)\) in variables \(x_1, \ldots, x_n\), if 
\[ \mathcal{A}_D \models (\exists x)(\phi)[x_1 := c_1, \ldots, x_n := c_n] \]
and \(c_1, \ldots, c_n \in \mathcal{A}_C\), then there is \(c \in \mathcal{A}_C\) such that 
\[ \mathcal{A}_D \models \phi[x := c, x_1 := c_1, \ldots, x_n := c_n]. \]
Note that if \(\mathcal{A}_D \models (\exists x)(\phi)[x_1 := c_1, \ldots, x_n := c_n]\), then there is a \(d\) in \(\mathcal{A}_D\) such that either 
\[ \mathcal{A}_D \models \phi[x := d, x_1 := c_1, \ldots, x_n := c_n] \]
\[ \land \neg S(d, c_1) \land \neg S(c_1, d) \land \cdots \land \neg S(d, c_n) \land \neg S(c_n, d), \]
or 
\[ \mathcal{A}_D \models \phi[x := d, x_1 := c_1, \ldots, x_n := c_n] \]
\[ \land (S(d, c_1) \lor S(c_1, d) \lor \cdots \lor S(d, c_n) \lor S(c_n, d)). \]
In the former case, choose \(c\) from \(\mathcal{A}_C\) outside of the \(S\)-chains of \(\mathcal{A}_C\) which \(c_1, \ldots, c_n\) belong to such that it satisfies \(Z\) and \(I\) in the same way \(d\) does. In the latter case, \(d\) is already part of an \(S\)-chain which is contained in \(\mathcal{A}_C\), and so an element of \(\mathcal{A}_C\). □

References


Department of Chemistry, University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z1

E-mail address: stephan@pepe.chem.ubc.ca