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HIGHER ORDER TURÁN INEQUALITIES

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ABSTRACT. The celebrated Turán inequalities $P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geq 0, \quad x \in [-1,1], \quad n \geq 1$, where $P_n(x)$ denotes the Legendre polynomial of degree n, are extended to inequalities for sums of products of four classical orthogonal polynomials. The proof is based on an extension of the inequalities $\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} \geq 0, \quad n \geq 1$, which hold for the Maclaurin coefficients of the real entire function ψ in the Laguerre-Pólya class, $\psi(x) = \sum_{n=0}^{\infty} \gamma_n x^n/n!$.

1. Introduction and statement of results

For any sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ the quantities $\Delta_n(p;x) := p_n^2(x) - p_{n-1}(x)p_{n+1}(x)$ are called Turán determinants, associated with $\{p_n\}_{n=0}^{\infty}$. Szegő [18] was the first to call attention to the following beautiful inequalities of P. Turán:

$$\Delta_n(P;x) = P_n^2(x) - P_{n-1}(x)P_{n+1}(x) \geq 0, \quad x \in [-1,1], \quad n \geq 1. \tag{1}$$

In the same paper Szegő obtained extensions of (1) to Gegenbauer (ultraspherical), Laguerre and Hermite polynomials. Karlin and Szegő [10] proved that certain higher order Turán determinants for the same classes of classical orthogonal polynomials do not change their sign in the interval of orthogonality. Gasper [9] proved the analog of (1) for a class of Jacobi polynomials. Askey's comments on [10] and [18] in Volume 3 of Szegő's collected papers survey further contributions and developments.

The reason for the recent interest in Turán determinants is that for the orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$ in a subclass of the class $M(0,1)$ the quantities $\Delta_n(p;x)$ converge uniformly on the compact subsets of $(-1,1)$ to $2(1-x^2)^{1/2}/(\pi\alpha'(x))$, where $\alpha'(x)$ is the absolutely continuous part of the measure, with respect to which the p_n are orthogonal [5, 6, 7, 12, 19].

Szegő [18] gives Turan's proof and three additional proofs of (1). The third proof is particularly ingenious and allows the extension of (1) to the ultraspherical, Laguerre and Hermite polynomials. Szegő atributes the idea of this proof to Pólya.

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A real entire function

$$\psi(x) = \sum_{n=0}^{\infty} \gamma_n \frac{x^n}{n!} \tag{2}$$

is said to belong to the Laguerre-Pólya class ($\psi \in \mathcal{L}$-$\mathcal{P}$) if

$$\psi(x) = cx^m e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} (1 + x/x_k) e^{-x/x_k},$$

where c, β, x_k are real, $\alpha \geq 0, m$ is a nonnegative integer and $\sum x_k^{-2} < \infty$. We have adopted the notations in [2, 4, 14], which one may consult for the important properties of the functions in Laguerre-Pólya class. Generally, $\mathcal{L}$-$\mathcal{P}$ consists of entire functions which are uniform limits on the compact sets of the complex plane of real polynomials with only real zeros. A necessary condition that $\psi \in \mathcal{L}$-$\mathcal{P}$ is that its Maclaurin coefficients satisfy (cf. [2, 4, 16])

$$\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} \geq 0, \quad n \geq 1. \tag{3}$$

Then, in order to prove the inequalities

$$\Delta_n(p; x) \geq 0, \quad n \geq 1,$$

where

a) $p_n(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$ for $x \in [-1, 1]$, $\lambda > -1/2$,
b) $p_n(x) = L_n^{(\alpha)}(x)/L_n^{(\alpha)}(0)$ for $x \in [0, \infty)$, $\alpha > -1$,

or

c) $p_n(x) = H_n(x)$ for $x \in (-\infty, \infty)$,

where $P_n^{(\lambda)}$, $L_n^{(\alpha)}$ and H_n denote the ultraspherical, Laguerre and Hermite polynomials, one uses (3) together with the fact that the generating functions which appear on the right-hand sides of

$$\sum_{n=0}^{\infty} \frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)} \frac{z^n}{n!} = 2^{\lambda-1/2}\Gamma(\lambda+1/2)e^{xz} \frac{J_{\lambda-1/2}((1-x^2)^{1/2}z)}{((1-x^2)^{1/2}z)^{\lambda-1/2}}, \quad \lambda > -1/2,$$

$$\sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} \frac{z^n}{n!} = \Gamma(\alpha+1)e^{z} \frac{J_{\alpha}(2(xz)^{1/2})}{(xz)^{\alpha/2}}, \quad \alpha > -1,$$

and

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz - z^2}$$

are in the Laguerre-Pólya class.

Another reason that inequalities (3) are interesting is their connection to the celebrated Riemann hypothesis [17] about the zeros of the Riemann ζ-function. It is well known and easy to see that the Riemann hypothesis holds true if and only if the Riemann ξ-function, defined by

$$\xi(iz) = \frac{1}{2}(z^2 - 1/4)\pi^{-z/2-1/4}\Gamma(z/2 + 1/4)\zeta(z + 1/2),$$

has only real zeros. It is known that ξ is a real entire function of order one. It can be represented in the form

$$\xi(x/2) = 8 \int_0^{\infty} \Phi(t) \cos xt \; dt,$$

where

$$\Phi(t)=\sum_{n=1}^{\infty}(2n^4\pi^2e^{9t}-3n^2\pi e^{5t})\exp(-n^2\pi e^{4t}). \tag{4}$$

Then

$$\frac{1}{8}\xi(x/2)=\sum_{k=0}^{\infty}(-1)^k\hat{b}_k\frac{x^{2k}}{(2k)!}$$

with

$$\hat{b}_k=\int_0^{\infty}t^{2k}\Phi(t)dt, \quad \text{for} \quad k\geq 0. \tag{5}$$

On setting $z=-x^2$ we obtain the entire function

$$\xi_1(z)=\sum_{k=0}^{\infty}\hat{\gamma}_k\frac{z^k}{k!}, \quad \hat{\gamma}_k=\frac{k!}{(2k)!}\hat{b}_k, \tag{6}$$

of order 1/2. Therefore, by the Hadamard theorem, the Riemann hypothesis is equivalent to the statement that $\xi_1\in\mathcal{L}\text{-}\mathcal{P}$ (cf. Pólya and Schur [16] and Boas [1, p. 24]). Hence the inequalities $\hat{\gamma}_n^2-\hat{\gamma}_{n-1}\hat{\gamma}_{n+1}\geq 0,\ \ n\geq 1$, which are equivalent to the inequalities $(2n+1)\hat{b}_n^2-(2n-1)\hat{b}_{n-1}\hat{b}_{n+1}\geq 0,\ \ n\geq 1$, are necessary conditions for the Riemann hypothesis to be true. Craven, Norfolk and Varga [3] proved the latter inequalities, thus verifying a conjecture of Pólya [15] (see also Varga [20, Chapter 3]).

In this paper we obtain, in a very simple way, new necessary conditions for a real entire function to belong to $\mathcal{L}\text{-}\mathcal{P}$. These condition are extensions of (3). Then the idea of Pólya, sketched above, immediately yields extensions of (1).

Theorem 1. *Let the real entire function ψ, defined by (2), be in the Laguerre-Pólya class. Then*

$$4(\gamma_n^2-\gamma_{n-1}\gamma_{n+1})(\gamma_{n+1}^2-\gamma_n\gamma_{n+2})-(\gamma_n\gamma_{n+1}-\gamma_{n-1}\gamma_{n+2})^2\geq 0 \qquad \textit{for} \quad n\geq 1.$$

Corollary 1. *Let $\hat{\gamma}_n$ be defined by (4), (5) and (6). A necessary condition that the Riemann hypothesis holds true is that the inequalities*

$$4(\hat{\gamma}_n^2-\hat{\gamma}_{n-1}\hat{\gamma}_{n+1})(\hat{\gamma}_{n+1}^2-\hat{\gamma}_n\hat{\gamma}_{n+2})-(\hat{\gamma}_n\hat{\gamma}_{n+1}-\hat{\gamma}_{n-1}\hat{\gamma}_{n+2})^2\geq 0, \quad n\geq 1, \tag{7}$$

hold.

Corollary 2. *The inequalities*

$$\begin{aligned}\delta_n(p;x) \quad := \quad & 4\left(p_n^2(x)-p_{n-1}(x)p_{n+1}(x)\right)\left(p_{n+1}^2(x)-p_n(x)p_{n+2}(x)\right)\\ & -\left(p_n(x)p_{n+1}(x)-p_{n-1}(x)p_{n+2}(x)\right)^2\geq 0, \qquad n\geq 1,\end{aligned}$$

hold for the classes of orthogonal polynomials a), b) and c), described above.

2. Proof of the theorem and remarks

Proof of the theorem. Let the real entire function ψ, defined by (2), be in the Laguerre-Pólya class. Then, for any positive integer n, the n-th associated Jensen polynomial

$$g_n(x):=\sum_{k=0}^{n}\binom{n}{k}\gamma_k x^k$$

has only real zeros (cf. [2, 4, 15]). Observe that for any $q \le n$

$$g_n^{(q)}(x) = \frac{n!}{(n-q)!} g_{n-q,q}(x),$$

where

$$g_{n,q}(x) := \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \gamma_{k+q} x^k, \qquad n = 0, 1, \dots ,$$

are the Jensen polynomials associated with $\psi^{(q)}$. Then Rolle's theorem implies that for any positive integer n and any nonnegative integer q the polynomial $g_{n,q}(x)$ has only real zeros. The latter follows also from the fact that the class $\mathcal{L}$-$\mathcal{P}$ is closed under differentiation (cf. Pólya and Schur [16]). Now the assertion of the theorem follows from a result of Mařík [11] (see also [13, Theorem 1.3.3 on p. 99]). It states that if the real polynomial

$$p(x) = \sum_{k=0}^{n} a_k x^k / (k!(n-k)!)$$

of degree $n \ge 3$ has only real zeros, then the inequalities

$$4(a_k^2 - a_{k-1}a_{k+1})(a_{k+1}^2 - a_k a_{k+2}) - (a_k a_{k+1} - a_{k-1} a_{k+2})^2 \ge 0, \qquad 1 \le k \le n-2,$$

hold. □

Corollary 1 is immediate. In order to prove Corollary 2 one uses the statement of Theorem 1 and the idea of Pólya, described in the first section.

A natural conjecture is that inequalities (7) hold true. Numerical calculations, based on the values of the first twenty coefficients $\hat{b}_n$, given in [3], support the conjecture.

It is interesting to see what is the limit of the quantities $\delta_n(p;x)$ for the class of orthogonal polynomials whose associated Jacobi matrix is a compact perturbation of the Jacobi matrix corresponding to the Chebyshev polynomials of the second kind. The above mentioned results on convergence of Turán determinants for the polynomials in the class $M(0,1)$ and their extension to the so-called shifted Turán (or Geronimo and Van Assche) determinants [8, Theorem 6] yield:

Proposition 1. *Let the sequence of orthogonal polynomials $\{p_n\}$ be defined by the three-term recurrence relation*

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \ge 0,$$

$$p_{-1}(x) = 0, \qquad p_0(x) = 1,$$

with real b_n and positive a_n. Suppose that the recurrence coefficients satisfy $a_n \to 1/2$ and $b_n \to 0$ as n diverges, and

$$\sum_{k=0}^{\infty} (|b_{k+1} - b_k| + |a_{k+2} - a_{k+1}|) < \infty.$$

Then the measure α, with respect to which the p_n are orthogonal, is absolutely continuous in $(-1,1)$, $\alpha'(x) > 0$ for all $x \in (-1,1)$, and α' is continuous in $(-1,1)$.

Moreover,

$$\lim_{n\to\infty} \delta_n(p;x) = \frac{8}{\pi^2}\frac{1-x^2}{[\alpha'(x)]^2}$$

uniformly on the compact subsets of $(-1,1)$.

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