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ON SCRAMBLED SETS AND A THEOREM OF KURATOWSKI ON INDEPENDENT SETS

HISAO KATO

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ABSTRACT. The measure of scrambled sets of interval self-maps $f : I = [0,1] \to I$ was studied by many authors, including Smítal, Misiurewicz, Bruckner and Hu, and Xiong and Yang. In this note, first we introduce the notion of "$*$-chaos" which is related to chaos in the sense of Li-Yorke, and we prove a general theorem which is an improvement of a theorem of Kuratowski on independent sets. Second, we apply the result to scrambled sets of higher dimensional cases. In particular, we show that if a map $f : I^k \to I^k$ ($k \geq 1$) of the unit k-cube I^k is $*$-chaotic on I^k, then for any $\epsilon > 0$ there is a map $g : I^k \to I^k$ such that f and g are topologically conjugate, $d(f, g) < \epsilon$ and g has a scrambled set which has Lebesgue measure 1, and hence if $k \geq 2$, then there is a homeomorphism $f : I^k \to I^k$ with a scrambled set S satisfying that S is an F_σ-set in I^k and S has Lebesgue measure 1.

1. Introduction

All spaces considered in this note are assumed to be separable and complete metric spaces. *Maps* are continuous functions. By a *compactum* we mean a compact metric space.

Let $f : X \to X$ be a map of a compactum X with metric d. A subset S of X is a *scrambled set* of f if there is a positive number $\tau > 0$ such that for any $x, y \in S$ with $x \neq y$,

1. $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \tau$,
2. $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

If there is an uncountable scrambled set S of f, then we say that f is *chaotic in the sense of Li-Yorke*. In the original paper [6] of Li and Yorke, there was the following one more condition: for any $x \in S$ and any periodic point $p \in X$, $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0$. But it is known that this condition is unnecessary.

In [6], it was proved that if a map $f : I = [0,1] \to I$ has a periodic point with period 3, then f is chaotic in the sense of Li-Yorke. The measure of scrambled sets of interval self-maps was studied by many authors, including Smítal [11], [12], Misiurewicz [7], Bruckner and Hu [1], and Xiong and Yang [13].

In this note, we prove a general theorem which is an improvement of a theorem [4] of Kuratowski on independent sets, and we apply this result to scrambled sets of higher dimensional cases.

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Let A be a closed subset of a compactum X. A map $f : X \to X$ is $*$-*chaotic* on A (in the sense of Li-Yorke) if the following conditions are satisfied:

1. there is $\tau > 0$ such that if U and V are any nonempty open subsets of A with $U \cap V = \emptyset$ and N is any natural number, then there is a natural number $n \geq N$ such that $d(f^n(x), f^n(y)) > \tau$ for some $x \in U$, $y \in V$, and
2. for any nonempty open subsets U, V of A and any $\epsilon > 0$ there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \epsilon$ for some $x \in U$, $y \in V$.

If $f : X \to X$ is $*$-chaotic on the total space X, we say that $f : X \to X$ is *everywhere* $*$-*chaotic*.

Note that if S is a scrambled set of f, then f is $*$-chaotic on $\mathrm{Cl}(S)$. A map $f : X \to X$ of a compactum X has *sensitive dependence on initial conditions* on a closed subset A of X if there is $\tau > 0$ such that if $x \in A$ and U is any neighborhood of x in A, then there is $y \in U$ such that $d(f^n(x), f^n(y)) > \tau$ for some n. Suppose that A is a closed subset of X and A has no isolated point ($= A$ is perfect). Then f is $*$-chaotic on A if and only if f has sensitive dependence on initial conditions on A and the above condition 2 is satisfied.

In the definition of chaos in the sense of Li-Yorke, we only assume the condition that there exists an uncountable scrambled set S of $f : X \to X$, and in general, S is an arbitrary subset of X. Note that if S is an uncountable set, then $\mathrm{Cl}(S)$ contains a Cantor set by the Cantor-Bendixon theorem (e.g., see [5, p. 253]).

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2. Independent sets and scrambled sets

In this section, first, we prove the following general theorem which is an improvement of a theorem of Kuratowski on independent sets (see [4]). Let X be a space and R be any subset of X^m ($m \geq 2$). A subset $F \subset X$ is said to be *independent in* R if for any different m points $x_1, \cdots, x_m$ of F (i.e., $x_i \neq x_j$ for $i \neq j$), we have $(x_1, x_2, \cdots, x_m) \in X^m - R$. For a subset $R \subset X^m$, consider the following property:

(*) If $U_1, \cdots, U_m$ are mutually disjoint nonempty sets of X, there are nonempty open sets $V_1, \cdots, V_m$ of X such that $V_i \subset U_i$ for each i and $V_1 \times \cdots \times V_m \subset X^m - R$.

Note that if X has no isolated point, $R \subset X^m$ has the property (*) if and only if R is nowhere dense (see Lemma 2.2).

A countable union of nowhere dense sets is called a set *of the first category*.

Theorem 2.1. *Suppose that X is a separable and completely metrizable space and $m \geq 2$ is a fixed natural number. Let $R \subset X^m$.*

1. *If $R = \bigcup_{n=1}^{\infty} R_n$ and each R_n has the property (*), then there is an F_σ-set S of X such that S is independent in R, and $\mathrm{Cl}(S) = X$.*
2. *If X has no isolated point and R is of the first category, then there is a subset S of X such that $S = \bigcup_{n=1}^{\infty} C_n$, where C_n are Cantor sets in X, S is independent in R, and $\mathrm{Cl}(S) = X$.*

To prove this theorem, we need the following lemma by Kuratowski [4, Lemma, p. 66].

Lemma 2.2. *Let X be an arbitrary space and let G be an open dense set in the space X^m ($m \geq 1$). Let $H_1, \cdots, H_n$, where $n \geq m$, be open non-empty sets in X.*

Then there exists a system of nonempty open sets $U_1, \cdots, U_n$ *such that* $U_i \subset H_i$ *for* $1 \leq i \leq n$, *and*

$$U_{i_1} \times \cdots \times U_{i_m} \subset G,$$

whenever $i_1, \cdots, i_m$ *are different* m *elements of* $\{1, 2, \cdots, n\}$.

Proof of Theorem 2.1. We prove the second case. The first case can be similarly proved.

We assume that X has no isolated points. Also, we may assume that $R = \bigcup_{k=1}^{\infty} R_k$, where R_k is nowhere dense and a closed subset of X^m for each k. Let $\{U_i\}_{i=1}^{\infty}$ be a countable open base of X. Let $i = 1$. By the above lemma, we can choose a family

$$\mathcal{V}_1 = \{V_{1,1}, V_{1,2}, \cdots, V_{1,n_1}\}$$

of mutually disjoint nonempty closed subsets of X such that $\mathrm{Int}(V_{1,j}) \neq \emptyset$ for each j, $(\bigcup \mathcal{V}_1) \cap U_1 \neq \emptyset$, and for any different m elements $V_{1,j_1}, \cdots, V_{1,j_m}$ of $\mathcal{V}_1$,

$$(V_{1,j_1} \times \cdots \times V_{1,j_m}) \cap (R_1) = \emptyset,$$

where $\bigcup \mathcal{V}_1 = \bigcup \{V | \, V \in \mathcal{V}_1\}$.

Inductively, by the above lemma we can choose a sequence

$$\mathcal{V}_i = \{V_{i,1}, \cdots, V_{i,n_i}\} \ (i \geq 1)$$

of families of mutually disjoint nonempty closed subsets of X such that

1. $\mathrm{Int}(V_{i,j}) \neq \emptyset$ and $\mathrm{mesh}(\mathcal{V}_i) < 1/i$,
2. for each $V_{i,j}$, there are at least two elements $V_{i+1,p}, V_{i+1,q}$ $(p \neq q)$ of $\mathcal{V}_{i+1}$ such that $V_{i+1,p} \cup V_{i+1,q} \subset \mathrm{Int}(V_{i,j})$,
3. $U_i \cap (\bigcup \mathcal{V}_i) \neq \emptyset$, and
4. for any different m elements $V_{i,j_1}, \cdots, V_{i,j_m}$ of $\mathcal{V}_i$,

$$(V_{i,j_1} \times \cdots \times V_{i,j_m}) \cap (\bigcup_{k=1}^{i} R_k) = \emptyset.$$

For each i, by induction we can obtain the sequence $\{_{i,j}$ $(j = i, i+1, i+2, \cdots)$ of subfamilies of $\mathcal{V}_j$ as follows:

$$\mathcal{F}_{i,i} = \mathcal{V}_i, \quad \mathcal{F}_{i,j+1} = \{V \in \mathcal{V}_{j+1} | \ V \text{ is contained in some element of } \mathcal{F}_{i,j}\} \ (j \geq i).$$

Put $C_i = \bigcap_{j=i}^{\infty} (\bigcup \mathcal{F}_{i,j})$. Then we see that C_i is a Cantor set and $\{C_i\}_{i=1}^{\infty}$ is increasing. Then by the condition 4, $S = \bigcup_{i=1}^{\infty} C_i$ is independent in $R = \bigcup_{k=1}^{\infty} R_k$ and by the condition 3, S is dense in X. This completes the proof.

Let $I = [0, 1]$ be the unit interval and I^k $(k \geq 1)$ the unit k-cube. A space homeomorphic to I^k is a *k-cell.* A k-cell B in $I^k - \partial I^k$ is *flat* if there is a homeomorphism $h : I^k \to I^k$ such that $h(B) = J^k$, where $J = [1/3, 2/3] \subset I$. A 0-dimensional compactum D in $I^k - \partial I^k$ is *flat* in I^k if for any neighborhood V of D in I^k, there is a neighborhood U of D in I^k such that $U \subset V$ and $U = B_1 \cup \cdots \cup B_p$, where B_i $(i = 1, 2, \cdots, p)$ are mutually disjoint k-cells. Then we may assume that B_i is flat in I^k for each i, because we can choose a k-cell B' in $\mathrm{Int}(B_i)$ such that $\partial B'$ is locally flat and hence B' is flat by Generalized Schoenflies Theorem (e.g. see [10, p. 48]). Note that if C and C' are flat Cantor sets in I^k and $k \geq 2$, then any homeomorphism $f : C \cup \partial I^k \to C' \cup \partial I^k$ can be extended to a homeomorphism $F : I^k \to I^k$ (e.g., see the proof of [8, p. 93, Theorem 7]). Also, note that any closed subset of a flat 0-dimensional compactum is also flat. It is well known that if $k \le 2$, any Cantor set in $I^k - \partial I^k$ is flat, but if $k \ge 3$, there are Cantor sets in $I^k - \partial I^k$ which are not flat (see [8, p. 127]).

Proposition 2.3. *Suppose that X is the unit k-cube I^k ($k \ge 1$). Let $m \ge 2$ be a fixed natural number and let $R \subset X^m$ be of the first category in X^m. Then there is a subset S of X such that $S = \bigcup_{n=1}^{\infty} C_n$, where C_n are flat Cantor sets in the unit k-cube X, S is independent in R, and $Cl(S) = X$.*

Proof. In the proof of the above theorem, we may assume that each $V_{i,j}$ is a k-cell (see Lemma 2.2).

Theorem 2.4. *If $f : X \to X$ is a map of a compactum X and f is $*$-chaotic on a closed set A, then there is an F_σ-set $S \subset A$ such that S is a scrambled set of f and $Cl(S) = A$. If A has no isolated points, we can choose S such that S is a countable union of Cantor sets C_n. Moreover if $X = I^k$, then the Cantor sets C_n can be chosen as flat Cantor sets in X.*

Proof. Suppose that τ is a positive number as in the definition of scrambled set S. Consider the following sets:

$$R_1 = \{(x, y) \in A^2 |\ \limsup_{i\to\infty} d(f^i(x), f^i(y)) < \tau\},$$

$$R_2 = \{(x, y) \in A^2 |\ \liminf_{i\to\infty} d(f^i(x), f^i(y)) > 0\}.$$

Let $\epsilon_n = 1/n$ $(n = 1, 2, \cdots)$. Then $R_1 = \bigcup_{n=1}^{\infty} T_n$, where

$$T_n = \{(x, y) \in A^2 |\ d(f^i(x), f^i(y)) \le \tau - \epsilon_n \text{ for every } i \ge n\ \}.$$

Also, $R_2 = \bigcup_{n=1}^{\infty} W_n$, where

$$W_n = \{(x, y) \in A^2 |\ d(f^i(x), f^i(y)) \ge \epsilon_n \text{ for every } i \ge n\ \}.$$

Note that $T_n, W_n \subset A^2$ are closed. Since f is $*$-chaotic on A, they have the property (*). By Theorem 2.1 and Proposition 2.3, we obtain a desired scrambled set S.

Corollary 2.5. *Let $f : X \to X$ be a map of a compactum X and S an uncountable scrambled set of f. Then there is an F_σ-set S' of X such that S' is a scrambled set of f, $Cl(S') = Cl(S)$, and S' contains a Cantor set. Moreover, if $Cl(S)$ has no isolated point, S' can be chosen so that S' is a countable union of Cantor sets C_n. Moreover if $X = I^k$, then the Cantor sets C_n can be chosen as flat Cantor sets in I^k.*

Let μ be the Lebesgue measure on I^k. Note that there are subsets E of $I^k - \partial I^k$ with $\mu(E) = 1$ which are countable union of flat Cantor sets in I^k.

Theorem 2.6. *Suppose that E is a countable union of flat Cantor sets of I^k ($k \ge 1$) such that $\mu(E) = 1$. If $f : I^k \to I^k$ is everywhere $*$-chaotic, then for any $\epsilon > 0$ there is a map $g : I^k \to I^k$ such that f and g are topologically conjugate, $d(f, g) < \epsilon$ and E is a scrambled set of g, where $d(f, g) = \sup\{d(f(x), g(x)) |\ x \in X\}$. In particular, g has a scrambled set which has Lebesgue measure 1.*

Proof. First, we assume $k \geq 2$. Note that the space $H(X)$ of all homeomorphisms of a compactum X has a complete metric, i.e., $\rho(f,g) = d(f,g) + d(f^{-1}, g^{-1})$. By Theorem 2.4, we can choose a scrambled set S of f such that S is a countable union of Cantor sets which are flat in I^k, and S is dense in I^k. Let $\epsilon_1 > \epsilon_2 > \cdots$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \epsilon_n < \epsilon$. Since any closed subset of a flat Cantor set is also flat, we can choose mutually disjoint flat Cantor sets D_n $(n = 1, 2, \cdots)$ such that $E = \bigcup_{n=1}^{\infty} D_n$. For $n = 1$, we choose mutually disjoint k-cells B_j $(j = 1, 2, \cdots, p)$ such that $U = \bigcup_{j=1}^{p} B_j$ is a neighborhood of D_1 and $\text{diam}(B_j) < \epsilon_1$. Also, since any closed subset of a flat Cantor set is flat, we can choose a flat Cantor set $C_1 \subset S \cap \text{Int}(U)$ such that $C_1 \cap B_j \neq \emptyset$ for each j. Then there is a homeomorphism $h_1 : I^k \to I^k$ such that $h_1(D_1) = C_1 \subset S, h_1 | I^k - U = id$ (see the proof of [8, p. 93, Theorem 7]). Then $\rho(id, h_1) < \epsilon_1$. Put $D' = h_1(D_2)$. Note that D' is flat. We can choose a neighborhood V of D' such that $V \cap h_1(D_1) = \emptyset$. By the above argument, we obtain a homeomorphism $h' : I^k \to I^k$ such that $h'(D') \subset S, h' | I^k - V = id, \rho(id, h') < \epsilon_2$. Put $h_2 = h' \cdot h_1$.

If we continue this procedure, we obtain a sequence $h_n : I^k \to I^k$ of homeomorphisms such that

1. $\rho(h_1, 1) < \epsilon_1$, $\rho(h_n, h_{n+1}) < \epsilon_n$ for each n,
2. $h_{n+1} | (\bigcup_{j=1}^{n} D_j) = h_n | (\bigcup_{j=1}^{n} D_j)$, and $h_n(D_n) \subset S$.

Then we obtain a homeomorphism $h = \lim_{n \to \infty} h_n$, since $H(X)$ is complete and the sequence $\{h_n\}_{n=1}^{\infty}$ is a Cauchy sequence. By the condition 2, we see that $h(E) \subset S$. Put $g = h^{-1} \cdot f \cdot h : I^k \to I^k$. Then g is a desired map. The case $k = 1$ is similarly proved by using Theorem 2.4 and [2]. This completes the proof.

Remark 2.7. For the proof of a weak version of Theorem 2.6, we can use the following theorem [9, Theorem 9] by Oxtoby and Ulam: for any subset B of any k-cube R $(k \geq 1)$, there is a homeomorphism $h : R \to R$ with $h | \partial R = id$ such that $\mu(h(B)) = 0$ if and only if $R - B$ contains a sequence of perfect sets whose union is dense in R. In fact, for any $\delta > 0$, we decompose I^k into k-cubes $R_1, \cdots, R_p$ such that $\text{Int}(R_i) \cap \text{Int}(R_j) = \emptyset$ $(i \neq j)$, $\text{diam}(R_i) < \delta$, and $I^k = R_1 \cup ... \cup R_p$. If S is a scrambled set of f such that S is dense in I^k and S is a countable union of Cantor sets, then for each R_i there is a homeomorphism $h_i : R_i \to R_i$ such that $h_i | \partial R_i = id, \mu(h_i(R_i - S)) = 0$. Define a map $h : I^k \to I^k$ by $h | R_i = h_i$. Put $S' = h(S)$ and $g = h \cdot f \cdot h^{-1}$. Then $\mu(S') = 1$. If we choose a sufficiently small positive number δ, then g has a scrambled set S' which has Lebesgue measure 1, and $d(f,g) < \epsilon$.

By [3, Corollary 3.3], every compact connected k-manifold $(k \geq 2)$ admits an everywhere $*$-chaotic homeomorphism. Hence we obtain

Corollary 2.8. *There is a homeomorphism $f : I^k \to I^k$ $(k \geq 2)$ such that there is a scrambled set S of f with $\mu(S) = 1$.*

Remark 2.9. There is no homeomorphism $f : I \to I$ with a nonempty scrambled set.

By the similar proof of Theorem 2.6, we obtain the following.

Theorem 2.10. *Let $f : I^k \to I^k$ be a map of the unit k-cube I^k and suppose that there is an uncountable scramble set S of f with $S \subset I^k - \partial I^k$. Then for any $\epsilon > 0$ there is a map $g : I^k \to I^k$ such that $d(f,g) < \epsilon$, g is topologically conjugate to*

f, and there is a scrambled set S' of g such that S' is a Cantor set and S' has a positive Lebesgue measure.

Let $f : X \to X$ be a map of a compactum X. A point $x \in X$ is *recurrent* if for any neighborhood U of x in X, there is a natural number $N > 0$ such that $f^N(x) \in U$. A point $x \in X$ is *nonwandering* if for any neighborhood U of x in X, there is a natural number $N > 0$ such that $f^N(U) \cap U \neq \emptyset$. If there is a point of X whose orbit is dense, then f is called *transitive*. It is well known that the set $\Omega(f)$ of all nonwandering points of f is closed, and the set $R(f)$ of all recurrent points of f is G_δ. Put $T(f) = \{x \in X \mid \text{the orbit of } x \text{ is dense in } X\}$. Then $T(f)$ is G_δ-dense if f is transitive. Note that if $X = \Omega(f)$, then $R(f)$ is dense in X. Note that, in general, if a set F is of the first category in X, then $(F \times X) \cup (X \times F)$ is of the first category in X^2. Hence by the same argument as above, we can add the following restriction to scrambled sets.

Corollary 2.11. *Let $f : I^k \to I^k$ be an everywhere $*$-chaotic map.*

1. *If f is transitive, then for any $\epsilon > 0$ there is a map $g : I^k \to I^k$ such that f and g are topologically conjugate, $d(f, g) < \epsilon$ and there is a scrambled set $S \subset T(g)$ of g which has Lebesgue measure 1.*
2. *If $\Omega(f) = X$, then for any $\epsilon > 0$ there is a map $g : I^k \to I^k$ such that f and g are topologically conjugate, $d(f, g) < \epsilon$ and there is a scrambled set $S \subset R(g)$ of g which has Lebesgue measure 1.*

A map $f : X \to X$ is *mixing* if for nonempty open sets U, V of X, there is a natural number N such that if $n > N$, then $f^n(U) \cap V \neq \emptyset$.

Corollary 2.12 (cf. [13])**.** *Let $f : I^k \to I^k$ be a mixing map. Then for any $\epsilon > 0$ there is a map $g : I^k \to I^k$ such that f and g are topologically conjugate, $d(f, g) < \epsilon$ and there is a scrambled set $S \subset T(g)$ of g which has Lebesgue measure 1.*

Remark 2.13 (cf. [4, Applications]).

1. There is a Vitali set in the space E of reals containing a countable union S of Cantor sets such that S is dense in E.
2. If X is an indecomposable continuum, then X contains a subset S such that S is a countable union of Cantor sets, S is dense in X, and no two of points of S belong to the same composant of X.

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Institute of Mathematics, University of Tsukuba, Ibaraki, 305 Japan
E-mail address: hisakato@sakura.cc.tsukuba.ac.jp