ENumerations, Countable Structures
And Turing Degrees

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Abstract. It is proven that there is a family of sets of natural numbers which
has enumerations in every Turing degree except for the recursive degree. This
implies that there is a countable structure which has representations in all but
the recursive degree. Moreover, it is shown that there is such a structure which
has a recursively represented elementary extension.

1. Introduction

In the following we are concerned with countable structures in a recursive lan-
guage. Researchers have investigated how one could measure the intuitive idea of
information content of such structures and tried to relate each one of them to a
Turing degree [2], [3], [1]. The natural starting point is to look at the collection of
representations. Let \( \mathfrak{A} \) be a structure. If \( \mathfrak{B} \) is an isomorphic structure with universe
\( \omega \), then \( \mathfrak{B} \) is called a representation of \( \mathfrak{A} \) (written \( \mathfrak{B} \simeq \mathfrak{A} \)). \( D(\mathfrak{B}) \), its open diagram,
can be regarded as a subset of \( \omega \) so that it has a Turing degree, and one can look
at the collection of degrees \( \{ \deg(D(\mathfrak{B})) : \mathfrak{B} \simeq \mathfrak{A} \} \). A first guess for capturing the
complexity of \( \mathfrak{A} \) would be to let its degree be the least element of this collection,
especially in the light of the following theorem [2, Theorem 4.1]:

**Theorem 1.1** (Knight). Let \( \mathfrak{A} \) be a structure in a relational language. Then ex-
actly one of the following holds: (1) For any \( d > \deg(D(\mathfrak{A})) \), there is a representa-
tion \( \mathfrak{B} \) of \( \mathfrak{A} \) such that \( \deg(D(\mathfrak{B})) = d \). (2) There is a finite subset \( S \) of the universe
of \( \mathfrak{A} \) such that all permutations of the universe which fix \( S \) are automorphisms of
\( \mathfrak{A} \).

But this idea fails. For example, Richter [3, Theorem 3.3] shows that for any
countable order \( \mathfrak{C} \) which has no recursive representation the collection \( \{ \deg(D(\mathfrak{B})) : \mathfrak{B} \simeq \mathfrak{C} \} \) has no least element. Therefore more involved concepts have been tried to
assign degrees to structures [1].

Now for the particular problems addressed in this paper. Steffen Lempp asked
(unpublished): Does a structure with representations in all non-recursive degrees
have a recursive representation? Julia Knight asked some related questions: With
a binary relation \( R \subseteq \omega^2 \) associate a family of subsets of \( \omega \) given by
\( \mathcal{F}_R := \{ R_n : n \in \omega \} \), where \( R_n := \{ x : (n, x) \in R \} \); say that \( R \) is an enumeration of \( \mathcal{F}_R \).
asked (also unpublished): If a family $F$ has the feature that for every non-recursive set $X$, $F$ has an enumeration recursive in $X$, does $F$ have a recursive enumeration? Similarly, if for every non-recursive set $X$, $F$ has an enumeration r.e. in $X$, does $F$ have an r.e. enumeration?

In the next section we give some positive results on Knight’s questions under extra hypotheses. In Section 3 we prove that the answer to Knight’s questions is negative, by constructing a single suitable family. This implies that Lempp’s question also has a negative answer, as is shown in the last section. The same finding is obtained in [4], but by another approach. We close by discussing the difference.

The notation is quite standard and follows [5]. All sets considered are subsets of $\omega$, the set of natural numbers. We call countable collections of subsets of $\omega$ families. Let $\varphi$ be a Gödel-numbering of the partial recursive functions (of varying arity) and $W_i := \text{Rng}(\varphi_i)$ be an enumeration of the recursively enumerable sets as usual. $W_{i,s}$ is the set of numbers enumerated in $W_i$ by stage $s$. We use recursive bijections $\langle \cdot, \cdot \rangle$ between $\omega$ and $\omega^2$, $\omega$ and $\omega^3$, respectively. We also use projections $(\cdot)_1$ and $(\cdot)_2$ so that, for example, $((a,b))_1 = a$. Let $\langle x, A \rangle$ be the set $\{\langle x, a \rangle : a \in A \}$. Similarly, $A_2 := \{(a)_2 : a \in A \}$. We let $A + x = \{a + x : a \in A \}$ and $A - x = \{b : b + x \in A \}$.

Fix an effective listing $\Omega$ of recursively enumerable enumerations of all families with r.e. enumerations:

$$\Omega^{(e)} := \{(i, x) : \langle i, x \rangle \in W_e \},$$

and write $\Omega_i^{(e)}$ for $\{x : \langle i, x \rangle \in W_e \}$, the $i$-th set of the enumeration $\Omega^{(e)}$. Define $C^{(e)} := \{\Omega_i^{(e)} : i \in \omega \}$ to be the family enumerated by $\Omega^{(e)}$.

$D : \omega \to 2^{\omega}$ denotes the canonical enumeration of the family of finite sets; write $D_n$ for the $n$-th finite set. Then the binary predicates $x \in D_n$ and $x = |D_n|$ are recursive.

2. Positive results

In this section we give conditions on a family which ensure that the implications of Knight’s two questions hold. These are given in Theorems 2.3 and 2.4 and are due to Julia Knight. Jockusch gave a proof of Theorem 2.4 which also showed the following: There is a property which families with a recursive (r.e.) enumeration share with families that have, for all non-recursive degrees $d$, an enumeration recursive (r.e.) in $d$. These seem to be the only positive statements possible about such families.

What is this property? By a rather straightforward forcing construction it follows that if a set of natural numbers is recursive (r.e.) in all non-recursive degrees, then it is recursive (r.e.). Therefore, the members of a family are recursive (recursively enumerable) if the family has, for all non-recursive degrees $d$, an enumeration recursive (r.e.) in $d$. Hence such families $F$ are fully described by the index set

$$I_r(F) := \{i : (\exists A \in F)(\varphi_i = \chi_A)\}$$

or, respectively,

$$I_{re}(F) := \{i : (\exists A \in F)(W_i = A)\}.$$
Both the index set $I_r$ of a family with a recursive enumeration and the index set $I_{re}$ of a family with a recursively enumerable enumeration are $\Sigma^0_3$ in the arithmetical hierarchy. Here is the coincidence:

**Theorem 2.1.** Let $\mathcal{F}$ be a family. If, for every non-recursive degree $d$, $\mathcal{F}$ has an enumeration recursive in $d$, then its index set $I_r(\mathcal{F})$ is $\Sigma^0_3$.

**Theorem 2.2** (Jockusch). Let $\mathcal{F}$ be a family. If, for every non-recursive degree $d$, $\mathcal{F}$ has an enumeration recursively enumerable in $d$, then its index set $I_{re}(\mathcal{F})$ is $\Sigma^0_3$.

Towards giving a sufficient condition under which the implication of her first question holds, Julia Knight defines an extension function for a family $\mathcal{F}$ to be a (possibly partial) function $f \colon 2^{<\omega} \to \omega$ such that if $\sigma \in 2^{<\omega}$ and there exists a set $A \in \mathcal{F}$ such that $\chi_A \supseteq \sigma$, then $\varphi_{f(\sigma)} = \chi_A$ for some such $A$. We mention two facts: Any family with a recursive enumeration has a partial recursive extension function, and so does a family containing all finite sets.

We prove Theorems 2.1 and 2.2 simultaneously with the following two.

**Theorem 2.3** (Knight). Let $\mathcal{F}$ be a family which has, for all non-recursive $d$, an enumeration recursive in $d$. If $\mathcal{F}$ has a partial recursive extension function, then $\mathcal{F}$ has a recursive enumeration.

**Proof (of Theorems 2.3 and 2.1).** Let $\mathcal{F}$ be a family which has, for any non-recursive set $X$, an enumeration recursive in $X$. We construct a generic set $D$, attempting to meet the following requirements and expecting to fail. Below, this failure will be exploited to prove the statements of the two theorems for $\mathcal{F}$ separately.

$$R_e \colon \varphi_e^D$$ is not the characteristic function of an enumeration of $\mathcal{F}$.

The set $D$ will be Cohen-generic. The set of forcing conditions is $2^{<\omega}$ and the partial order is given by $\subseteq$. We use the old-fashioned notion of a complete forcing sequence (c.f.s.), where $p_{n+1} \supseteq p_n$, with $p_{n+1}$ entering the $n$th dense set in some countable collection. $D$ is the set with characteristic function $\bigcup_{n \in \omega} p_n$.

Fix a condition $p \in 2^{<\omega}$ and $e \in \omega$. We consider the following four possibilities for $p$ and $e$, showing how extensions of $p$ may force satisfaction of $R_e$ in each case.

**P1.** For some $q \supseteq p$, $n, x \in \omega$, $q \vdash \varphi_e^D(n, x) \upharpoonright 0 \neq 0, 1$, or $q \vdash \varphi_e^D(n, x) \downarrow$.

We include $q$ in the c.f.s., thereby satisfying $R_e$.

**P2.** For some $q \supseteq p$ and some $n$, for all $q' \supseteq q$ there exist $x$ and $r_0, r_1 \supseteq q'$ such that $r_i \vdash \varphi_e^D(n, x) = i$.

For each $A \in \mathcal{F}$ the set

$$D^1_A := \{ r : r \supseteq q \Rightarrow (\exists x)(r \vdash \varphi_e^D(n, x) \neq \chi_A(x)) \}$$

is dense. We add $q$ to the c.f.s. and enter the sets $D^1_A$. Then the requirement $R_e$ is satisfied.

We write $q \vdash E_n = A$ if for all $x$ and all $q' \supseteq q$ there is an $r \supseteq q'$ such that $r \vdash \varphi_e^D(n, x) \downarrow = \chi_A(x)$. Note that for $q$ and $n$ there is at most one set $A$ such that $q \vdash E_n = A$.

**P3.** For some $q \supseteq p$, $n \in \omega$ and $B \notin \mathcal{F}$ we have $q \vdash E_n = B$.

By putting $q$ into the c.f.s., we meet $R_e$.

**P4.** Not P2 but there exist $A \in \mathcal{F}$ and $q \supseteq p$ such that for all $q' \supseteq q$ and all $n$, if $q' \vdash E_n = B$ then $A \neq B$. 


Since P2 does not hold, the set
\[
D_n^2 := \{q' : q' \supseteq q \Rightarrow (\exists B)(q' \vdash E_n = B)\}
\]
is dense for each \(n\). By including \(q\) in the c.f.s. and entering the sets \(D_n^2\), we meet \(R_n\).

If for all \(p \in 2^{<\omega}\) and \(e \in \omega\) one of the cases P1, ..., P4 holds, then the forcing construction yields a generic (and so non-recursive) set \(D\), in which \(\mathcal{F}\) has no recursive enumeration, contrary to the assumption on \(\mathcal{F}\).

So let \(p \in 2^{<\omega}\) and \(e \in \omega\) be such that none of the cases P1, ..., P4 hold. It follows that if \(q \vdash E_n = A\), then \(A \in \mathcal{F}\) for all \(q \supseteq p\), \(n\) and \(A \subseteq \omega\). Moreover, all elements of \(\mathcal{F}\) occur in this way.

We first complete the proof of Theorem 2.3. Let \(f\) be a partial recursive extension function for \(\mathcal{F}\). Fix \(n \in \omega\) and \(p \subseteq q \in 2^{<\omega}\). Say that \(\sigma \in 2^{<\omega}\) has a \(q\)-computation if there is an \(r \supseteq q\) such that \(r \vdash \varphi^D_e(n, x) = \sigma(x)\) for all \(x \in \text{dom}(\sigma)\). We make the following observations:

- \(q \vdash E_n = A\) and \(q' \supseteq q\) implies \(q' \vdash E_n = A\).
- The set \(\{\sigma \in 2^{<\omega} : \sigma\ \text{has a q-computation}\}\) is r.e. (uniformly in \(q\)).
- By definition, \(q \vdash E_n = A\) if and only if any \(\sigma \in 2^{<\omega}\) has a \(q\)-computation if and only if \(\sigma \subseteq \chi_A\).
- Since \(p, e\) do not satisfy P1, for any \(q' \supseteq q\) there is a \(\sigma \in 2^{<\omega}\) which has a \(q'\)-computation.
- The previous two items imply that the predicate \(q \supseteq p \Rightarrow (\forall A)(q \not\vdash E_n = A)\) in \(q\) and \(n\) is r.e.
- Since \(p, e\) do not satisfy P3, if \(q \supseteq p\) and \(q \not\vdash E_n = A\) for any \(A \in \mathcal{F}\), then for any \(\sigma \in 2^{<\omega}\) with a \(q\)-computation there is a set \(A \in \mathcal{F}\) such that \(\sigma \subseteq \chi_A\).

We form a recursive enumeration \(R\) of \(\mathcal{F}\), using pairs \((q, n)\) as indices, where \(q \in 2^{<\omega}\) with \(q \supseteq p\) and \(n \in \omega\). Since there is a recursive bijection between \(P \times \omega\) and \(\omega\), where \(P = \{q \in 2^{<\omega} : q \supseteq p\}\), this suffices. Let \((\sigma^{(n)}_q)_{n \in \omega}\) be an effective enumeration of \(\{\sigma \in 2^{<\omega} : \sigma\ \text{has a q-computation}\}\) by the second observation. We choose this enumeration so that, additionally, for every \(\sigma\) with a \(q\)-computation there are infinitely many \(i\) such that \(\sigma^{(i)}_q = \sigma\). Define a partial recursive function \(a_q\) by \(a_q(0) := 0\) and \(a_q(n + 1) := \mu m > n.\sigma^{(m)}_q \supseteq \sigma^{(a_q(n))}_q\). By the fourth observation, let \(h : \omega \rightarrow 2^{<\omega} \times \omega\) be a recursive function with range \(\{(q, n) : q \supseteq p\} \subseteq (\forall A)(q \not\vdash E_n = A)\).

\(R_{(q, n)}\) is defined by
\[
R_{(q, n)} := \begin{cases} 
\bigcup_{0 \leq i \leq m} \sigma^{(a_q(i))}_q \cup \varphi_{f(\sigma^{(a_q(n))}_q)} & \text{if } m \text{ is least such that } h(m) = (q, n), \\
\bigcup_{i \in \omega} \sigma^{(a_q(i))}_q & \text{otherwise.}
\end{cases}
\]
Fix \(q \supseteq p\) and \(n \in \omega\). By definition of the functions \(h\), there is \(m\) satisfying the first case if and only if \((\forall A)(q \not\vdash E_n = A)\). If \(q \vdash E_n = A\), then the choice of the enumeration \((\sigma^{(n)}_q)_{n \in 2^{<\omega}}\) and the function \(a\) guarantees \(R_{(q, n)} = A\). If \(q \not\vdash E_n = A\) for any \(A\), then by the fact that \(f\) is an extension function for \(\mathcal{F}\), it follows that \(R_{(q, n)} \in \mathcal{F}\).

Since for every \(A \in \mathcal{F}\) there are \(q\) and \(n\) such that \(q \vdash E_n = A\), it follows that \(R\) is an enumeration of \(\mathcal{F}\), and Theorem 2.3 is proved.
We turn to the proof of Theorem 2.1. As mentioned above, if none of the cases holds for $p$ and $e$, then we have

$$\mathcal{F} = \{ X : (\exists q \supseteq p)(\exists n)(q \vdash E_n = X) \}.$$  

The ternary relation $(q \vdash E_n = X) \land (\forall X = \varphi_i)$ (in $q$, $n$ and $i$), when restricted to $i$ such that $\varphi_i$ is the characteristic function of a set, is $\Pi^0_2$ so that $I_e(\mathcal{F})$ is $\Sigma^0_3$. □

**Theorem 2.4** (Knight). Let $\mathcal{F}$ be a family such that for all non-recursive $X$, $\mathcal{F}$ has an enumeration r.e. in $X$. If $\mathcal{F}$ contains all finite sets, then $\mathcal{F}$ has an r.e. enumeration.

**Proof of Theorems 2.4 and 2.2.** As above, we construct a generic set $D$, attempting to meet the following requirements and expecting to fail.

$R_e$. $W^D_e$ is not an enumeration of $\mathcal{F}$.

We use the same forcing notion as was used for the previous proof. Fix $p \in 2^{<\omega}$ and $e \in \omega$. We consider the following three cases.

**C1.** For some $q \supseteq p$ and some $n \in \omega$, for all $q' \supseteq q$, there exist $x \in \omega$ and $r_0, r_1 \supseteq q'$ such that $r_0 \vdash (n, x) \in W^D_e$ and $r_1 \vdash (n, x) \not\in W^D_e$.

For each $A \in \mathcal{F}$ the set

$$D^1_A := \{ r : r \supseteq q \Rightarrow (\exists x)[(r \vdash (n, x) \in W^D_e \land x \not\in A) \lor (r \vdash (n, x) \not\in W^D_e \land x \in A)] \}$$

is dense. We put $q$ into the c.f.s. and enter the sets $D^1_A$. Requirement $R_e$ is satisfied.

We write $q \vdash E_n = A$ if for all $x$, if $x \in A$, then for all $q' \supseteq q$ there is $r \supseteq q'$ such that $r \vdash (n, x) \in W^D_e$, and if $x \not\in A$ then $q \vdash (n, x) \not\in W^D_e$.

**C2.** Case C1 does not hold, but $q \vdash E_n = B$ for some $q \supseteq p$, $n \in \omega$ and $B \not\in \mathcal{F}$.

Requirement $R_e$ is met by putting $q$ in the c.f.s.

**C3.** Case C1 does not hold, but there exists $A \in \mathcal{F}$ such that for all $q' \supseteq q$ and all $n$, if $q' \vdash E_n = B$, then $A \not= B$.

Since Case C1 does not hold, the set

$$D^2_n := \{ q' : q' \supseteq q \Rightarrow (\exists B)(q' \vdash E_n = B) \}$$

is dense for all $n$. We meet the requirement $R_e$ by including $q$ in the c.f.s. and entering all sets $D^2_n$.

If for all $p \in 2^{<\omega}$ and $e \in \omega$ one of the cases C1,C2, or C3 holds, then the forcing construction yields a generic (and hence non-recursive) set $D$, in which $\mathcal{F}$ has no r.e. enumeration. This contradicts the assumption of both Theorem 2.2 and Theorem 2.4. So let $p \in 2^{<\omega}$ and $e \in \omega$ be such that none of the cases C1,C2,C3 holds. We show that the index set $I_{re}$ of $\mathcal{F}$ is $\Sigma^0_3$. As in the proof of Theorem 2.3, for all $q \supseteq p$, $n$, and $A \subseteq \omega$, if $q \vdash E_n = A$, then $A \in \mathcal{F}$, and all members of $\mathcal{F}$ occur in this way. Therefore we have

$$\{ i : W_i \in \mathcal{F} \} = \{ i : (\exists q \supseteq p)(\exists n)(q \vdash E_n = W_i) \}.$$ 

The relation “$q \vdash E_n = W_i$” is $\Pi^0_2$, and so the index set of $\mathcal{F}$ is $\Sigma^0_3$. This completes the proof of Theorem 2.2. To complete the proof of Theorem 2.4, note that since $\mathcal{F}$ includes all finite sets and only contains r.e. sets, by a theorem of Yates [7, Theorem 8], $\mathcal{F}$ has an r.e. enumeration. □
3. A FAMILY OF FINITE SETS

In this section we first define a family $\mathcal{C}$ which has no r.e. enumeration. Then we show that for every non-recursive set $X$ there is an enumeration of $\mathcal{C}$ which is recursive in $X$, and finally that every non-recursive degree contains an enumeration of $\mathcal{C}$. This corrects an earlier statement in [6, p 187]. In particular, we apologize for connecting the error with Martin Kummer.

Let $r$ be the partial recursive function defined by

$$r(e) := (\mu(i,x,s). \langle e, x \rangle \in \Omega_{i,s}^{(e)}).$$

Informally, the value of $r(e)$ is the first index $i$ to be found such that there is a number of the form $\langle e, x \rangle \in \Omega_{i,s}^{(e)}$; if there is no such $i$ then $r(e)$ is not defined. Let the family $\mathcal{C}$ be defined by

$$\mathcal{C} := \{ \langle e, A \rangle : A \text{ is finite, } e \in \omega \} - \{ \langle e, \omega \rangle \cap \Omega_{r(e)}^{(e)} : r(e) \downarrow \}.$$ 

$\mathcal{C}$ does not have an r.e. enumeration; for suppose $\Omega^{(e_0)}$ is an enumeration of $\mathcal{C}$. Then $r(e_0)$ is defined, and the set $\Omega_{r(e_0)}^{(e_0)} = \langle e, \omega \rangle \cap \Omega_{r(e_0)}^{(e_0)}$ is not a member of $\mathcal{C}$.

Let $X$ be an arbitrary non-recursive set. To see how to construct an enumeration $S^X$ of $\mathcal{C}$ such that $S^X \leq_T X$ we use the following lemma.

**Lemma 3.1.** Uniformly in $i$ and recursively in $X$ there is a finite set $A_i^X \leq_T X$ such that $W_i \neq A_i^X$.

Let $g$ be a partial recursive function such that $g(e) \downarrow$ if and only if $r(e) \downarrow$ and if $r(e) \downarrow$ then $W_{g(e)} = (\langle e, \omega \rangle \cap \Omega_{r(e)}^{(e)}).2$. Let $h$ be a partial recursive function such that $h(e,a) \downarrow$ if and only if $g(e) \downarrow$ and if $g(e) \downarrow$ then $W_{h(e,a)} = W_{g(e)} - a$.

Define

$$S^X_{(n,t,e)} := \langle e, B^X(n,t,e) \rangle,$$

where

$$B^X(n,t,e) := \begin{cases} D_n \cup (s_0 + A^X_{h(e,s_0)}) & \text{if there is } s > t, \max(D_n) + 1 \text{ such that } g_s(e) \downarrow \text{ and } W_{g(e),s} = D_n, \\ D_n & \text{and } s_0 \text{ is the least such,} \\ \text{otherwise.} \end{cases}$$

By the lemma, $S^X$ is recursive in $X$. We claim that $S^X$ is an enumeration of $\mathcal{C}$.

"$\subseteq$". First of all, it follows from the lemma that all sets $B^X(n,t,e)$ are finite and so the family enumerated by $S^X$ is contained in $\{ \langle e, A \rangle : A \text{ is finite} \}$.

Suppose $r(e) \downarrow$ and $S^X_{(n,t,e)} = \langle e, \omega \rangle \cap \Omega_{r(e)}^{(e)}$. It follows that $B^X(n,t,e) = W_{g(e)}$. There has to be a stage $s > t, \max(D_n) + 1$ such that $W_{g(e),s} = D_n$, because otherwise $B^X(n,t,e) = D_n \neq W_{g(e)}$, a contradiction. Let $s_0$ be the least such $s$. Then $B^X(n,t,e) = D_n \cup (s_0 + A^X_{h(e,s_0)}) = W_{g(e)}$, so that $A^X_{h(e,s_0)} = W_{h(e,s_0)}$, a contradiction.

"$\supseteq$". Let $e, n \in \omega$ such that $C = \langle e, D_n \rangle \in \mathcal{C}$. We want to find a number $z$ such that $S^X_z = C$. Let $t$ be such that $W_{g(e),s} \neq D_n$ for all $s > t$. (If there is no such $t$ then $C \notin \mathcal{C}$.) By definition, $B^X(n,t,e) = D_n$, so that $S^X_{(n,t,e)} = C$.

We have constructed an enumeration of $\mathcal{C}$ which is recursive in $X$. Simple coding is sufficient to obtain an enumeration $T$-equivalent to $X$: Choose $A, B \in \mathcal{C}$ so that
A - B \neq \emptyset. Define another enumeration of \( C \),

\[
P^X_{2n} := S^X_n, \quad P^X_{2n+1} := \begin{cases} A & \text{if } n \in X, \\ B & \text{if } n \notin X. \end{cases}
\]

Clearly, \( P^X \) is recursive in \( X \) and enumerates \( C \). Let \( x \in A - B \). Then \( z \in X \) if and only \( x \in P^X_{2n+1} \), so that \( X \) is recursive in \( P^X \).

**Proof of Lemma 3.1.** Let \( y, n : \omega^2 \to \omega \) be two recursive one-one functions such that their ranges are disjoint and cover \( \omega \). Let

\[
\alpha(s,x) := \begin{cases} y(s,x) & \text{if } x \in X, \\ n(s,x) & \text{otherwise,} \end{cases}
\]

so that \( \alpha \leq_T X \). We construct the set \( A^X_i \) in stages uniformly in \( i \) and recursively in \( X \). In the course of the construction, numbers \( a_x \) may become defined.

Stage 0. Set \( x := 0. \ A^X_{i,0} \) is empty.

Stage \( s + 1 \). If \( W_i,s \neq A^X_{i,s} \), pass to the next stage. Otherwise enumerate

\[
\alpha(s,x) \in A^X_i, \text{ let } a_x := s \text{ and increase } x \text{ by one.}
\]

End of construction.

The set \( A^X_i \) is (by construction) r.e. in \( X \). It is also recursive in \( X \): By inspection, \( A^X_i \) only contains numbers \( \alpha(a,b) \). If \( y(s,x) \in A^X_i \), then \( y(s,x) \in A^X_{i,s+1} \) and the same holds for numbers \( n(s,x) \). This together with the choice of \( y \) and \( n \) is sufficient.

**Claim 1.** The set \( A^X_i \) is finite.

Suppose \( A^X_i \) is infinite. During the construction of \( A^X_i \) infinitely many numbers \( a_x \) are defined. By induction on \( x \), it follows that \( A^X_{i,a_x} = \{ \alpha(a_j,j) : j \leq x \} \), and therefore \( W_{i,a_x} = \{ \alpha(a_j,j) : j < x \} \) for all \( x \in \omega \). Thus, \( W_i = \{ \alpha(a_x,x) : x \in \omega \} \).

Now, \( x \in X \) if and only if \( (\exists t)(y(t,x) \in W_i) \), and \( x \notin X \) if and only if \( (\exists t)(n(t,x) \in W_i) \). This means that \( X \) is recursive, a contradiction.

**Claim 2.** \( A^X_i \) is different from \( W_i \).

By the previous claim it is sufficient to consider the case when \( W_i \) is finite. Let \( s_0 \) be the least number such that \( W_{i,s_0} = W_i \). Then either \( W_{i,s_0} \neq A^X_{i,s_0} \), in which case \( W_{i,s_0} \neq A^X_{i,s_0} = A^X_i \); or \( W_{i,s_0} = A^X_{i,s_0} \), so that at stage \( s_1 = s_0 + 1 \) a new number is enumerated in \( A^X_i \), whence \( W_i = W_{i,s_1} = A^X_i \neq A^X_i \).

Lemma 3.1 is proved.

4. **APPLICATION TO LEMPP’S QUESTION**

Let \( \mathcal{F} \) be a family. With \( \mathcal{F} \) we associate the following countable structure \( \mathfrak{A}_\mathcal{F} \) in the language \( L = (S, Z, I) \), where \( S \) is a binary predicate symbol and \( Z \) and \( I \) are unary predicate symbols. The universe of \( \mathfrak{A}_\mathcal{F} \) is \( \mathcal{F} \times \omega \times \omega \). For every \( A \in \mathcal{F} \), set \( Z((A,x,0)) \) and \( S((A,x,n), (A,x,n+1)) \). Set \( I((A,x,n)) \) if and only if \( n \in A \).

Thus, countably many \( S \)-chains \( (A,x,0), (A,x,1), (A,x,2), \ldots \) are associated with every \( A \in \mathcal{F} \), and in every chain \( I \) holds of the \( n \)-th member if \( n \in A \).

**Theorem 4.1.** Let \( d \) be a Turing degree and \( \mathcal{F} \) be a family. Then \( \mathcal{F} \) has an enumeration recursive in \( d \) if and only if the structure \( \mathfrak{A}_\mathcal{F} \) has a representation recursive in \( d \).
Proof. “⇒”. Let \( Q^X \) be an enumeration of \( F \) which is recursive in \( X \in d \). Define \( R^X \) to be another \( X \)-recursive enumeration of \( F \) by \( R^X_{(n,i)} := Q^X_i \). A representation of \( \mathfrak{A}_F \) is given by \( \mathfrak{B} \), where \( Z_\mathfrak{B}(x,0) \), \( S_\mathfrak{B}(x,n), \langle x, n+1 \rangle \), and \( I_\mathfrak{B}(x,n) \) if and only if \( x \in R^X_n \). These predicates are recursive in \( R^X \), and therefore \( D(\mathfrak{B}) \) is recursive in \( X \).

“⇐”. Let \( \mathfrak{B} \) be a representation of \( \mathfrak{A}_F \) whose open diagram is recursive in \( X \in d \).

The set \( Z := \{ x : Z_\mathfrak{B}(x) \} \) is recursive in \( X \). With each \( x \in Z \) we associate the set \( S_x := \{ n : I_\mathfrak{B}(f^{(n)}(x)) \} \), where \( f(x) \) is the unique \( y \) such that \( S_\mathfrak{B}(x,y) \), and \( f^{(n)}(x) \) denotes the \( n \)-fold application of \( f \) to \( x \). The sets \( S_x \) are uniformly in \( Z \) recursive in \( X \), and, by definition of \( \mathfrak{A}_F \), \( F = \{ S_x : x \in Z \} \). This suffices.

\[ \square \]

**Corollary 4.2 (Slaman [4]).** There is a structure which has representations only in the non-recursive degrees.

**Proof.** Apply the theorem to the family \( \mathcal{C} \) defined in the previous section to obtain representations of \( \mathfrak{A}_C \) below any non-recursive T-degree. Apply Theorem 1.1 to obtain representations in all non-recursive T-degrees.

Slaman remarked (private communication) that the construction given in [4] yields a structure which is not elementarily equivalent to any recursively represented structure. Informally, the reason for this is as follows. Essentially, the construction proceeds in such a way that the final outcome of the actions taken to diagonalize against the recursive representations can be read off the theory of the structure (which is called \( \mathfrak{M} \)).

He writes ([4, Section 2.1]): “We will ensure that either \( R^{-1}(T_1)^{\mathfrak{M}} = \emptyset \), or \( \langle T_1, <_L | T_1 \rangle^{\mathfrak{M}} \) is not isomorphic to \( \langle T_1, <_L | T_1 \rangle_{\mathfrak{M}} \), or there is a \( p \in R^{-1}(T_1)^{\mathfrak{M}} \) such that \( \zeta(p)_{\mathfrak{M}} \) is not maximal, or there is a \( p \in R^{-1}(T_1)^{\mathfrak{M}} \) such that \( \zeta(p)_{\mathfrak{M}} \) is infinite. Since none of these disjuncts apply to \( \mathfrak{M} \), we will thus ensure that \( \mathfrak{M} \) has no recursive presentation.”

Fix \( i \in \omega \). The first and third disjunct can be directly formalized in the language \( \mathcal{L} \) of the structure. If the second disjunct holds, then the construction [4, Section 2.2.1] yields a tree \( \langle T_1, <_L | T_1 \rangle^{\mathfrak{M}} \) which is finite. Therefore this tree is described by a sentence in the theory of \( \mathfrak{M} \). The fourth disjunct also cannot be formalized in \( \mathcal{L} \), but, provided the first three disjuncts are not true, the strategy used in [4, Section 2.2.2] results in

\[ (\exists p)(R(p,s^1(0)) \land (\forall x)(R(p,x) \rightarrow (\exists y)(x <_T y \land R(p,y)))) \]

being true in \( \mathfrak{R}_i \). This is a formula in \( \mathcal{L} \), and (by the same strategy) not true in \( \mathfrak{M} \). Hence there is no recursively represented structure which is elementarily equivalent to \( \mathfrak{M} \).

The structure obtained from the family \( \mathcal{C} \) by Theorem 4.1 is of a different kind:

**Theorem 4.3.** There is a structure which has representations only in the non-recursive degrees and has a recursively represented elementary extension.

**Proof.** Let us look at the following family:

\[ D := \{ (e,A) : A \text{ is finite}, e \in \omega \}. \]

Obviously, \( D \) has a recursive enumeration, and \( \mathfrak{A}_D \) has a recursive representation. At the same time, the structure \( \mathfrak{A}_C \) from the proof of Corollary 4.2 is contained in \( \mathfrak{A}_D \), and they are elementarily equivalent:
It suffices to show that for any formula \((\exists x)(\phi)\) in variables \(x_1, \ldots, x_n\), if 
\[ \mathcal{A}_D \models (\exists x)(\phi)[x_1 := c_1, \ldots, x_n := c_n] \]
and \(c_1, \ldots, c_n \in \mathcal{A}_C\), then there is \(c \in \mathcal{A}_C\) such that 
\[ \mathcal{A}_D \models \phi[x := c, x_1 := c_1, \ldots, x_n := c_n]. \]
Note that if \(\mathcal{A}_D \models (\exists x)(\phi)[x_1 := c_1, \ldots, x_n := c_n]\), then there is a \(d \in \mathcal{A}_D\) such that either
\[ \mathcal{A}_D \models \phi[x := d, x_1 := c_1, \ldots, x_n := c_n] \]
\[ \land \lnot S(d, c_1) \land \lnot S(c_1, d) \land \ldots \land \lnot S(d, c_n) \land \lnot S(c_n, d), \]
or
\[ \mathcal{A}_D \models \phi[x := d, x_1 := c_1, \ldots, x_n := c_n] \]
\[ \land (S(d, c_1) \lor S(c_1, d) \lor \ldots \lor S(d, c_n) \lor S(c_n, d)). \]
In the former case, choose \(c\) from \(\mathcal{A}_C\) outside of the \(S\)-chains of \(\mathcal{A}_C\) which \(c_1, \ldots, c_n\) belong to such that it satisfies \(Z\) and \(I\) in the same way \(d\) does. In the latter case, \(d\) is already part of an \(S\)-chain which is contained in \(\mathcal{A}_C\), and so an element of \(\mathcal{A}_C\). 

References


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