PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 7, July 1998, Pages 1989–1997
S 0002-9939(98)04739-X

A NOTE ON NORM ATTAINING FUNCTIONALS

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(Communicated by Palle E. T. Jorgensen)

Abstract. We are concerned in this paper with the density of functionals which do not attain their norms in Banach spaces. Some earlier results given for separable spaces are extended to the nonseparable case. We obtain that a Banach space X is reflexive if and only if it satisfies any of the following properties: (i) X admits a norm $\|\cdot\|$ with the Mazur Intersection Property and the set $NA_{\|\cdot\|}$ of all norm attaining functionals of X^* contains an open set, (ii) the set $NA^1_{\|\cdot\|}$ of all norm one elements of $NA_{\|\cdot\|}$ contains a (relative) weak* open set of the unit sphere, (iii) X^* has C^*PCP and $NA^1_{\|\cdot\|}$ contains a (relative) weak open set of the unit sphere, (iv) X is WCG, X^* has $CPCP$ and $NA^1_{\|\cdot\|}$ contains a (relative) weak open set of the unit sphere. Finally, if X is separable, then X is reflexive if and only if $NA^1_{\|\cdot\|}$ contains a (relative) weak open set of the unit sphere.

1. Notation and preliminaries

Given a real Banach space $(X, \|\cdot\|)$ with dual $(X^*, \|\cdot\|^*)$, we denote by $B_{\|\cdot\|}$ its closed unit ball and by $S_{\|\cdot\|}$ its unit sphere. A functional $f \in X^*$ *attains its norm* if there exists $x \in S_{\|\cdot\|}$ such that $f(x) = \|f\|^*$. We denote by $NA_{\|\cdot\|}$ or simply by NA (if there is no ambiguity on the norm) the set of all norm attaining functionals of X^*, and by $NA^1_{\|\cdot\|}$ or NA^1 the set of norm one elements of $NA_{\|\cdot\|}$.

The classical James' Theorem [14] asserts that a Banach space X is nonreflexive if and only if $NA \neq X^*$. On the other hand, the Bishop-Phelps Theorem [23] tells us that NA is always dense in X^*. The structure of the set NA has been studied by many authors. Petunin and Plichko [22] proved that a separable Banach space X is isometric to a dual space whenever there is a closed and weak* dense subspace $H \subset NA$. Bourgin and Stegall [2] characterized Banach spaces with the Radon-Nikodým property as those spaces such that, for every bounded and closed set C, the functionals attaining their supremum on C constitute a residual set of X^*. Moors [19] showed the above characterization of the Radon-Nikodým property considering only balls of equivalent norms. Recently, Debs, Godefroy and Saint Raymond [4] obtained that if X is separable and nonreflexive, then NA is not a weak*-G_δ subset of X^*.

Received by the editors December 2, 1996.
1991 *Mathematics Subject Classification.* Primary 46B20.
Key words and phrases. Reflexive spaces, Mazur Intersection Property, (Weak*) Convex Point of Continuity Property.
Partially supported by DGICYT PB 96-0607.

2. The problem

It is our purpose in this note to discuss some topological properties of NA implying reflexivity. More precisely, we are motivated by the following question: is a Banach space reflexive provided the set NA has nonempty interior? It can be easily seen, using a quite simple renorming (see [1], [21]), that the answer is negative when the topology is the norm topology. This situation suggests, at least, two possibilities. First, to investigate which additional geometric conditions imply a positive answer. Second, to consider the same problem for the weak or the weak* topology.

There exist previous results in both directions. For a separable and nonreflexive Banach space X, Acosta and Galán [1] proved the (norm) density of $X^* \setminus NA$ whenever the norm either is very smooth or has the Mazur Intersection Property. Debs, Godefroy and Saint Raymond showed in [4] that $S_{\|\cdot\|^*} \setminus NA$ is weak* dense, again for X separable and nonreflexive. Our aim is to remove, whenever possible, the condition of separability, thus improving the preceding results. The technique employed is different from those exhibited in [1] and [4], although they have in common that they all rely on James' Theorem. Similar results with other geometrical conditions, namely Hahn-Banach smoothness and a new ball separation property introduced by Chen and Lin, are obtained.

When considering the weak instead of the norm topology, often the results do not depend on the norm of the space. We prove that if the dual of a Banach space X has C^*PCP, then either X is reflexive or NA^1 contains no (relative) weak open sets of the unit sphere. The same conclusion holds for every separable Banach space. Finally, we present a renorming result for weakly compactly generated Banach spaces and deduce some consequences in the line of previous results.

Closely related to this note is the remarkable paper of G. Godefroy [10].

3. The results

We begin our discussion by showing that a simple convexity property of NA^1 implies reflexivity. Recall that a slice of the ball $B_{\|\cdot\|}$ is a set of the form $S(B_{\|\cdot\|}, f, \delta) = \{x \in B_{\|\cdot\|} : \ f(x) > \delta\}$ with $f \in X^* \setminus \{0\}$ and $\delta < ||f||^*$. Throughout, int C and ∂C denote respectively the interior and the border of a set $C \subset X$.

Lemma 3.1. *A Banach space X is reflexive provided its dual unit ball $B_{\|\cdot\|^*}$ contains a slice of norm attaining functionals.*

Proof. By assumption, there exist $T \in X^{**}$ and $0 < \delta < \|T\|^{**} = 1$ such that each functional of the slice $S \equiv S(B_{\|\cdot\|^*}, T, \delta)$ attains its norm. We take $x_0^* \in S$ with $1 - T(x_0^*)$ small enough so that

$$B = (S - x_0^*) \cap (-S + x_0^*) = (B_{\|\cdot\|^*} - x_0^*) \cap (B_{\|\cdot\|^*} + x_0^*).$$

Thus, B is the unit ball of an equivalent dual norm $|\cdot|^*$ on X^* and every functional of $S_{|\cdot|^*}$ attains its norm. Indeed, if $x^* \in S_{|\cdot|^*}$, then $x^* + x_o^* \in NA^1_{\|\cdot\|}$ or $-x^* + x_o^* \in NA^1_{\|\cdot\|}$. Suppose the first assertion: there is $x \in S_{\|\cdot\|}$ such that $(x^* + x_0^*)(x) \geq y^*(x)$, for every $y^* \in B_{\|\cdot\|^*}$. Therefore, $x^*(x) \geq y^*(x)$, for every $y^* \in S - x_0^*$. Thus, $x^*(x/|x|) \geq y^*(x/|x|)$ for every $y^* \in B_{|\cdot|^*}$ and x^* attains its norm at $x/|x|$. James' Theorem now allows us to conclude that X is reflexive. $\square$

The following result shows the weak* density of $S_{\|\cdot\|^*} \setminus NA$ in $S_{\|\cdot\|^*}$ for every nonreflexive Banach space, thus extending the result given in [4, Lemma 11] for the separable case.

Proposition 3.2. *Given a Banach space X, either the set $C = S_{\|\cdot\|^*} \setminus NA$ is weak* dense in $S_{\|\cdot\|^*}$ or X is reflexive. Also, in the first case $\overline{\text{conv}}\,(C) = B_{\|\cdot\|^*}$.*

Proof. If we assume that $C = S_{\|\cdot\|^*} \setminus NA$ is not weak* dense, there exist $\{x_i\}_{i=1}^n \subset S_{\|\cdot\|}$ and $\{\delta_i\}_{i=1}^n \subset (0,1)$ such that the set $W = \{x^* \in B_{\|\cdot\|^*} : x^*(x_i) \geq \delta_i,\ i = 1,\dots,n\}$ has (relative) nonempty interior in $B_{\|\cdot\|^*}$ and $V = W \cap S_{\|\cdot\|^*} \subset NA$. Also, the set W is a bounded, weak* closed and convex set in X^*. Since $\partial W \subset V \cup \bigcup_{i=1}^n \{x^* \in W : x^*(x_i) = \delta_i\}$, we deduce that for every $x^* \in \partial W$ there is $x \in X \setminus \{0\}$ satisfying $x^*(x) \geq y^*(x)$ for every $y^* \in W$. It is not difficult to observe that the same property holds for the weak* closed, convex, bounded and symmetric set

$$U = (W - y_o^*) \cap (y_o^* - W),$$

with $0 \in \text{int } U$, whenever $y_0^* \in \text{int } W$. Hence, the set U is the unit ball of an equivalent dual norm $|\cdot|^*$ on X^* and, using James' Theorem again, we obtain that X is reflexive.

Suppose now that X is nonreflexive and $\overline{\text{conv}}\,(C)$ is a proper subset of $B_{\|\cdot\|^*}$. Then, there is a slice S of $B_{\|\cdot\|^*}$ such that $S \cap \overline{\text{conv}}\,(C) = \emptyset$. Thus, Lemma 3.1 implies that X is reflexive, a contradiction. □

With Lemma 3.1 in hand, we are ready to prove the next result concerning two well known geometrical properties. To do so, we start by recalling some definitions. A point $x \in S_{\|\cdot\|}$ is said to be a *denting point* of $B_{\|\cdot\|}$ if for every $\varepsilon > 0$ there exist $f \in X^*$ and $0 < \delta < f(x)$ such that diam $S(B_{\|\cdot\|}, f, \delta) < \varepsilon$. A point $f \in S_{\|\cdot\|^*}$ is said to be a *weak* denting point* of $B_{\|\cdot\|^*}$ if for every $\varepsilon > 0$ there exist $x \in X$ and $0 < \delta < f(x)$ such that diam $S(B_{\|\cdot\|^*}, x, \delta) < \varepsilon$. A Banach space is said to have the *Mazur Intersection Property* [18] if every bounded closed convex set is an intersection of closed balls. Analogously, a dual Banach space has the Weak* Mazur Intersection Property [8] if every weak* compact convex set is an intersection of closed dual balls. A Banach space $(X, \|\cdot\|)$ has the Mazur Intersection Property if, and only if, the set of weak* denting points of $B_{\|\cdot\|^*}$ is dense in $S_{\|\cdot\|^*}$. Similarly, a dual Banach space $(X^*, \|\cdot\|^*)$ has the weak* Mazur Intersection Property if, and only if, the set of denting points of $B_{\|\cdot\|}$ is dense in $S_{\|\cdot\|}$ [8].

Proposition 3.3. *Let X be a Banach space. The following are equivalent:*

(i) *X is reflexive.*
(ii) *X admits an equivalent norm $\|\cdot\|$ with the Mazur Intersection Property and $NA_{\|\cdot\|}$ has nonempty interior.*
(iii) *X admits an equivalent norm $|\cdot|$ such that $(X^{**}, |\cdot|^{**})$ has the Weak* Mazur Intersection Property and $NA_{|\cdot|}$ has nonempty interior.*

Proof. Every reflexive Banach space can be equivalently renormed with a dual locally uniformly rotund norm [25], and thus with a Fréchet differentiable norm. It is well known that every Fréchet differentiable norm has the Mazur Intersection Property [6], so (i) ⇒ (ii). Using the characterization mentioned above, it is clear that (ii) ⇒ (iii), so it only remains to prove (iii) ⇒ (i). Indeed, if the set of denting points of $B_{|\cdot|^*}$ is dense in $S_{|\cdot|^*}$ and $NA_{|\cdot|}$ contains an open set, we can find a slice of the unit ball of norm attaining functionals and apply Lemma 3.1. □

Denote by $N(X)$ the metric space of all equivalent norms on X endowed with the uniform convergence on bounded sets.

Corollary 3.4. *Let X be a nonreflexive Banach space with the Mazur Intersection Property. Then, the set $C = \{ |\cdot| \in N(X) : X^* \setminus NA_{|\cdot|^*}$ is dense in $X^* \}$ is residual.*

Proof. Note that the set of equivalent norms with the Mazur Intersection Property in a Banach space X, which is included in C, is empty or residual [7]. □

It seems to be unknown if having an equivalent norm with the Mazur Intersection Property is in fact more restrictive than having an equivalent norm so that its bidual norm has the Weak* Mazur Intersection Property. On the other hand, it is natural to ask if there exist other geometrical properties which might play the role of the Mazur Intersection Property in the previous results. Let us recall some definitions. A *point of continuity* of $C \subset X$ is a point at which the relative norm topology and the weak topology coincide on C. Analogously, a *point of weak* continuity* of $C \subset X^*$ is a point at which the relative norm topology and weak* topology agree on C. Finally, a *point of weak*-weak continuity* of $C \subset X$ is a point at which the relative weak* topology and weak topology coincide on C. Denote by $C(w, \|\cdot\|)$, $C(w^*, \|\cdot\|^*)$ and $C(w^*, w)$ the set of points of continuity, weak* continuity and weak*-weak continuity of $B_{\|\cdot\|}$ and $B_{\|\cdot\|^*}$, respectively. The following result is both a generalization and an alternative proof of Proposition 3.3.

Proposition 3.5. *Let X be a Banach space. Suppose that the set $C(w^*, \|\cdot\|^*)$ ($C(w^*, w)$) is dense in the norm (weak) topology in $S_{\|\cdot\|^*}$. Then, either X is reflexive or $S_{\|\cdot\|^*} \setminus NA$ is dense (weak dense, respectively) in $S_{\|\cdot\|^*}$.*

Proof. In the first case, suppose that $S_{\|\cdot\|^*} \setminus NA$ is not dense. Then, there is an $x_0^* \in C(w^*, \|\cdot\|^*)$ and a (relative) open neighbourhood U of x_0^* in $S_{\|\cdot\|^*}$ of norm attaining functionals. Since $x_0^* \in C(w^*, \|\cdot\|^*)$ there is a (relative) weak* open neighbourhood $V \subseteq U$ of x_0^* in $S_{\|\cdot\|^*}$. Now, apply Proposition 3.2. The second case is analogous. □

We turn now to a brief look at a new ball separation property introduced by Chen and Lin [3] which generalizes the Mazur Intersection Property. By definition, a Banach space X has *Property (II)* if for every bounded, closed and convex subset B in X, $B = \bigcap_{i\in I} K_i$, where, for every $i \in I$, $K_i = \overline{\text{conv}}\,(\bigcup_{i=1}^n B_i)$ for some closed balls $B_1, B_2, ..., B_n$ in X. They showed that a Banach space X has Property (II) if and only if $C(w^*, \|\cdot\|^*)$ is norm dense in $S_{\|\cdot\|^*}$. Also, they showed that the set of equivalent norms with Property (II) is either empty or residual.

Corollary 3.6. *A Banach space X is reflexive if, and only if, X admits an equivalent norm $\|\cdot\|$ with Property (II) and NA contains an open set.*

A Banach space X is *Hahn-Banach smooth* [24] if for every $x^* \in X^*$ there is a unique Hahn-Banach extension in X^{***}, that is, if $y \in X^{***}$ and $y_{|X} = x^*$ then $y = x^*$. This condition is equivalent to $S_{\|\cdot\|^*} = C(w^*, w)$ [9], so the possibility of applying Proposition 3.5 arises again. Notice also that it has been proved in [3] that the set of equivalent Hahn-Banach smooth norms is either empty or residual.

Corollary 3.7. *A Banach space X is reflexive if and only if it admits an equivalent Hahn-Banach smooth norm $\|\cdot\|$ so that $NA^1_{\|\cdot\|}$ contains a (relative) weak open set in the unit sphere.*

In what follows, we deal with conditions on a Banach space implying the weak density of $S_{\|\cdot\|^*} \setminus NA$ in the unit sphere, for every equivalent norm. Recall that a Banach space X has the *Convex Point of Continuity Property* ($CPCP$, for short) if every bounded closed convex subset of X has a point of continuity. Analogously, a dual Banach space X^* has the *Weak* Convex Point of Continuity Property* (C^*PCP, for short) provided every weak* compact subset of X^* has a point of weak* continuity.

Proposition 3.8. *Let X be a Banach space such that X^* has the C^*PCP. Then either $S_{\|\cdot\|^*} \setminus NA$ is weak dense in the unit sphere, or X is reflexive.*

Proof. Suppose that there is a (relative) weak open and convex set $U \subset B_{\|\cdot\|^*}$ such that its (norm) closure W satisfies $W \cap S_{\|\cdot\|^*} \subset NA^1$. We may assume that $0 \in U$. If not, take $x^* \in U$, $\|x^*\|^* < 1$, and consider the new equivalent dual norm $|||\cdot|||^*$ whose unit ball is $B = \left(-x^* + B_{\|\cdot\|^*}\right) \cap \left(x^* + B_{\|\cdot\|^*}\right)$. Observe that $W_1 = (-x + W) \cap (x - W)$ satisfies $W_1 \cap S_{|||\cdot|||^*} \subset NA^1_{|||\cdot|||^*}$. We denote this new norm $|||\cdot|||$ also by $\|\cdot\|$ and W_1 by W. We may assume that W is of the form

$$W = \left\{x^* \in B_{\|\cdot\|^*} : |F_i(x^*)| \le \delta_i,\ i = 1, ..., n\right\},$$

where $\{F_i\}_{i=1}^n \subset X^{**}$ and $\{\delta_i\}_{i=1}^n$ are positive numbers. By assumption, there is a point of weak* continuity y^* in the weak* closure of W. Since W is (norm) closed, y^* lies in W. Now if we knew that $|F_i(y^*)| < \delta_i$ for every $i = 1, \ldots, n$, then y^* would be a point of weak* continuity of $B_{\|\cdot\|^*}$, letting us conclude, by Proposition 3.2, that X is reflexive.

Otherwise we may assume that, for instance, $F_n(y^*) = \delta_n$. There exist $\{x_i\}_{i=1}^k \subset X$ and positive numbers $\{\gamma_i\}_{i=1}^k$ such that, if we define $V = \{x^* \in B_{\|\cdot\|^*} : x_i(x^*) \ge \gamma_i,\ i = 1, ..., n\}$, then $V \cap W$ has nonempty interior and $s = \inf_{V \cap W} F_n > -\delta_n$. Take $z^* \in \operatorname{int}(V \cap W)$ satisfying $F_n(z^*) < \frac{1}{3}\delta_n + \frac{2}{3}s$, and consider the equivalent dual norm $|\cdot|^*$ whose unit ball is

$$B_{|\cdot|^*} = (-z^* + V) \cap (z^* - V)\ .$$

We claim that the set

$$W_1 = \left\{x^* \in B_{|\cdot|^*} : |F_i(x^*)| \le \delta_i - |F_i(z^*)|,\ i = 1, \ldots, n-1\right\}$$

satisfies $W_1 \cap S_{|\cdot|^*} \subset NA^1_{|\cdot|}$. Iterating this argument, after at most n times, we obtain that the space is reflexive. We now carry out the details of the claim.

Clearly, for each $x^* \in W_1$, we have $z^* \pm x^* \in V$ and $|F_i(z^* \pm x^*)| \le \delta_i$, $i = 1, \ldots, n-1$. Also, it is easy to see that $\inf_{W_1} F_n \ge s - F_n(z^*)$ and, by symmetry, $\sup_{W_1} F_n \le F_n(z^*) - s$. Then

$$\begin{aligned} s = F_n(z^*) + (s - F_n(z^*)) &\le F_n(z^* \pm x^*) \\ &\le F_n(z^*) + (F_n(z^*) - s) < \frac{2}{3}\delta_n + \frac{1}{3}s\,, \end{aligned}$$

thus implying that $z^* \pm x^* \in V \cap W$. Finally, if $x^* \in W_1 \cap S_{|\cdot|^*}$, then $z^* + x^* \in \partial V \cap W$ (or $z^* - x^* \in \partial V \cap W$) and V is supported at the point $z^* + x^*$ (or $z^* - x^*$) by a functional of X. Therefore, $B_{|\cdot|^*}$ is supported at the point x^* by a functional of X. □

Remark 3.9. The above proof yields a result similar to the one given in Proposition 3.8, replacing C^*PCP by the following apparently weaker condition: every closed, convex and bounded set of X^* has a point of weak*-weak continuity.

Proposition 3.10. *Let X be a separable Banach space. Then either $S_{\|\cdot\|^*} \setminus NA$ is weak dense in $S_{\|\cdot\|^*}$, or X is reflexive.*

Proof. Suppose there is a weak open set $W \subset X$ such that $W \cap S_{\|\cdot\|^*} \neq \emptyset$ and $W \cap S_{\|\cdot\|^*} \subset NA^1$. We may assume, as in Proposition 3.8, that $0 \in W$. Then, there is a closed subspace $H \subset X^*$ of finite codimension such that $H \subset W$, and thus $H \subset NA$. Suppose that X is not reflexive and $S_{\|\cdot\|^*} \setminus NA$ is not weak dense. By Proposition 3.8, X is not Asplund, and hence X^* and so X^{**} are not separable. If $H = \bigcap_{i=1}^n \operatorname{Ker} x_i^{**}$, where $x_i^{**} \in X^{**}$, and $E = \overline{\operatorname{span}}(X, \{x_i^{**}\}_1^n)$, there is $F \in S_{\|\cdot\|^{**}}$ so that $0 < \theta < \operatorname{dist}(F, E)$. Let $\{x_k\}$ be dense in X, and for each k choose $x_k^* \in B_{\|\cdot\|^*}$ with

(a) $F(x_k^*) \geq \theta$,
(b) $x_k^*(x_i) = 0$ if $i \leq k$,
(c) $x_i^{**}(x_k^*) = 0$, $i = 1, \ldots, n$.

For every $x^* \in \overline{\operatorname{conv}}(x_k^*)$ we have $F(x^*) \geq \theta$, so $\|x^*\|^* \geq \theta$, and $\lim_k x_k^*(x) = 0$ if $x \in X$. If we apply Simons' inequality we obtain $y^* \in \overline{\operatorname{conv}}(x_k^*)$ that does not attain its norm. Notice that $y^* \in H \subset NA$, a contradiction. □

Corollary 3.11. *Let X be a Banach space such that the set $S_{\|\cdot\|^*} \setminus NA$ is not weak dense in $S_{\|\cdot\|^*}$. Then every separable quotient of X is reflexive.*

Proof. Let $Y \subset X$ be a closed subspace so that X/Y is separable. Consider the quotient norm on X/Y. Recall that if $\pi : X \longrightarrow X/Y$ is the canonical projection, then

$$\pi^* : (X/Y)^* \longrightarrow Y^\perp \subset X^*$$

is an isometry, where $Y^\perp = \{x^* \in X^* : x^*(y) = 0 \text{ for every } y \in Y\}$. Let W be a weak open and convex set such that $V = W \cap S_{\|\cdot\|^*}$ is non-empty and is included in NA^1. First, assume that $Y^\perp \cap V \neq \emptyset$. Note that if $y^* \in S_{\|\cdot\|^*} \cap Y^\perp$ attains its norm at $x \in S_{\|\cdot\|}$, then $y^*(\pi(x)) = y^*(x) = 1$. That is, y^* attains its norm at $\pi(x) \in S_{X/Y}$, $S_{X/Y}$ the unit sphere of X/Y. Now, we apply Proposition 3.10 to the separable space X/Y and deduce that it is reflexive.

If $Y^\perp \cap V = \emptyset$, we work with $Z = Y \cap \operatorname{Ker} x^*$, where $x^* \in V$. It is clear that X/Z is separable and $Z^\perp = \overline{\operatorname{span}}(Y^\perp, x^*)$, so $Z^\perp \cap V \neq \emptyset$. □

The following results concern weakly compactly generated spaces (WCG spaces, for short). Recall that a Banach space X is *WCG* if there is a weakly compact set in X which spans a dense set of X. These spaces can be renormed in a suitable manner, keeping the same set of norm attaining functionals. We use here an idea from [4].

Lemma 3.12. *Let X be a WCG Banach space and $x^* \in S_{\|\cdot\|^*}$. Then, for every $\lambda > 0$, there is a λ-isometric norm $|\cdot|$ such that*

(i) $NA_{|\cdot|} = NA_{\|\cdot\|}$,
(ii) *x^* is a extreme point of $B_{|\cdot|^*}$,*
(iii) *x^* is a point of continuity of the norm $|\cdot|^*$ whenever x^* is a point of continuity of the norm $\|\cdot\|^*$.*

Proof. By assumption, X is WCG, and thus the closed hyperplane $\operatorname{Ker} x^* = \{x \in X : x^*(x) = 0\}$ is WCG, too. Then, there is a weakly compact set $W \subset B_{\|\cdot\|^*}$, which we may suppose convex by the Krein-Šmulian Theorem, such that $\overline{\operatorname{span}}(W) = \operatorname{Ker} x^*$. Denote $C = \overline{\operatorname{conv}}(-K \cup K)$, which is convex, symmetric and weakly compact, too. Let $|\cdot|$ be the norm whose unit ball is $B_{|\cdot|} = B_{\|\cdot\|} + \lambda C$. Since C is weakly compact, we clearly have (i). To prove (ii), observe that the dual norm $|\cdot|^*$ has the following expression, for every $y^* \in X^*$:

$$
\begin{aligned}
|y^*|^* &= \sup\{y^*(y) + y^*(y') : y \in B_{\|\cdot\|},\, y' \in \lambda C\} \\
&= \sup\{y^*(y) : y \in B_{\|\cdot\|}\} + \sup\{y^*(y') : y' \in \lambda C\} \qquad (3.1)\\
&= ||y^*||^* + \lambda \sup_C y^*.
\end{aligned}
$$

Suppose now that there are $y^*,\, z^* \in X^*$ so that $\frac{1}{2}(y^* + z^*) = x^*$ and $|y^*|^* + |z^*|^* = 2|x^*|^*$. By convexity arguments and (3.1), we have

$$\sup_C y^* + \sup_C z^* - 2\sup_C x^* = 0.$$

Since $\sup_C x^* = 0$, $\sup_C y^* \geq 0$ and $\sup_C z^* \geq 0$, we deduce that $\sup_C y^* = \sup_C z^* = 0$. Thus, $y^* = z^* = x^*$.

Finally, let $\{x^*_\alpha\}$ be a net in X^* so that $|x^*_\alpha|^* = |x^*|^*$ and $\lim_\alpha x^*_\alpha = x^*$ in the weak topology. We just need prove that $\lim_\alpha ||x^*_\alpha||^* = ||x^*||^*$ to obtain $\lim_\alpha ||x^*_\alpha - x^*||^* = 0$. Since $|x^*|^* = ||x^*||^*$, the first assumption on $\{x^*_\alpha\}$ implies $||x^*_\alpha||^* \leq ||x^*||^*$ for every α. On the other hand, the second assumption on $\{x^*_\alpha\}$ yields $||x^*||^* \leq \liminf_\alpha ||x^*_\alpha||^*$. Hence, $\lim_\alpha ||x^*_\alpha||^* = ||x^*||^*$. □

With the above lemma, we gain insight into the very special nature of the set NA and the geometry of WCG Banach spaces. As a consequence, we derive the following proposition.

Proposition 3.13. *Let X be a WCG Banach space. If $C(w, \|\cdot\|^*)$ is dense (weak dense) in $S_{\|\cdot\|^*}$, then either $S_{\|\cdot\|^*} \backslash NA$ is dense (weak dense, respectively) in $S_{\|\cdot\|^*}$, or X is reflexive.*

Proof. Suppose there is a point $x^* \in C(w, \|\cdot\|^*)$ with a relative neighbourhood (weak neighbourhood) of norm attaining functionals in $S_{\|\cdot\|^*}$. By Lemma 3.12, we modify the norm so that x^* is also an extreme point of the unit ball $B_{|\cdot|^*}$ of the new norm $|\cdot|$, and thus, by [17], x^* is a denting point of $B_{|\cdot|^*}$. Now, observe that x^* has a relative neighbourhood (weak neighbourhood) of norm attaining functionals in $S_{|\cdot|^*}$, too. Hence, there is a slice of $B_{|\cdot|^*}$ of norm attaining functionals containing x^*. Now, apply Lemma 3.1 to obtain that X is reflexive. □

Proposition 3.14. *Let X be a WCG Banach space such that X^* has the CPCP. Then either the set $S_{\|\cdot\|^*} \setminus NA$ is weak dense in $S_{\|\cdot\|^*}$, or X is reflexive.*

To prove the proposition, it suffices to see that $C(w, \|\cdot\|^*)$ is weak dense in $S_{\|\cdot\|^*}$. This is a consequence of the following assertion, which is a generalization of [5, Lemma 3], later rediscovered in [13, Lemma 8], given for slices. As it seems to have some independent interest, we decide to isolate the argument in a lemma.

Lemma 3.15. *Consider $\{f_i\}_{i=1}^n \subset S_{\|\cdot\|^{**}}$, $\{\delta_i\}_{i=1}^n \subset (0,1)$ and $y \in B_{\|\cdot\|^*}$ such that $f_i(y) > 1 - \delta_i$. Then every point of continuity of $W = \{x^* \in B_{\|\cdot\|^*} : f_i(x^*) \geq 1 - \delta_i,\ i = 1\ldots, n\}$ is a point of continuity of $B_{\|\cdot\|^*}$.*

Proof. Assume that x^* is a point of continuity of W and let $\{x^*_\alpha\} \subset B_{\|\cdot\|^*}$ be a net such that $\lim_\alpha x^*_\alpha = x^*$ in the weak topology. Consider, for every α,

$$\varepsilon^i_\alpha = \max\left\{1 - \delta_i - f_i(x^*_\alpha)\,,\, 0\right\},$$

for $i = 1, \ldots, n$, and define

$$\varepsilon = \min \{ f_i(y) - 1 + \delta_i \, ; \; i = 1, \ldots, n \},$$
$$\lambda_\alpha = \max \{ \varepsilon_\alpha^i / \varepsilon \, ; \; i = 1, \ldots, n \},$$
$$z_\alpha^* = \lambda_\alpha y^* + (1 - \lambda_\alpha) \, x_\alpha^* \, .$$

It is clear that $\lim_\alpha \varepsilon_\alpha^i = 0$, for $i = 1, \ldots, n$. We may assume that $0 \leq \lambda_\alpha \leq 1$, since $\lim_\alpha \lambda_\alpha = 0$. It is straightforward to verify that $z_\alpha^* \in W$ and $\lim_\alpha z_\alpha^* = x^*$ in the weak topology. Thus, $\lim_\alpha ||z_\alpha^* - x^*|| = 0$ and $\lim_\alpha ||x_\alpha^* - x^*|| = 0$. □

4. Remarks

As far as we know, neither the proof of James' Theorem (for the nonseparable case) nor that of the Josefson–Nissenzweig Theorem seem to give insight into proving the weak density of $X^* \setminus NA$ in every nonreflexive Banach space. On the other hand, a wide class of Banach spaces can be renormed in such a way that NA contains an *norm dense open* set of the dual space:

(i) *Banach spaces with property* α. Is has been proved in [21] that every norm $\| \cdot \|$ with property α is such that $NA_{\|\cdot\|}$ contains a dense open set. The class of Banach spaces which can be renormed with property α includes those Banach spaces admitting a biorthogonal system with cardinality the density of the space [11]. Therefore, many Banach spaces failing the Radon-Nikodým Property are in this situation—for instance, the space $\ell_\infty(\Gamma)$ for every Γ.

(ii) *Banach spaces with the Mazur Intersection Property*. In this case, it is necessary to renorm the space as it is done in [20]. This class includes every Banach space X whose dual X^* has a fundamental biorthogonal system in $X^* \times X$ [15]. Let us mention also that the dual of a Banach space with the Mazur Intersection Property can fail the C^*PCP and, moreover, any hereditary isomorphic property, since it was proved in [16] that every Banach space can be complementably embedded into a Banach space with the Mazur Intersection Property.

These examples show the necessity of assuming further geometrical properties on nonreflexive Banach spaces to ensure that NA contains no open sets. They also point out the difficulty of keeping the structure of the set NA under renormings. However, Debs, Godefroy and Saint Raymond proved that in a separable Banach space $(X, \| \cdot \|)$ there is an equivalent Gâteaux differentiable norm $| \cdot |$ such that $NA_{\|\cdot\|} = NA_{|\cdot|}$ [4].

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