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A NEW PROOF OF THE SOLOMON-TITS THEOREM

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ABSTRACT. We give a new proof of the Solomon-Tits Theorem which asserts that the Tits building of a finite group of Lie type has the homotopy type of a bouquet of spheres.

1. Introduction

Let G be a finite group of Lie type of rank $n \geqslant 3$, (G, B, N, S) a Tits system for G, $H = B \cap N$, $B = UH$ where $U = O_p(B)$ and p is the defining prime for G. $\mathbf{F} = \{G_i;\ 1 \leqslant i \leqslant n\}$ is the set of maximal parabolics of G over B and $\mathcal{B} = \mathcal{C}(G, \mathbf{F})$ the simplicial complex whose vertex set is the union of coset spaces G/G_i, $i \in I$, and whose simplices are collection of vertices with nonempty intersection. $\mathcal{B}$ is called the Tits building of G. The following theorem is well known:

Theorem (Solomon-Tits)**.** *The Tits building $\mathcal{B}$ has the homotopy type of a wedge of $|U|$ $(n-1)$-spheres.*

In his seminar paper [S] Solomon first mentioned the theorem and sketched a proof. Later Garland ([G]) gave a proof which is essentially the same as Solomon's, and Curtis and Lehrer ([CL]) gave a proof of a related result based on the computation of the $\mathbb{Q}G$-endomorphism algebra of $H_*(\mathcal{B})$. We provide a new proof here which should help to better understand the structure of the Tits building. Namely, $\mathcal{B}$ is the union of the $|U|$ apartments containing a fixed chamber, each having the homotopy type of an $(n-1)$-sphere as it is a Coxeter complex. We show that the intersection of any k $(k \geqslant 2)$ of these subcomplexes is contractible. Then it follows from an easy lemma in topology that $\mathcal{B}$ has the homotopy type of a wedge of these spheres.

2. Preliminary lemmas

Let us begin by recording a well known lemma from topology.

Lemma 1. *Suppose X is a simplicial complex and $X_1, X_2, \ldots, X_m$, $m > 1$, are subcomplexes of X with $X = \bigcup_{1 \leqslant i \leqslant m} X_i$ such that for any $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m$, $k > 1$, the set $\bigcap_{1 \leqslant j \leqslant k} X_{i_j}$ is contractible. Then X has the homotopy type of $X_1 \vee X_2 \vee \cdots \vee X_m$.*

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Proof: By induction. Note that if B is a contractible subcomplex of a simplicial complex A, then the quotient map $q : A \to A/B$ is a homotopy equivalence. If $m = 2$, as $Y = X_1 \cap X_2$ is contractible, $X/Y = X_1/Y \cup X_2/Y$ and $X_1/Y \cap X_2/Y = Y/Y$ is a point, the lemma follows easily in this case.

Assume the lemma holds for $< m$. Then $X_1 \cap (\bigcup_{i>1} X_i) = \bigcup_{i>1}(X_1 \cap X_i)$ is homotopy equivalent to $\bigvee_{i>1}(X_1 \cap X_i)$ by induction and hence contractible. Therefore X is homotopy equivalent to $X_1 \vee (\bigcup_{i>1} X_i)$, which is in turn homotopy equivalent to $X_1 \vee X_2 \vee \cdots \vee X_m$. □

We refer readers who are unfamiliar with the geometry of groups of Lie type and the theory of Coxeter groups to [A] or [B]. From time to time by abuse of notation we identify an element of N with its image in the Weyl group $W = N/H$. We may also use the same notation for a simplex of a complex and the subcomplex consisting of its faces.

Lemma 2. *Let $W_J = G_J \cap N/H$ be a parabolic subgroup of W, where $G_J = \bigcap_{j \in J \subseteq I} G_i$. Then each coset $W_J d$ has a unique element d_J of minimal length such that $l(wd_J) = l(w) + l(d_J)$ for all $w \in W_J$. Here l is the length function on W with respect to S.*

Proof. This is exercise 3, page 37 in [B]. □

Define a partial ordering $\leqslant$ on W by $v \leqslant w$ if and only if $w = uv$ and $l(w) = l(u) + l(v)$ for some $u \in W$. This is a well defined ordering, and w_0, the longest element of W, is the unique maximal element with respect to this order. See for instance exercise 6, page 155 in [A].

Lemma 3. *Let Φ be the Coxeter complex (see page 210 in* [A] *for the definition) for W, $\mathbf{C} \in \Phi$ the fundamental chamber and $\sigma \in \Phi$ a simplex. Then there is a unique $w_\sigma \in W$ with $\sigma \subseteq \mathbf{C}w_\sigma$ such that for any chamber $\mathbf{C}w$ containing σ, $w_\sigma \leqslant w$. Moreover, there is a sequence of chambers $\mathbf{C}w_\sigma = \mathbf{C}w_1, \mathbf{C}w_2, \ldots, \mathbf{C}w_n = \mathbf{C}w$, each containing σ, such that $w_i \leqslant w_{i+1}$ with $l(w_{i+1}) = l(w_i) + 1$, $1 \leqslant i < n$.*

Proof. This is a direct consequence of Lemma 2. □

3. Proof of the Theorem

Let $\mathbf{C} = \{G_i; 1 \leqslant i \leqslant n\}$ be the fundamental chamber of $\mathcal{B}$ and $\Sigma = \bigcup_{w \in W} \mathbf{C}w$. Then Σ is an apartment stabilized by N isomorphic to the Coxeter complex Φ. Hence, as Φ has the homotopy type of a $(n-1)$-sphere,

Proposition 1. *Σ has the homotopy type of a $(n-1)$-sphere.*

As $G = BNB = BNU$, it is obvious that

Proposition 2. *$\mathcal{B} = \bigcup_{u \in U} \Sigma^u$, where $\Sigma^u = \bigcup_{w \in W} \mathbf{C}wu$.*

Notice that for $u_i \in U$, $1 \leqslant i \leqslant k$, $\bigcap_{1 \leqslant i \leqslant k} \Sigma^{u_i}$ is isomorphic to

$$(\bigcap_{1 \leqslant i \leqslant k} \Sigma^{u_i})^{u_1{}^{-1}} = \bigcap_{1 \leqslant i \leqslant k} \Sigma^{u_i u_1^{-1}}.$$

So by the argument in the introduction, a proof of the following proposition will complete the proof of the Solomon-Tits Theorem.

Proposition 3. *For distinct $u_1, u_2, \ldots, u_k \in U - \{1\}$, $k \geqslant 1$, $\Sigma \cap \bigcap \Sigma^{u_k}$ is contractible.*

The rest of paper is dedicated to the proof of Proposition 3. For simplicity, we deal only with the case $k = 1$. The same proof works for $k > 1$.

Lemma 4. (1) *For $1 \neq u \in U$, $\mathcal{T} = \Sigma \cap \Sigma^u$ is the subcomplex of Σ consisting of simplices fixed by u. Moreover, $\mathcal{T}$ is a union of chambers.*

(2) *If $\mathbf{C}w \in \mathcal{T}$ and $v \leqslant w$, then $\mathbf{C}v \in \mathcal{T}$.*

Proof. Let $\mathbf{C}_J = \{G_j; j \in J \subseteq I\} \subseteq \mathbf{C}$ and assume $\mathbf{C}_J v_1 u = \mathbf{C}_J v_2$ for some $v_1, v_2 \in W$. Then $G_J v_1 u = G_J v_2$. As $N \cap G_J = N_J$ is the preimage of W_J in N, we may choose v_1 and v_2 as in Lemma 2. Let $l(v_k) = r_k$, $k = 1, 2$. Without loss assume $r_1 \leqslant r_2$. Otherwise we consider u^{-1} instead. Set $v_1 = x_1 x_2 \dots x_{r_1}$ with x_k's from S. As $sBw \subseteq BwB \cup BswB$ for $s \in S$ and $w \in W$, we have

$$v_1 u v_2^{-1} \in B v_2^{-1} B \cup \bigcup\nolimits_{1 \leqslant k_1 < k_2 < \cdots < k_t \leqslant r_1} B x_{k_1} x_{k_2} \dots x_{k_t} v_2^{-1} B.$$

Then as $v_1 u v_2^{-1} \in G_J = BN_J B$, either $v_1 u v_2^{-1} \in B v_2^{-1} B \subseteq G_J$ or there are $1 \leqslant k_1 < k_2 < \cdots < k_t \leqslant r_1$ such that $v_1 u v_2^{-1} \in B x_{k_1} \dots x_{k_s} v_2^{-1} B \subseteq G_J$. In the first case, $v_2 \in G_J$ and by Lemma 2 $v_2 = 1$. Consequently $v_1 \in G_J v_2 u^{-1} = G_J$, and again by Lemma 2 $v_1 = 1$. In the second case, set $v_0 = x_{k_1} \dots x_{k_t}$. Then $v_0 \in G_J v_2$ with $l(v_0) \leqslant t \leqslant r_1 \leqslant r_2 = l(v_2)$; hence by Lemma 2, all equalities must hold and $t = r_1$, $r_1 = r_2$, $v_0 = v_1 = v_2$. Therefore $v_1 u v_2^{-1} \in B v_0 v_2^{-1} B = B$, i.e. $u \in B \cap B^{v_1}$. In either case, $v_1 = v_2$, $u \in B \cap B^{v_1}$ and $\mathbf{C}_J v_1 \subseteq \mathbf{C} v_1 \subseteq \mathcal{T}$. Part (1) is proved.

For $v \leqslant w \in W$ and $\mathbf{C}w \in \mathcal{T}$, assume $w = xv$ with $l(w) = l(x) + l(v)$. Let $x = x_1 x_2 \dots x_r$ with $r = l(x)$ and $x_i \in S$. For $t \leqslant r$, set $w_t = x_t \dots x_r v$. Now for any $s \in S$ and any $w \in W$ with $sw \leqslant w$, we have $B \cap B^w \leqslant B^{sw}$. Applying the observation to x_1 and w, we have $u \in B \cap B^w \in B^{w_2}$, so $\mathbf{C}w_2 \in \mathcal{T}$. Now part (2) follows inductively. □

Lemma 5. *$\mathcal{T}$ is contractible.*

Proof. As $\mathbf{C} \in \mathcal{T}$, $\mathcal{T}$ is nonempty. $\mathcal{T}$ is connected by Lemma 3 and Lemma 4.2. Recall that w_0 is the longest element in W. Notice that $\mathbf{C}w_0 \notin \mathcal{T}$, as $u \notin B \cap B^{w_0} = H$.

Pick a $\bar{w} \in W$ with $\mathbf{C}\bar{w} \subseteq \mathcal{T}$ such that $l(\bar{w})$ is maximal. Set

$$\mathcal{T}_1 = \bigcup_{w \neq \bar{w}, u \in B \cap B^w} \mathbf{C}w \subseteq \mathcal{T}.$$

For a simplex $\sigma \in \mathcal{T}_1 \cap \mathbf{C}\bar{w}$, choose $v \in W$ as in Lemma 3 such that $\sigma \in \mathbf{C}v$. Then by Lemma 3, $v \leqslant \bar{w}$. So $\mathbf{C}v \in \mathcal{T}$ by Lemma 4.2. But $\sigma \in \mathcal{T}_1$, and consequently there is a $w \in W$ with $\sigma \in \mathbf{C}w \in \mathcal{T}_1$. This forces $v \neq \bar{w}$ and $\mathbf{C}v \in \mathcal{T}_1$. Therefore by Lemma 3 there is an $s \in S$ with $v \leqslant s\bar{w} \leqslant \bar{w}$ and $\sigma \in \mathbf{C}s\bar{w} \subseteq \mathcal{T}_1$. So $\sigma \subseteq \mathbf{C}\bar{w} \cap \mathbf{C}s\bar{w}$, an $(n-2)$-simplex of $\mathbf{C}\bar{w}$.

On the other hand, for any $s \in S$ with $s\bar{w} \leqslant \bar{w}$, it is obvious that $\mathbf{C}s\bar{w} \in \mathcal{T}_1$ and $\mathbf{C}\bar{w} \cap \mathbf{C}s\bar{w} \subseteq \mathcal{T}_1 \cap \mathbf{C}\bar{w}$. We have shown that $\mathcal{T}_1 \cap \mathbf{C}\bar{w}$ is the union of $(n-2)$-simplices $\mathbf{C}\bar{w} \cap \mathbf{C}s\bar{w}$ of $\mathbf{C}\bar{w}$ with $s\bar{w} \leqslant \bar{w}$. As $\bar{w} \neq w_0$, there is an $s \in S$ with $s\bar{w} > \bar{w}$. Therefore there are at most $n-1$ $(n-2)$-simplices of $\mathbf{C}\bar{w}$ in $\mathcal{T}_1$, and their intersection contains a vertex; hence $\mathcal{T}_1 \cap \mathbf{C}\bar{w}$ is contractible. Applying Lemma 1, we conclude that $\mathcal{T}$ is homotopy equivalent to $\mathcal{T}_1$.

Notice that $\mathcal{T}_1$ is also a union of chambers and satisfies Lemma 3, part (2). Therefore, repeating the above procedure gives that $\mathcal{T}$ is homotopy equivalent to $\mathbf{C}$, the fundamental chamber, hence contractible. □

References

[A] M. Aschbacher, *Finite Group Theory*, Cambridge University Press, Cambridge,1986. MR **95b:**20002

[B] Bourbaki, N., *Groupes et algebras de Lie*, 4, 5, 6, Masson, Paris, 1981. MR **83g:**17001

[CL] C.W. Curtis and G. I. Lehrer, A new proof of a theorem of Solomon-Tits, *Proc. of AMS v. 82, No. 2* (1982) 154-156. MR **83j:**20048

[G] H. Garland, p-adic curvature and the cohomology of discrete subgroups of p-adic groups, *Ann. Math.(2)* 97 (1973) 375-393. MR **47:**8719

[S] L. Solomon, The Steinberg character of a finite group with a BN-pair, Theory of Finite Groups, R. Brauer and C.H. Sah, Eds., Benjamin, New York, 1969, pp. 213-221. MR **40:**220

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